Cauchy and his modern rivals

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In 1879, the Oxford mathematician Charles Dodgson (1832-1898) published *Euclid and his modern rivals*. This book compared the work of contemporary geometers to that of the Greek master from two millennia earlier. Dodgson, best known by his pen name of 'Lewis Carroll', much preferred Euclid. For him, the ancients outshone the moderns.

The present article also discusses mathematical works from different eras, and so a tweak of Dodgson's title seemed appropriate. Unlike him, however, we shall not take sides, for all the mathematics we consider – spoiler alert! – is worthy of celebration.

In the firmament of mathematical analysis, no star shines brighter than that of Augustin-Louis Cauchy (1789-1857). It was he who brought logical rigour to the calculus and in the process changed the character of analysis for ever. This change was best reflected in *Cours d'analyse*, his groundbreaking text of 1821.

In what follows, we shall look at two selections from that book, a pair of inequalities listed as theorems 16 and 17 of Note II. This 'Note', which we might call an Appendix or Addendum, was titled 'On formulas that result from the use of the signs > or <, and on averages among several quantities'. The title may be awkward, but the inequalities are timeless [1].

Many would argue that the greatest inequality of all dates back to the Greeks. As Proposition 20 in Book I of his *Elements*, Euclid proved:

> In any triangle, two sides taken together in any manner
> are greater than the remaining one.

We know this as the triangle inequality, and its significance is impossible to overstate. Euclid's proof, built from a modest collection of postulates and common notions, is a beautiful exercise in logic. The triangle inequality was a geometric one, established two thousand years before 'modern' mathematics arose in the decades after 1600.

Perhaps the most important inequality from the 17th century was due to Jakob Bernoulli (1654-1705) and appeared as Proposition IV of his *Tractatus de seriebus infinitis* [2]. Although barely recognisable in its original form, the result is now stated as:

$$\text{If } x > 1 \text{ and } n \text{ is a whole number, then } (1 + x)^n \geqslant 1 + nx.$$

This is easily proved by mathematical induction. Note that, in contrast to the triangle inequality, Bernoulli's was an analytic one.

Nonetheless, stand-alone inequalities were hardly the rage before Cauchy's time. For instance, Leonhard Euler (1707-1783) included

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relatively few among his thousands of published pages. Euler preferred *equalities*, especially those of cosmic significance like

$$e^{ix} = \cos x + i \sin x \text{ or } 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = \frac{\pi^2}{6} \text{ or } V + F = E + 2.$$

By contrast, modern analysts have at their fingertips a host of inequalities: Hölder's, Jensen's, Hardy's, Minkowski's, Callebaut's and more. Many are discussed in *Inequalities* by G. H. Hardy, J. E. Littlewood and George Pólya, a remarkable book by three of the 20th century's most famous analysts [3]. But among these many results, none stands taller than the two from Cauchy that are the focus of this Article:

Cauchy's inequality: If a_i and b_i are real numbers, then

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2},$$

with equality if, and only if, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots = \frac{a_n}{b_n}$.

Arithmetic/geometric mean inequality: If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \leqslant \frac{a_1 + a_2 + \ldots + a_n}{n},$$

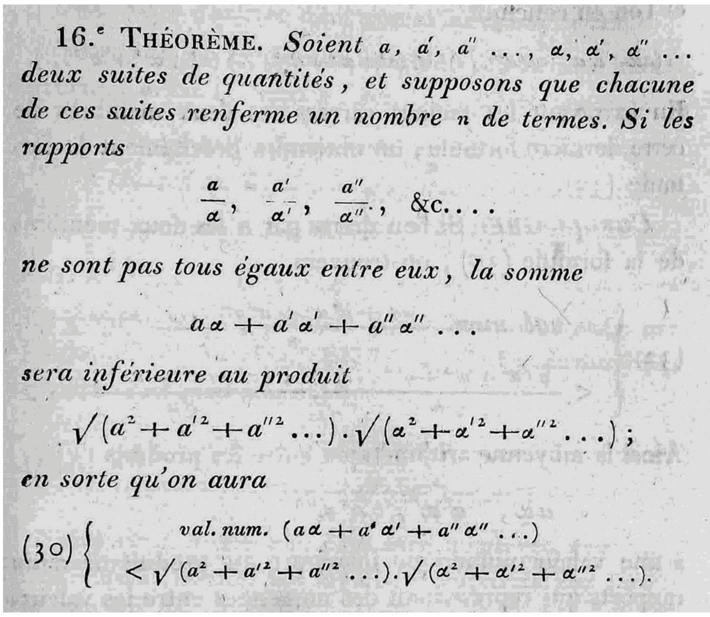
with equality, if and only if, $a_1 = a_2 = \ldots = a_n$.

