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ON MULTIPLICATIVE PROPERTIES OF FAMILIES OF COMPLEXES OF CERTAIN LOOPS

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1. Introduction. It is well-known that if $\mathscr{F}$ is a quotient group of a group G, then (i) $\mathscr{F}$ is a partition of G, and (ii) the usual complex product (in the usual sense of multiplication of complexes) of every pair of members of $\mathscr{F}$ is a member of $\mathscr{F}$. There arises the question whether, conversely, (i) and (ii), perhaps in a weaker form, suffice for $\mathscr{F}$ to be a quotient group of G. Actually, the answer is affirmative (see [**8**, p. 30, Exercise 10], also Corollary 6.1 in this paper) if, instead of (i) and (ii):

(I) $\mathscr{F}$ is a family of complexes of G such that every element of G belongs to a member of $\mathscr{F}$, and every member of $\mathscr{F}$ is not a proper subset of another member of $\mathscr{F}$;

(II) the usual complex product of every pair of members of $\mathscr{F}$ is contained in a third member of $\mathscr{F}$.

The object of this paper is analysis of conditions (I) and (II) with reference to two quite wide classes of loops: those with the right inverse property and those with the weak inverse property (see [**6**]). For clarity, results for each type are derived separately, in spite of a certain similarity between them. It should be noted that there is a duality between the loops with the right inverse property and the loops with the left inverse property, leading to analogous results.

§ 2 deals with preliminary considerations for the subsequent sections.

In § 3, G is a loop with the right inverse property. It is shown essentially that (I) and (II) combined form a sufficient condition for $\mathscr{F}$ to have the following properties: it is a right coset expansion of G modulo a subloop E of G; it is a left division system by the usual multiplication of complexes with E as identity element and with two-sided inverses; it satisfies the right inverse property. If, in the particular case, the members of $\mathscr{F}$ are finite complexes, one deduces that E is a normal subloop and $\mathscr{F}$ a quotient loop of G modulo E.

In § 4, G is a weak inverse property loop. In this case the main result states that (I) and (II) combined suffice for $\mathscr{F}$ to be a quotient loop.

In § 5 a necessary and sufficient condition is discussed, slightly different from the above and leading to the same results as in §§ 3,4.

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In § 6 some of the above results are applied to certain special loops, and conclusions are drawn concerning simple loops and homomorphic mappings in the cases of weak inverse property or finite right inverse property loops.

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2. Preliminaries. Let G be a groupoid with a (two-sided) identity element denoted throughout this paper by 1.

A *left* or λ-*sided* (*right* or ρ-*sided*) *inverse* a^λ (a^ρ) of $a \in G$ is an element in G such that $a^\lambda a = 1$ ($aa^\rho = 1$). Denote:

$$\bar{a}^\lambda = \{a^\lambda | a^\lambda a = 1\}, \quad \bar{a}^\rho = \{a^\rho | aa^\rho = 1\}, \quad X^i = \bigcup_{x \in X \subset G} \bar{x}^i \quad (i = \lambda, \rho).$$

If each $a \in G$ has a unique a^λ and a unique a^ρ, then clearly $(a^\lambda)^\rho = (a^\rho)^\lambda = a$ and therefore $(A^\lambda)^\rho = (A^\rho)^\lambda = A$ for all $A \subset G$.

Throughout this paper, $\mathscr{F}$ denotes a family of non-empty sets. If $\mathscr{F}$ is a family of subsets of a set S, then it is a *covering* of S, or equivalently, S is said to be *covered* by $\mathscr{F}$ if and only if $S = \bigcup_{A \in F} A$. $\mathscr{F}$ is a *partition* of S if and only if it is a covering of S and every pair of its members are mutually disjoint.

The *usual product* AB *of complexes* (non-empty subsets) A and B of a groupoid, defined by $AB = \{ab | a \in A, b \in B\}$, is in the following sometimes denoted also by $A \cdot B$. For a set $\{a\}$ consisting of a single element a of G we usually denote $\{a\}B = aB$ or $B\{a\} = Ba$.

