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ON THE LND CONJECTURE

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Abstract

Let k be a field of characteristic zero and $k^{[n]}$ the polynomial algebra in n variables over k. The LND conjecture concerning the images of locally nilpotent derivations arose from the Jacobian conjecture. We give a positive answer to the LND conjecture in several cases. More precisely, we prove that the images of rank-one locally nilpotent derivations of $k^{[n]}$ acting on principal ideals are MZ-subspaces for any $n \geq 2$, and that the images of a large class of locally nilpotent derivations of $k^{[3]}$ (including all rank-two and homogeneous rank-three locally nilpotent derivations) acting on principal ideals are MZ-subspaces.

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1. Introduction

Throughout the paper, k is a field of characteristic zero and $k^{[n]} := k[x_1, x_2, \ldots, x_n]$ the polynomial algebra in n variables over k. The well-known Jacobian conjecture asserts that any polynomial map of $k^{[n]}$ with nonzero constant determinant is invertible (see [1, 11]).

In the last decade, it was found that the Jacobian conjecture is closely related to the study of the images of derivations and differential operators of polynomial algebras. More precisely, the Jacobian conjecture is associated with the problem of whether the images of some derivations or differential operators are MZ-subspaces (see [14–17]). Now we recall the notion of MZ-subspaces, which is a natural generalisation of ideals.

DEFINITION 1.1 [18, 19]. Let A be a commutative k-algebra. A k-subspace M of A is called a Mathieu–Zhao subspace (MZ-subspace for short) if for each pair $f, g \in A$ with $f^m \in M$, for all $m \geq 1$, we have $gf^m \in M$, for all $m \gg 0$, that is, for all sufficiently large m.

The notion of MZ-subspace was first introduced by Zhao in [18] (named after Mathieu [8]) when he investigated the Jacobian conjecture, and originally named Mathieu

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subspace. Later, van den Essen proposed the change of name to Mathieu–Zhao subspace (see [12]).

In 2011, van den Essen *et al.* [15] asked the following question (called the EWZ problem): for a k-derivation D of $k^{[2]}$ with divergence zero, is the image $\operatorname{Im} D := D(k^{[2]})$ an MZ-subspace of $k^{[2]}$? They proved that the two-dimensional Jacobian conjecture is equivalent to the assertion that the EWZ problem has an affirmative answer when $1 \in \operatorname{Im} D$.

In 2017, the first author [9] proved that the EWZ problem has a negative answer in general by investigating monomial preserving derivations, and subsequently van den Essen and the first author [13] completely solved the problem of whether the images of monomial preserving derivations of $k^{[n]}$ are MZ-subspaces.

Since locally nilpotent derivations of $k^{[n]}$ are of divergence zero, it is natural to ask if the images of locally nilpotent derivations of $k^{[n]}$ are MZ-subspaces. In fact, Zhao [20] formulated the following LND conjecture.

Conjecture 1.2 (LND conjecture). *Let D be a locally nilpotent derivation of $k^{[n]}$. Then for any ideal I of $k^{[n]}$, the image $D(I)$ is an MZ-subspace of $k^{[n]}$.*

A more general conjecture called the LNED conjecture was proposed in [20]. It asserts that the same conclusion holds even if we replace $k^{[n]}$ by any commutative k-algebra, and includes a variant of the conjecture where derivations are replaced by ϵ-derivations. However, even the LND conjecture is still open for any $n \geq 2$.

We focus on the LND conjecture. In 2017, Zhao [20] proved that the LND conjecture holds for $k^{[1]}$. Van den Essen *et al.* [15] proved that the image $\operatorname{Im} D := D(k^{[2]})$ of any locally finite derivation D of $k^{[2]}$ is an MZ-subspace using the technique of Newton polytopes. Liu, Zeng and the first author [7] proved that the LND conjecture holds for principal ideals and some other ideals of $k^{[2]}$. In 2020, Liu and the first author [6] proved that the images of linear locally nilpotent derivations of $k^{[3]}$ are MZ-subspaces using the technique of integrals. In 2021, Liu and the first author [10] proved that $\operatorname{Im} D$ is an MZ-subspace of $k^{[3]}$ for any rank-two or homogeneous rank-three locally nilpotent derivation D of $k^{[3]}$, by improving some results on local slice constructions. In conclusion, the LND conjecture was only verified for some special cases and remains unsolved in general for any dimension $n \geq 2$.