The first of these involves sums and products of two finite sequences. The second features a single finite sequence of positive terms and is always accompanied by this terminology: the right-hand expression is the 'arithmetic mean', and the left-hand one is the 'geometric mean'. The result is thus known as the 'AM/GM inequality'.

We shall consider Cauchy's derivations of these results. Neither is difficult. Both are classic. Over time, other mathematicians concocted their own proofs, and so Cauchy acquired 'modern rivals'. It is in this spirit that we present a second, entirely different demonstration of each inequality.

Cauchy's inequality

In his book, *The Cauchy-Schwarz master class*, J. Michael Steele took a deep look at Cauchy's inequality. Steele's enthusiasm was evident in his introductory observation: 'There is no doubt that this is one of the most widely used and most important inequalities in all of mathematics' [4]. In Figure 1 we see its debut in *Cours d'analyse*.



16.^e THÉORÈME. *Soient* $a, a', a'' \ldots, \alpha, \alpha', \alpha'' \ldots$ *deux suites de quantités, et supposons que chacune de ces suites renferme un nombre* n *de termes. Si les rapports*

$$\frac{a}{\alpha}, \frac{a'}{\alpha'}, \frac{a''}{\alpha''}, \&c. \ldots$$

ne sont pas tous égaux entre eux, la somme

$$a\alpha + a'\alpha' + a''\alpha'' \ldots$$

sera inférieure au produit

$$\sqrt{(a^2 + a'^2 + a''^2 \ldots)} \cdot \sqrt{(\alpha^2 + \alpha'^2 + \alpha''^2 \ldots)};$$

en sorte qu'on aura

$$(30) \left\{ \begin{array}{l} val.\ num.\ (a\alpha + a'\alpha' + a''\alpha'' \ldots) \\ < \sqrt{(a^2 + a'^2 + a''^2 \ldots)} \cdot \sqrt{(\alpha^2 + \alpha'^2 + \alpha''^2 \ldots)}. \end{array} \right.$$

FIGURE 1

Before considering Cauchy's argument, we offer a few words about his notation and terminology. First, mathematicians in 1821 used neither sigmas nor subscripts (as we did above). Rather, Cauchy introduced his two sequences as

$$a, a', a'', a''', \ldots \text{ and } \alpha, \alpha', \alpha'', \alpha''', \ldots,$$

where each of these 'contains a number n of terms'. Although his reasoning was sound, his notation was problematic. After all, the twenty-fifth term of such a sequence would be written as $a^{\prime}$, a notation likely to induce eye-strain in an eagle.

Second, the critical inequality, labelled (30) in Figure 1, started with the abbreviation '*val. num*'. This meant 'numerical value', the equivalent of our 'absolute value'. Such terminology, although unfamiliar, holds a certain archaic charm.

Notational issues aside, we can get to the heart of Cauchy's argument by limiting ourselves to the case where $n = 3$. That is, we introduce sequences of real numbers a, b, c and A, B, C. Cauchy's inequality then becomes

Theorem 1: $\left| aA + bB + cC \right| \leqslant \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{A^2 + B^2 + C^2}.$