Definition 2.1. If in a family $\mathscr{F}$ of complexes of a groupoid G to each pair $A, B \in \mathscr{F}$ there corresponds a unique $C \in \mathscr{F}$ such that $AB \subset C$, one gets a groupoid $(\mathscr{F}, \circ)$, denoting $A \circ B = C$. This single-valued binary operation "$\circ$" is called the *general* (*single-valued*) *multiplication of complexes.*

Let G be a groupoid and $\mathscr{F}$ a partition of G and a groupoid by "$\circ$" or "$\cdot$". It is readily seen that the mapping $a \to a\theta = A$, where $a \in A \in \mathscr{F}$, is a homomorphism of G onto $(\mathscr{F}, \circ)$ or onto $(\mathscr{F}, \cdot)$. Therefore we can state:

LEMMA 2.1. *Let G be a groupoid with an identity element and with unique a^λ, a^ρ for all $a \in G$. If a partition $\mathscr{F}$ of G is a groupoid by "$\circ$" or "$\cdot$", then the mapping $a \to a\theta = A$, $a \in A \in \mathscr{F}$, is a homomorphism of G onto $(\mathscr{F}, \circ)$ or onto $(\mathscr{F}, \cdot)$ such that $\mathscr{F}$ has in both cases an identity element which is the kernel of the homomorphism; furthermore, $a^i\theta$ is an i-sided inverse in $\mathscr{F}$ of $a\theta \in \mathscr{F}$ for all $a \in G (i = \lambda, \rho)$.*

If $A^i \in \mathscr{F}$ for $A \in \mathscr{F}$ $(i = \lambda, \rho)$, then A^i is an i-sided inverse of A in $(\mathscr{F}, \circ)$ or in $(\mathscr{F}, \cdot)$.

If G satisfies any identical relation, then $(\mathscr{F}, \circ)$ or $(\mathscr{F}, \cdot)$ also satisfies the relation.

Definition 2.2. A member A of a family $\mathscr{F}$ of (non-empty) sets is called a

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minimal set of $\mathscr{F}$ if and only if $B \in \mathscr{F}$, $B \subset A$ imply $B = A$. If the members of $\mathscr{F}$ are subsets of a set S and each $A \in \mathscr{F}$ is a minimal set of $\mathscr{F}$ (i.e., no member of $\mathscr{F}$ contains any other member of $\mathscr{F}$), then $\mathscr{F}$ is called a *family of minimal subsets of S.*

We shall frequently use the following condition concerning a family $\mathscr{F}$ of complexes of a groupoid G:

$$(\alpha) \qquad A, B \in F \Rightarrow (\exists\ C \in \mathscr{F})(AB \subset C).$$

We shall sometimes make use of a special case of (α): Let E be a fixed member of $\mathscr{F}$, then

$$(\alpha^*) \qquad A \in \mathscr{F} \Rightarrow (\exists\ B, C \in \mathscr{F})(AE \subset B \quad \text{and} \quad EA \subset C).$$

LEMMA 2.2. (i) *Let G be a groupoid with an identity element* 1, *and $\mathscr{F}$ a family of minimal complexes of G with $1 \in E \in \mathscr{F}$. If $\mathscr{F}$ satisfies (α^*) with this E, then*

$$(1) \qquad AE = EA = A \text{ for all } A \in \mathscr{F}.$$

(ii) *If, in addition to* (i), *$\mathscr{F}$ is a covering of G, then*

$$(2) \qquad E^i \subset E \quad (i = \lambda, \rho).$$

Proof. In case (i), assuming $A \in \mathscr{F}$, $a \in A$, we have $a = a1 \in AE$, therefore $A \subset AE$. By (α^*), $AE \subset B \in \mathscr{F}$, i.e., $A \subset AE \subset B$, hence, by the minimal property, $A = AE$. In an analogous way we obtain $A = EA$.