In this paper, we prove in Section 2 that the LND conjecture holds for rank-one locally nilpotent derivations acting on principal ideals of $k^{[n]}$ for any $n \geq 2$ using the technique of integrals. For higher-rank locally nilpotent derivations, the problem becomes more complicated. We prove by the technique of local slices in Section 3 that the images of a large class of locally nilpotent derivations of $k^{[3]}$ (including all rank-two and homogeneous rank-three locally nilpotent derivations) acting on principal ideals are MZ-subspaces; the crucial point is that for these derivations, we reduce the discussion successfully to a local slice. Whether the LND conjecture can be treated in this way for general locally nilpotent derivations deserves study, but this question seems hard.

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2. Images of rank-one locally nilpotent derivations acting on principal ideals

We begin by recalling some basic concepts and properties of locally nilpotent derivations.

Let A be a commutative k-algebra over a field k of characteristic zero. A k-derivation D of A is a k-linear map $D : A \to A$ that satisfies $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. A derivation D is locally nilpotent if for each $a \in A$, there exists some $m_a \geq 1$ such that $D^{m_a}(a) = 0$. For a derivation D, the kernel of D is $\ker D := \{a \in A \mid D(a) = 0\}$. For brevity, we use the following notation:

(1) $\mathrm{Der}_k(A) = \{D \mid D \text{ is a } k\text{-derivation of } A\}$;
(2) $\mathrm{LND}(A) = \{D \in \mathrm{Der}_k(A) \mid D \text{ is locally nilpotent}\}$.

When A is a k-domain and $0 \neq a \in A$, a derivation aD_1 is locally nilpotent if and only if $a \in \ker D_1$ and D_1 is locally nilpotent (see [11, Corollary 1.3.34]).

Now we focus on the LND conjecture (Conjecture 1.2). It is a standard technique (using Lefschetz's principle) to reduce the base field k to the complex field $\mathbb{C}$. For completeness, we give a proof here.

LEMMA 2.1. (1) *If the LND conjecture holds over $\mathbb{C}$, then it holds over any field k of characteristic zero.*
(2) *If the LND conjecture holds for principal ideals over $\mathbb{C}$, then it holds for principal ideals over k.*

PROOF. (1) Suppose the LND conjecture holds over $\mathbb{C}$. Let $k^{[n]} = k[x_1, \ldots, x_n]$, $D \in \mathrm{LND}(k^{[n]})$ and $I = (u_1, \ldots, u_s)$ be any ideal of $k^{[n]}$. Given $f, g \in k^{[n]}$ with $f^m \in D(I)$, for all $m \geq 1$, it suffices to show that $gf^m \in D(I)$, for all $m \gg 0$.

Let L be the extension field over $\mathbb{Q}$ generated by the coefficients of f, g, $D(x_i)$ for $1 \leq i \leq n$, and u_j for $1 \leq j \leq s$. By Lefschetz's principle (see [11, Lemma 1.1.13]), L can be seen as a subfield of $\mathbb{C}$. Then D induces a locally nilpotent derivation $\bar{D}$ on $\mathbb{C}^{[n]} := \mathbb{C}[x_1, \ldots, x_n]$ by $\bar{D}(x_i) = D(x_i)$ for each i. Denote by $\bar{I}$ the ideal of $\mathbb{C}^{[n]}$ generated by $u_1, u_2, \ldots, u_s$. Both D and $\bar{D}$ restrict to $L^{[n]} := L[x_1, \ldots, x_n]$.