Proof: Cauchy began by looking at the *square* of the left-hand side, $(aA + bB + cC)^2$. To this he added 'the numerators of the fractions which represent the squares of the differences between the ratios … combined in all possible manners'. By this he meant that we first subtract all pairs of fractions

$$\frac{a}{A} - \frac{b}{B} = \frac{aB - bA}{AB}, \qquad \frac{a}{A} - \frac{c}{C} = \frac{aC - cA}{AC}, \qquad \frac{b}{B} - \frac{c}{C} = \frac{bC - cB}{BC} \qquad (1)$$

and add the squares of their numerators to $(aA + bB + cC)^2$. This generates an inequality, which is then expanded and simplified as follows:

$$(aA + bB + cC)^2 \leqslant (aA + bB + cC)^2 + (aB - bA)^2 + (aC - cA)^2 + (bC - cB)^2 \qquad (2)$$

$$= a^2A^2 + b^2B^2 + c^2C^2 + a^2B^2 + b^2A^2 + a^2C^2 + c^2A^2 + b^2C^2 + c^2B^2$$

$$= \left(a^2 + b^2 + c^2\right) \cdot \left(A^2 + B^2 + C^2\right).$$

Taking square roots of both sides yields

$$|aA + bB + cC| \leqslant \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{A^2 + B^2 + C^2},$$

which is Cauchy's inequality for $n = 3$. This argument generalises to any n.

Often inequalities are accompanied by an 'equality condition'. That is, they establish in general that $P \leqslant Q$ and then give a necessary and sufficient condition under which $P = Q$. This provides a kind of logical completion to the matter at hand.

To his credit, Cauchy found the condition under which his inequality becomes an equation. In the original notation of Figure 1, this occurs if, and only if, the ratios $\frac{a'}{\alpha'}, \frac{a''}{\alpha''}, \frac{a'''}{\alpha'''}, \ldots$ are all equal.

To see why (reverting to our notation above), we first assume that the fractions $\frac{a}{A}, \frac{b}{B}$ and $\frac{c}{C}$ are not all equal. Then in (1) at least one pair of fractional differences is non-zero, and so, in (2), at least one of the three addends is positive. Hence the inequality is strict.

On the other hand, suppose all these fractions are equal. In that case, let $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = r$. Then the left-hand side of Cauchy's inequality becomes

$$|aA + bB + cC| = |rA^2 + rB^2 + rC^2| = |r|\left(A^2 + B^2 + C^2\right),$$

and its right-hand side is

$$\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{A^2 + B^2 + C^2} = \sqrt{r^2A^2 + r^2B^2 + r^2C^2} \cdot \sqrt{A^2 + B^2 + C^2}$$

$$= |r|\left(A^2 + B^2 + C^2\right) \text{ as well.}$$

So we have equality.

In this way, Cauchy not only established the general inequality but identified the condition under which it becomes an equation. It was splendid work for 1821.

Over time, the result would be extended to other realms. It has a counterpart in the complex numbers, where the role of absolute value is

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played by ‘modulus’. Generalised to n-dimensional space, and indeed to any inner-product space, it appears as a key theorem of linear algebra.

A modern rival

Later in the nineteenth century, Hermann Amandus Schwarz (1843-1921) sought to extend Cauchy's inequality from the realm of sums to that of integrals. In the process, he hit upon a new line of attack, one that transferred easily to the numerical world of Cauchy's original result [5]. Largely because of this, Schwarz's name has been hyphenated with that of his illustrious predecessor, so that analysts now talk of the ‘Cauchy-Schwarz inequality’ even though (as we have seen) Cauchy proved it long before Schwarz was born.

We begin with two elementary observations.

Lemma 1: If $A > 0$ and the quadratic polynomial $P(x) = Ax^2 + Bx + C$ has two different real roots, $r_1 < r_2$, then $P\left(\frac{r_1 + r_2}{2}\right) < 0$.

Proof: We factorise the quadratic as $P(x) = A(x - r_1)(x - r_2)$. Then

$$P\left(\frac{r_1 + r_2}{2}\right) = A\left(\frac{r_2 - r_1}{2}\right)\left(\frac{r_1 - r_2}{2}\right) < 0$$

because the fraction in the first parentheses is positive and that in the second is negative.

Lemma 2: If $P(x) = Ax^2 + Bx + C$ is as above, then $P(x) = 0$ has fewer than two different real solutions if, and only if, $B^2 - 4AC \leqslant 0$.

Proof: This is just the quadratic formula in action.

With these two preliminaries, we prove our theorem in the spirit of Schwarz.

Theorem 2: (Cauchy's inequality revisited) If a_i and b_i are real numbers, then

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2},$$

with equality if, and only if, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots = \frac{a_n}{b_n}$.

Proof: If $a_1 = a_2 = \dots = a_n = 0$ the result is immediate. Otherwise, define

$$P(x) = \sum_{i=1}^{n} (a_i x - b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)x^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right)x + \left(\sum_{i=1}^{n} b_i^2\right).$$

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In the notation of the lemmas above, we have $P(x) = Ax^2 + Bx + C$, where

$$A = \left(\sum_{i=1}^{n} a_i^2\right) > 0,\ B = -2\left(\sum_{i=1}^{n} a_i b_i\right) \text{ and } C = \left(\sum_{i=1}^{n} b_i^2\right).$$

From Lemma 1, the quadratic P cannot have two different real roots, for by definition $P(x) \geqslant 0$. Thus, by Lemma 2, we know that $B^2 - 4AC \leqslant 0$, i.e.

$$4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \leqslant 0.$$

Therefore $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$, and so $\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$. Moreover, we have equality precisely when $B^2 - 4AC = 0$, which implies that $P(x) = 0$ has a unique solution, say $x = r$. But then

$$0 = P(r) = \sum_{i=1}^{n} (a_i r - b_i)^2.$$

As a consequence, $a_i r = b_i$ and $\frac{b_i}{a_i} = r$ for all i. This of course guarantees that $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots = \frac{a_n}{b_n}$, which is the equality condition.

Cauchy's inequality has become a direct consequence of the quadratic formula.

The arithmetic mean/geometric mean inequality

In Figure 2 we see Cauchy's statement as it appeared in *Cours d'analyse*.

> 17.e THÉORÈME. *La moyenne géométrique entre plusieurs nombres* A, B, C, D, . . . *est toujours inférieure à leur moyenne arithmétique.*

FIGURE 2

His proof rested upon repeated use of

Lemma 3: If $x > 0$ and $y > 0$, then

$$xy \leqslant \left(\frac{x+y}{2}\right)^2. \tag{3}$$

Proof: This follows because

$$xy = \frac{x^2 + 2xy + y^2 - x^2 + 2xy - y^2}{4} = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 \leqslant \left(\frac{x+y}{2}\right)^2.$$

It is obvious that we have equality if, and only if, $\frac{1}{2}(x - y) = 0$, i.e. if, and only if, $x = y$.

Having established Lemma 3, we now move to Cauchy's proof.

Theorem 3: (The arithmetic mean/geometric mean inequality)
Proof: For $A > 0$ and $B > 0$, we take square roots of (3) to get $\sqrt{AB} \leqslant \frac{1}{2}(A + B)$.

This shows that the geometric mean of two positive numbers is less than or equal to their arithmetic mean.

Next, Cauchy moved to the case $n = 4$. He reasoned that

$$ABCD = (AB)(CD) = \sqrt{AB}^2 \cdot \sqrt{CD}^2$$

$$\leqslant \left(\frac{A + B}{2}\right)^2 \left(\frac{C + D}{2}\right)^2 \text{ by the case for } n = 2$$

$$= \left[\left(\frac{A + B}{2}\right)\left(\frac{C + D}{2}\right)\right]^2$$

$$\leqslant \left[\left(\frac{\frac{A+B}{2} + \frac{C+D}{2}}{2}\right)^2\right]^2 \text{ using (3) with } x = \tfrac{1}{2}(A + B) \text{ and } y = \tfrac{1}{2}(C + D)$$

$$= \left[\frac{A + B + C + D}{4}\right]^4.$$

Taking fourth roots yields $\sqrt[4]{ABCD} \leqslant \frac{1}{4}(A + B + C + D)$, the AM/GM inequality for $n = 4$.