One may verify that $D(I) \cap L^{[n]} = \bar{D}(\bar{I}) \cap L^{[n]}$. In fact, for any $u \in D(I) \cap L^{[n]}$, there exist $w_i = \sum_{\alpha_i} c_{\alpha_i} x^{\alpha_i} \in k^{[n]}$, $i = 1, 2, \ldots, s$, such that

$$u = D\Big(\sum_{i=1}^{s} u_i w_i\Big) = \sum_{i=1}^{s} D(u_i) w_i + \sum_{i=1}^{s} u_i D(w_i) \in L^{[n]},$$

and this equality is equivalent to a system of linear equations over k with solution c_{α_i}. Noticing that all coefficients of the system are in L, we may take the solution c_{α_i} in L, which implies that $u \in \bar{D}(\bar{I}) \cap L^{[n]}$. Hence, $D(I) \cap L^{[n]} \subseteq \bar{D}(\bar{I}) \cap L^{[n]}$. Similarly, $\bar{D}(\bar{I}) \cap L^{[n]} \subseteq D(I) \cap L^{[n]}$.

Since $f^m \in D(I) \cap L^{[n]} \subseteq \bar{D}(\bar{I})$, for all $m \geq 1$, and $\bar{D}(\bar{I})$ is an MZ-subspace, we have $gf^m \in \bar{D}(\bar{I}) \cap L^{[n]} \subseteq D(I)$, for all $m \gg 0$, as desired.

(2) The conclusion follows from the fact that when I is a principal ideal of $k^{[n]}$, $\bar{I}$ is a principal ideal of $\mathbb{C}^{[n]}$. □

LEMMA 2.2. *Let B be a commutative k-domain, $D = aD_1 \in \mathrm{LND}(B)$ and I an ideal of B. If $D_1(I)$ is an MZ-subspace of B, then so is $D(I)$.*

PROOF. Note that $D(I) = aD_1(I) = D_1(aI) \subseteq D_1(I)$. Let $f \in B$ be such that $f^m \in D(I)$ $(\subseteq D_1(I))$, for all $m \geq 1$. Certainly, $f \in D(I) = aD_1(I)$, say $f = af_1$ where $f_1 \in D_1(I)$. For any $g \in B$, since $D_1(I)$ is an MZ-subspace, $(gf_1)f^m \in D_1(I)$, for all $m \gg 0$, and thus $gf^{m+1} = agf_1f^m \in aD_1(I) = D(I)$, for all $m \gg 0$. Hence, $D(I)$ is an MZ-subspace. □

LEMMA 2.3. *Let $A[x]$ be a polynomial algebra in one variable over a k-domain A, I an ideal of $A[x]$ and $D = \partial_x$. If $f(x) \in D(I)$, then*

$$x \int_0^1 f(xt)\, dt \in A + I.$$

PROOF. Write $f(x) = \sum_i a_i x^i$, where $a_i \in A$. Since $\int x^i\, dx = x^{i+1}/(i+1) = x\int_0^1 (xt)^i\, dt$ for any $i \in \mathbb{N}$, we have

$$w := \int f(x)\, dx = \int \sum_i a_i x^i\, dx = x \int_0^1 \sum_i a_i (xt)^i\, dt = x \int_0^1 f(xt)\, dt.$$

Since $D(w) = f(x) \in D(I)$, it follows that $w \in \ker D + I$, that is, $x\int_0^1 f(xt)\, dt \in A + I$. □

We also need the following property of moments of polynomial functions.

LEMMA 2.4 [3]. *Let $a \neq b \in k$. If $f \in k^{[1]} = k[t]$ is such that $\int_a^b f(t)^m\, dt = 0$, for all $m \geq 1$, then $f = 0$.*

An n-tuple $f_1, f_2, \ldots, f_n$ in $k^{[n]} = k[x_1, \ldots, x_n]$ is called a system of coordinates if $k^{[n]} = k[f_1, f_2, \ldots, f_n]$. For a derivation D of $k^{[n]}$, the rank of D, denoted by $\mathrm{rank}(D)$, is the least integer $r \geq 0$ for which there exists a system of coordinates $f_1, f_2, \ldots, f_n$ of B such that $k[f_{r+1}, f_{r+2}, \ldots, f_n] \subseteq \ker D$. When $k = \bar{k}$, for a polynomial $f \in k^{[n]}$, we denote by $V(f)$ the set of zero points of f in the affine space $\mathbb{A}_k^n$.