Next, Cauchy moved up to $n = 8$ and followed a similar path:

$$ABCDEFGH = (ABCD)(EFGH) = \left(\sqrt[4]{ABCD}\right)^4 \cdot \left(\sqrt[4]{EFGH}\right)^4$$

$$\leqslant \left(\frac{A+B+C+D}{4}\right)^4 \left(\frac{E+F+G+H}{4}\right)^4 \text{ by the case for } n = 4$$

$$= \left[\left(\frac{A + B + C + D}{4}\right)\left(\frac{E + F + G + H}{4}\right)\right]^4$$

$$\leqslant \left[\left(\frac{\frac{A + B + C + D}{4} + \frac{E + F + G + H}{4}}{2}\right)^2\right]^4$$

by (3) with $x = \dfrac{A + B + C + D}{4}$ and $y = \dfrac{E + F + G + H}{4}$

$$= \left[\frac{A + B + C + D + E + F + G + H}{8}\right]^8.$$

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We extract eighth roots to conclude

$$\sqrt[8]{ABCDEFGH} \leqslant \frac{A + B + C + D + E + F + G + H}{8}.$$

At this point, Cauchy wrote '&c.…' and stated that generally

$$ABCD\ldots \leqslant \left(\frac{A + B + C + D + \ldots}{2^m}\right)^{2^m}.$$

In Cauchy's time, our induction protocol had not yet been formalized. But from the specific cases addressed, Cauchy had satisfied the 'informal induction' of the day to establish the AM/GM inequality … provided n is a power of 2 .

But what of other values of n? Cauchy found a proof for these as well. We shall describe it using the specific example of $n = 5$. That is, beginning with five positive numbers, A, B, C, D and E, we must show that

$$\sqrt[5]{ABCDE} \leqslant \frac{A + B + C + D + E}{5}.$$

First, Cauchy let $K = \frac{1}{5}(A + B + C + D + E)$ be the arithmetic mean of these five numbers. He then inserted this as often as necessary to bring the length of his list up to the next power of 2. In our case, the list would be extended to the *eight* numbers A, B, C, D, E, K, K, K. From the work above, Cauchy knew that

$$ABCDEKKK \leqslant \left(\frac{A + B + C + D + E + K + K + K}{8}\right)^8$$

$$= \left(\frac{5K + 3K}{8}\right)^8 = K^8 \text{ because } A + B + C + D + E = 5K.$$

Then $ABCDE \leqslant K^5 = \left(\frac{A + B + C + D + E}{5}\right)^5$.

Taking fifth-roots of both sides yields the AM/GM inequality for $n = 5$. And in exactly the same manner, the truth of the result for powers of 2 (the first step of his proof) established its truth for any whole number (as shown in the second). This strategy, which Cauchy employed brilliantly, is sometimes called 'forwards/backwards induction'.

The proof was complete, but Cauchy was not quite finished. Cross-multiplying the inequality, he observed that for n positive numbers $A, B, C, \ldots$, we have

$$A + B + C + D + \ldots \geqslant n\sqrt[n]{A \cdot B \cdot C \cdot D \cdot \ldots}.$$

This yields a string of beautiful, although hardly obvious, formulas for positive numbers:

$$A + B \geqslant 2\sqrt{AB},\ A + B + C \geqslant 3\sqrt[3]{ABC},\ A + B + C + D \geqslant 4\sqrt[4]{ABCD},\ \ldots. \quad (4)$$

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The observation can be used to establish a curious result known as Nesbitt's inequality.

Theorem 4: (Nesbitt's inequality)

If a, b and c are positive numbers. then

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geqslant \frac{3}{2}.$$

Proof: First note that

$$\begin{aligned} 2(a+b+c) &= (b+c) + (a+c) + (a+b) \\ &\geqslant 3\sqrt[3]{(b+c)(a+c)(a+b)} \qquad \text{by (4)}. \end{aligned}$$

Likewise

$$\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b} \geqslant 3\sqrt[3]{\frac{1}{(b+c)(a+c)(a+b)}}, \text{ also by (4)}.$$

Multiplying these inequalities gives

$$2(a+b+c)\left[\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right] \geqslant 3 \cdot 3 \cdot \sqrt[3]{1} = 9.$$

Thus

$$\frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} \geqslant \frac{9}{2}.$$

From this we see that

$$\left(\frac{a}{b+c} + 1\right) + \left(\frac{b}{a+c} + 1\right) + \left(\frac{c}{a+b} + 1\right) \geqslant \frac{9}{2},$$

and so
$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geqslant \frac{3}{2},$$

which is Nesbitt's inequality, cleverly proved.

A modern rival

For a very different derivation of the AM/GM inequality, we look to the 20th century mathematician George Pólya (1887-1985) [3, p. 103].

Assume we are given positive numbers $a_1, a_2, \ldots, a_n$ having geometric mean $G = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}$ and arithmetic mean $M = \frac{1}{n}(a_1 + a_2 + \ldots + a_n)$. The goal, of course, is to show $G \leqslant M$. Pólya began with

Lemma 4: For any real number x, we have $e^x \geqslant x + 1$.

Proof: A differential calculus argument is straightforward. Alternatively, we might graph the functions $y = e^x$ and $y = x + 1$, as shown in Figure 3.

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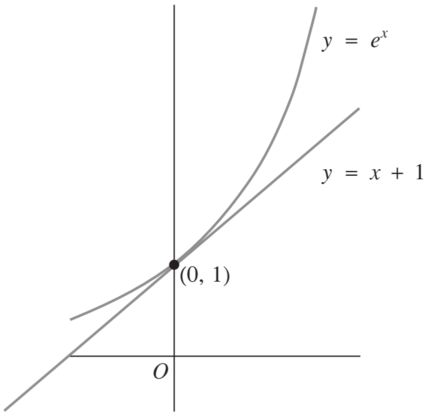

FIGURE 3

Theorem 5: (The AM/GM inequality, revisited)

Proof: For each a_i apply Lemma 4 to get $e^{(a_i/M) - 1} \geqslant \left(\frac{a_i}{M} - 1\right) + 1 = \frac{a_i}{M}$.

Thus

$$e^{\Sigma\left[\frac{a_i}{M} - 1\right]} = \prod_{i=1}^{n} e^{\left[\frac{a_i}{M} - 1\right]} \geqslant \prod_{i=1}^{n} \frac{a_i}{M} = \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{M^n} = \frac{G^n}{M^n}.$$

But

$$\sum_{i=1}^{n} \left[\frac{a_i}{M} - 1\right] = \left(\frac{1}{M} \sum_{i=1}^{n} a_i\right) - n = \frac{1}{M} \cdot nM - n = 0.$$

It follows that $e^0 \geqslant \frac{G^n}{M^n}$, which means $G^n \leqslant M^n$ and thus $G \leqslant M$.

Very elegant. Very nice.

Conclusion

We end with two observations.

First, even after two centuries the genius of Augustin-Louis Cauchy comes through loud and clear. In 1821, only a perceptive, and nimble, mathematician could have done what we have just seen.

Of course, Cauchy is celebrated for so much more than a pair of inequalities. He rebuilt the calculus upon the foundation of limits. He gave us the Cauchy integral theorem, the Cauchy residue theorem, and other results that serve as the bedrock of *complex* analysis. He presented the first proof of the polygonal number theorem (any whole number is the sum of three triangular numbers, of four square numbers, of five pentagonal numbers, etc.), thereby earning himself a spot in the 'Number theory hall of fame'. His name shows up in the theory of groups (Cauchy's theorem), in probability (Cauchy distribution), and in branches of applied mathematics like mechanics, optics, and the theory of elasticity. There are few who are his equal.

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Our second observation is that mathematics is enriched when the same result generates radically different proofs. The variants we have seen here, both for Cauchy's inequality and for the AM/GM inequality, are but a sampling of the multiple ways these can be derived. Again, anyone interested should consult Steele's book as a starting point.

Artists are not satisfied with a single landscape. Musicians are not content with a single love song. Likewise, mathematicians do not settle for a single proof of a great theorem. Rather, they employ their creative powers to revisit familiar territory, devising new approaches to old results and in the process exhibiting their own style, their own talent. As this Article should make clear, we are all the better for it.

So, kudos to Cauchy and his modern rivals.

Augustin-Louis Cauchy

Hermann Amandus Schwarz

George Pólya

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