THEOREM 2.5. *Let I be a principal ideal of $k^{[n]} = k[x_1, x_2, \ldots, x_n]$ and $D \in \mathrm{LND}(k^{[n]})$ with $\mathrm{rank}\, D = 1$. Then DI is an MZ-subspace of $k^{[n]}$.*

PROOF. It is well known that D is conjugate by an automorphism of $k^{[n]}$ to $a\partial_{x_1}$, where $a \in k[x_2, \ldots, x_n]$. So we may assume that $D = a\partial_{x_1}$. From Lemma 2.1, we may assume that $k = \mathbb{C}$, and from Lemma 2.2 that $D = \partial_{x_1}$.

Let $I = (u(x_1, \ldots, x_n))$ be a nonzero principal ideal of $k^{[n]}$. We may assume that $u(x_1, \ldots, x_n) \notin k[x_2, \ldots, x_n]$ for otherwise, $\partial_{x_1}(I) = I$ is an MZ-subspace.

If $u(x_1, \ldots, x_n)$ has a divisor in $k[x_2, \ldots, x_n]\backslash k$, say $u(x_1, \ldots, x_n) = b\tilde{u}(x_1, \ldots, x_n)$, $b \in k[x_2, \ldots, x_n]\backslash k$, then

$$\partial_{x_1} I = \partial_{x_1}(u(x_1, \ldots, x_n)k^{[n]}) = b\partial_{x_1}(\tilde{u}(x_1, \ldots, x_n)k^{[n]}).$$

So we may assume $u(x_1, \ldots, x_n)$ has no divisor in $k[x_2, \ldots, x_n]\backslash k$ due to Lemma 2.2.

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(i) Suppose that $u(x_1,\dots,x_n)$ has at least two distinct roots over $\overline{k(x_2,\dots,x_n)}$, say a and b. For any $h\in\partial_{x_1}I$, there exists $v(x_1,\dots,x_n)\in k^{[n]}$ such that $h=\partial_{x_1}(u(x_1,\dots,x_n)v(x_1,\dots,x_n))$, and then

$$\int_a^b h\,dx_1=\int_a^b \partial_{x_1}(u(x_1,\dots,x_n)v(x_1,\dots,x_n))\,dx_1=u(x_1,\dots,x_n)v(x_1,\dots,x_n)\Big|_a^b=0.$$

Let $f\in k^{[n]}$ be such that $f^m\in\partial_{x_1}I$, for all $m\ge 1$. Then, $\int_a^b f^m\,dx_1=0$, for all $m\ge 1$, and by Lemma 2.4, we have $f=0$. Then for any $g\in k^{[n]}$, we have $0=gf^m\in\partial_{x_1}I$, for all $m\ge 1$. Thus, $\partial_{x_1}I$ is an MZ-subspace of $k^{[n]}$.

(ii) Suppose that $u(x_1,\dots,x_n)$ does not have a nonzero root over $\overline{k(x_2,\dots,x_n)}$. Then, $u(x_1,\dots,x_n)=cx_1^n$, where $c\in k\backslash\{0\}$. In this case, $\partial_{x_1}I=x_1^{n-1}k^{[n]}$ is an ideal and thus an MZ-subspace.

Following items (i) and (ii), we now assume that $u(x_1,\dots,x_n)$ has a unique root over $\overline{k(x_2,\dots,x_n)}$ and the root is nonzero. Then one may verify that

$$u(x_1,\dots,x_n)=(c_1x_1-c_2)^p,$$

where $c_1,c_2\in k[x_2,\dots,x_n]\backslash 0$ and c_1,c_2 are coprime. When $c_1\in k$, we have $\partial_{x_1}I=(x_1-c_1^{-1}c_2)^{p-1}k^{[n]}$ is an MZ-subspace, so we assume that $c_1\notin k$.

Let c_3 be the square-free part of c_1. Then for $f\in k^{[n]}\backslash\{0\}$,

$$\tilde{s}(f):=\min\{i\in\mathbb{Z}\mid c_3^if\in k[c_1x_1,x_2,\dots,x_n]\}$$

exists.

Claim 1. If $f\in k^{[n]}$ such that $f^m\in\partial_{x_1}I$, for all $m\ge 1$, then $\tilde{s}(f)<0$.

Suppose that $f\in k^{[n]}$ such that $f^m\in\partial_{x_1}I$, for all $m\ge 1$, and that $t:=\tilde{s}(f)\ge 0$. Let $\tilde{f}=c_3^tf$. Then, $\tilde{f}^m\in\partial_{x_1}I$, for all $m\ge 1$, and $\tilde{s}(\tilde{f})=0$. As $\tilde{s}(\tilde{f})\le 0$, $\tilde{f}$ has the form $\sum_i a_i(c_1x_1)^i$, where $a_i\in k[x_2,\dots,x_n]$.

By Lemma 2.3, since $\tilde{f}^m\in\partial_{x_1}I$, for all $m\ge 1$,

$$x_1\int_0^1 \tilde{f}(x_1t)^m\,dt\in k[x_2,\dots,x_n]+I=k[x_2,\dots,x_n]+(c_1x_1-c_2)^pk^{[n]}.$$

Taking $x_1=c_2/c_1$, we obtain $c_2\int_0^1\tilde{f}(c_2/c_1t)^m\,dt\in c_1k[x_2,\dots,x_n]$, that is,

$$c_2\int_0^1\Big(\sum_i a_i(c_2t)^i\Big)^m dt\in c_1k[x_2,\dots,x_n].$$

For any $(\xi_2,\dots,\xi_n)\in V(c_1)\backslash V(c_2)$,

$$\int_0^1\Big(\sum_i a_i(\xi_2,\dots,\xi_n)(c_2(\xi_2,\dots,\xi_n)t)^i\Big)^m dt=0\quad\text{for all } m\ge 1.$$

By Lemma 2.4, $\sum_i a_i(\xi_2,\dots,\xi_n)(c_2(\xi_2,\dots,\xi_n)t)^i=0$, and thus $a_i(\xi_2,\dots,\xi_n)=0$ for all i.

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Hence, $V(c_1)\backslash V(c_2) \subseteq V(a_i)$. From $\gcd(c_1, c_2) = 1$, it follows that $V(c_1)\backslash V(c_2)$ is dense in $V(c_1)$ for the Zariski topology. Therefore,

$$V(c_1) = \overline{V(c_1)\backslash V(c_2)} \subseteq V(a_i).$$

Consequently, $c_3 \mid a_i$ for all i. This contradicts $\tilde{s}(\tilde{f}) = 0$, so Claim 1 has been proved.

Claim 2. If $h \in k^{[n]}$ such that $x_1 h \in k[c_1x_1, x_2, \ldots, x_n]$ and $(c_1x_1 - c_2)^{p-1} \mid h$, then $h \in \partial_{x_1} I$.

Suppose that $h \in k^{[n]}$ such that $x_1 h \in k[c_1x_1, x_2, \ldots, x_n]$. Then,

$$h \in c_1 k[c_1x_1, x_2, \ldots, x_n] = c_1 k[c_1x_1 - c_2, x_2, \ldots, x_n].$$

Suppose in addition that $(c_1x_1 - c_2)^{p-1} \mid h$. Then,

$$h \in c_1(c_1x_1 - c_2)^{p-1} k[c_1x_1 - c_2, x_2, \ldots, x_n].$$

Note that when $j \geq p - 1$,

$$c_1(c_1x_1 - c_2)^j = \partial_{x_1}\Big(\frac{1}{j+1}(c_1x_1 - c_2)^{j+1}\Big) \in \partial_{x_1} I.$$

It follows that $h \in \partial_{x_1} I$. Thus, Claim 2 has been proved.

Finally, given $f, g \in k^{[n]}$ with $f^m \in \partial_{x_1} I$, for all $m \geq 1$, we obtain by Claim 1 that $x_1 g f^m \in k[c_1x_1, x_2, \ldots, x_n]$, for all $m \geq \max\{\tilde{s}(x_1 g), 1\}$. Combining this with

$$f \in \partial_{x_1} I \subseteq (c_1x_1 - c_2)^{p-1} k^{[n]},$$

we obtain by Claim 2 that $g f^m \in \partial_{x_1} I$, for all $m \gg 0$. So $\partial_{x_1} I$ is an MZ-subspace of $k^{[n]}$. □

3. Images of locally nilpotent derivations acting on ideals of $k^{[3]}$

To study the LND conjecture for higher-rank locally nilpotent derivations, we recall the notion and basic properties of local slices for locally nilpotent derivations (see [4, Sections 1.1, 2.2] for details).

Let B be a commutative k-domain over a field k of characteristic zero, $D \in \mathrm{Der}_k(B)$ and $A = \ker D$. An element $r \in B$ with $Dr \neq 0$ and $D^2 r = 0$ is called a local slice of D. Any nonzero $D \in \mathrm{LND}(B)$ has a local slice. A local slice r of D is called minimal if $A[r]$ is a maximal element in $\{A[r'] \mid r' \text{ is a local slice of } D\}$ with respect to inclusion. Denote by $\min D$ the set of all minimal local slices of D.

If B satisfies the ascending chain condition (ACC) on principal ideals, then any nonzero $D \in \mathrm{LND}(B)$ has a minimal local slice. Take any $r \in \min(D)$. Let $B_0 = A[r]$ and $s = r/Dr$. Then we have $B_{Dr} = A_{Dr}[s] = (B_0)_{Dr}$. Thus, for any $f \in B$, there exists some $i \in \mathbb{N}$ such that $(Dr)^i f \in B_0$. For $f \in B\backslash\{0\}$, define

$$s(f) := \min\{i \in \mathbb{Z} \mid (Dr)^i f \in B_0\}.$$

One may verify easily the following lemma.

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LEMMA 3.1. *Set* $D_0 := D|_{B_0}$. *Then,*

$$s(f) \le 0 \Longleftrightarrow f \in B_0,$$
$$s(f) < 0 \Longleftrightarrow f \in (Dr)B_0 = \mathrm{Im} D_0,$$

and

$$s(fg) \le s(f) + s(g),$$
$$s(f+g) \le \max\{s(f), s(g)\},$$
$$s((Dr)^j f) = s(f) - j, \quad j \in \mathbb{N}.$$

REMARK 3.2. Let B be a commutative k-domain. Given $D \in \mathrm{LND}(B)$, we consider the condition:

(*) If $f \in B$ is such that $f^m \in D(B)$, for all $m \ge 1$, then $s(f^l) < 0$ for some $l \ge 1$.

It was proved in [10, Theorems 3.9, 5.3] that most locally nilpotent derivations of $k^{[3]}$ satisfy the condition $(*)$; more precisely, all rank-one, rank-two and homogeneous rank-three derivations in $\mathrm{LND}(k^{[3]})$ satisfy condition $(*)$.

We will prove that the LND conjecture holds for all locally nilpotent derivations of $k^{[3]}$ with the condition $(*)$. The first step is to show that, for these derivations, we may reduce the LND conjecture to a local slice.

THEOREM 3.3. *Suppose that B is a commutative k-domain and I is an ideal of B. Let $0 \ne D \in \mathrm{LND}(B)$ which satisfies the condition $(*)$ and let $A = \ker D$. Take a minimal local slice r of D and let $B_0 = A[r]$, $D_0 := D|_{B_0}$ and $I_0 := I \cap B_0$. If $D_0 I_0$ is an MZ-subspace of B_0, then DI is an MZ-subspace of B.*

PROOF. Note that D_0 acts as $(Dr)\partial_r$ on $B_0 = A[r]$ and $\mathrm{Im}\, D_0 = (Dr)B_0$. First, we show that if $h \in \mathrm{Im}\, D_0$, then $h \in DI$ if and only if $h \in D_0 I_0$. The if part follows from the fact that $D_0 I_0 \subseteq DI$. Write $h = D_0(w)$, $w \in B_0$. Since $h \in DI$, we have $h = D_0(w) = D(v)$ for some $v \in I$. It follows that

$$v - w \in \ker D = A \subseteq B_0$$

and thus $v \in B_0 \cap I = I_0$. Hence, $h \in D_0 I_0$. The only if part is proved.

Given $f, g \in B$ with $f^m \in DI$, for all $m \ge 1$, it suffices to show that $gf^m \in DI$, for all $m \gg 0$.

Since $f^m \in DI \subseteq \mathrm{Im}\, D$, for all $m \ge 1$ and $D \in \mathrm{LND}(B)$ satisfies condition $(*)$, $s(f^{n_0}) < 0$ for some $n_0 \ge 1$ and thus $s((f^{n_0})^m) < 0$, for all $m \ge 1$. It follows that $(f^{n_0})^m \in (Dr)B_0 = \mathrm{Im}\, D_0$. Then by the discussion in the first paragraph, $(f^{n_0})^m \in D_0 I_0$, for all $m \ge 1$.

For $g \in B$, take an $n_1 \ge 1$ such that $s(gf^{n_0 n_1}) < 0$, and so $gf^{n_0 n_1} \in (Dr)B_0 \subseteq B_0$. Then, because $D_0 I_0$ is an MZ-subspace of B_0, we have

$$g(f^{n_0})^{m+n_1} = (gf^{n_0 n_1})(f^{n_0})^m \in D_0 I_0 \subseteq DI, \quad \text{for all } m \gg 0.$$

Therefore, there exists $m_0 \ge 1$ such that $g(f^{n_0})^m \in DI$, for all $m \ge m_0$.

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For each $j = 1, 2, \ldots, n_0 - 1$, replacing g by gf^j, we see that there exists $m_j \geq 1$ such that $gf^j(f^{n_0})^m \in DI$, for all $m \geq m_j$. Let $\overline{m} := \max\{m_j \mid 0 \leq j \leq n_0 - 1\}$. Then,

$$gf^{n_0 m + j} = gf^j(f^{n_0})^m \in DI, \quad \text{for all } m \geq \overline{m} \text{ and } 0 \leq j \leq n_0 - 1.$$

Then $gf^m \in DI$, for all $m \geq n_0\overline{m}$. Therefore, $\operatorname{Im} D$ is an MZ-subspace of B. □

A gcd-domain is a commutative domain for which any two elements have a greatest common divisor (gcd), that is, there is a unique minimal principal ideal containing the ideal generated by the two elements. In a gcd-domain, the gcd of finitely many elements exists and is unique up to a unit factor. A polynomial ring $A[x]$ over a gcd-domain A is still a gcd-domain (see [5, page 172]). A gcd-domain is a Schreier domain (see [2, Theorem 2.4]). A Schreier domain is an integrally closed commutative domain of which every element f is primal: whenever $f \mid gh$, f can be written as $f = f_1 f_2$ such that $f_1 \mid g$ and $f_2 \mid h$.

THEOREM 3.4. *Suppose that B is a k-gcd-domain and $I = Bu$ is a principal ideal of B. Let $0 \neq D \in \mathrm{LND}(B)$ and $A = \ker D$. Suppose that r is a minimal local slice of D, and let $B_0 = A[r]$. If $\gcd(u, Dr) = 1$, then $I_0 := I \cap B_0$ is also a principal ideal of B_0.*

PROOF. Suppose that $\gcd(u, Dr) = 1$. The case $u = 0$ is trivial, so assume that $u \neq 0$. Since $A = \ker D$ is factorially closed, it does not matter if we take the gcd of elements of A over A or over B: the result will be the same. Furthermore, A is a gcd-domain as well. Consequently, $B_0 = A[r]$ is also a gcd-domain. From $\gcd(u, Dr) = 1$, it follows that $t := s(u) \geq 0$. As $(Dr)^t u \in B_0$, we can write $(Dr)^t u = \sum_i \tilde{a}_i r^i$ with $\tilde{a}_i \in A$. Since $(Dr)^t \in A$ as well, we infer that $f := \gcd((Dr)^t, \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots) \in A$. We can write $\hat{u} := f^{-1}(Dr)^t u = \sum_i a_i r^i$ with $a_i = f^{-1}\tilde{a}_i \in A$.

We show that $\gcd(\hat{u}, Dr) = 1$ over $B_0 = A[r]$. Suppose that $d \mid \gcd(\hat{u}, Dr)$ over $A[r]$. As $Dr \in A$ and A is factorially closed, it follows that $d \in A$. So $d \mid \hat{u}$ over $A[r]$ gives $d \mid a_i$ over A for all i, and therefore $df \mid \tilde{a}_i$ over A for all i. Since $d \mid Dr$ and $\gcd(u, Dr) = 1$ over B, we infer that $\gcd(u, d) = 1$ over B. So by way of primality of d, we infer from $d \mid \hat{u} = (f^{-1}(Dr)^t)u$ over B that $d \mid f^{-1}(Dr)^t$ and $df \mid (Dr)^t$ over B. As $(Dr)^t \in A$ and A is factorially closed, $df \mid (Dr)^t$ over A. So $df \mid \gcd((Dr)^t, \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots) = f$ and $d \mid 1$ over A. Hence, $\gcd(\hat{u}, Dr) = 1$ over B_0.

Let $b \in I_0 = I \cap B_0$, and write $b = uv$ with $v \in B$. Take $t' \in \mathbb{N}$ such that $t' \geq s(v)$. Then, $(Dr)^t u = \hat{u} f$ and $(Dr)^{t'} v \in B_0$, so $(Dr)^{t+t'} b \in \hat{u} f B_0$. Hence, $\hat{u} \mid (Dr)^{t+t'} b$ over B_0. Since $\gcd(\hat{u}, Dr) = 1$ over B_0, we infer by way of primality of $\hat{u}$ and induction that $\hat{u} \mid b$ over B_0. So $I_0 \subseteq B_0\hat{u}$. However, $B_0\hat{u} \subseteq B_0$ and $B_0\hat{u} = B_0(f^{-1}(Dr)^t)u \subseteq I$. So $I_0 = B_0\hat{u}$ is a principal ideal of B_0. □

THEOREM 3.5. *Suppose that $B = k^{[3]} = k[x, y, z]$, I is a principal ideal of B, and $D \in \mathrm{LND}(B)$ satisfies the condition $(*)$. Then DI is an MZ-subspace of B.*

PROOF. By Miyanishi's theorem [4, Theorem 5.1], there exist $F, G \in B$ such that $A := \ker D = k[F, G] \cong k^{[2]}$. Take a minimal local slice r of D and let $B_0 = A[r]$ and $D_0 := D|_{B_0}$. Then, $B_0 = k[F, G, r] \cong k^{[3]}$ and D_0 acts as $(Dr)\partial_r$ on $B_0 = A[r]$, where $Dr \in A$.

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Let $I = Bu$ and $I_0 = I \cap B_0$. If u has a divisor in $A \backslash k$, say $u = au_1$ for some $a \in k[F, G] \backslash k$ and $u_1 \in B$. Then,

$$DI = D(uB) = D(au_1B) = aD(u_1B).$$

By Lemma 2.2, we may assume that u has no divisor in $A \backslash k$. Since A is factorially closed and $Dr \in A$, we have $\gcd(u, Dr) \in A$. Hence, $\gcd(u, Dr) = 1$. Then by Theorem 3.4, I_0 is a principal ideal of B_0. It follows by Theorem 2.5 that D_0I_0 is an MZ-subspace of B_0. Therefore, from Theorem 3.3, DI is an MZ-subspace of B. □

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