In case (ii) we have: $a \in E$, $a^\rho \in G \Rightarrow (\exists\ A \in \mathscr{F})(a^\rho \in A) \Rightarrow 1 = aa^\rho \in EA = A$ (by (1)) $\Rightarrow 1 \in A \Rightarrow E = E1 \subset EA = A \Rightarrow E = A$ (by the minimal property), hence $a^\rho \in E$ and therefore $E^\rho \subset E$. $E^\lambda \subset E$ holds analogously.

In the following, let τ be a correspondence (i.e., a many-valued function) of a set S onto itself. We denote by $a\tau$ the image of $a \in S$ under τ, and by τ^{-1} the inverse correspondence of τ, i.e., $a \in b\tau^{-1} \Leftrightarrow b \in a\tau$. Let

$$X\theta = \bigcup_{x \in X \subset S} x\theta, \quad \theta = \tau \text{ or } \tau^{-1}.$$

If $\mathscr{F}$ is a family of subsets of S, and if $\theta = \tau$ or τ^{-1}, let $\mathscr{F}\theta = \{A\theta | A \in \mathscr{F}\}$. We shall further denote the following property by

$$(\mathscr{F}, \tau)\colon A \in \mathscr{F} \Rightarrow (\exists\ B \in \mathscr{F})(A\tau \subset B).$$

LEMMA 2.3. *If $\mathscr{F}$ is a family of minimal subsets of a set S and if properties $(\mathscr{F}, \tau)$ and $(\mathscr{F}, \tau^{-1})$ both hold, then $\mathscr{F}\tau = \mathscr{F}\tau^{-1} = \mathscr{F}$.*

Proof. Let $X \subset S$. Clearly $X \subset X\tau\tau^{-1}$, $X \subset X\tau^{-1}\tau$. Therefore, if $A \in \mathscr{F}$, we have $A \subset A\tau\tau^{-1} \subset B\tau^{-1}$ (for some $B \in \mathscr{F}$) $\subset C \in \mathscr{F}$, whence, by the minimal property, $A = C$, and this implies

$$(3) \qquad A = B\tau^{-1}.$$

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From (3) we get $A\tau = B\tau^{-1}\tau \supset B$. On the other hand $A\tau \subset B$, therefore $A\tau = B$ and hence $\mathscr{F}\tau \subset \mathscr{F}$. (3) signifies $\mathscr{F} \subset \mathscr{F}\tau^{-1}$. Thus, because of the duality between τ and τ^{-1}, we get $\mathscr{F}\tau^{-1} \subset \mathscr{F}$ and $\mathscr{F} \subset \mathscr{F}\tau$. Hence $\mathscr{F}\tau = \mathscr{F}\tau^{-1} = \mathscr{F}$.

3. Families of complexes of right inverse property loops as right coset expansions. We recall that a loop G has the *right inverse property* (R.I.P. for short) or the *left inverse property* (L.I.P. for short) if and only if $(ab)b^{\rho} = a$ or $a^{\lambda}(ab) = b$ for all $a, b \in G$ respectively. The conjunction of both properties is called the *inverse property*. It is well-known that if G is an R.I.P. or L.I.P. loop then $a^{\lambda} = a^{\rho}$ for all $a \in G$. Therefore a^{-1} is defined by $a^{-1} = a^{\lambda} = a^{\rho}$ and is called the *inverse* of a. Consequently, for $X \subset G$, we have $X^{\lambda} = X^{\rho}$ and denote it by X^{-1}.

LEMMA 3.1. *If an R.I.P. loop G is covered by a family $\mathscr{F}$ of minimal complexes such that $1 \in E \in \mathscr{F}$ and $\mathscr{F}$ satisfies (α^*) with this E, then E is a subloop of G and is disjoint from each other member of $\mathscr{F}$.*

Proof. By (1) of Lemma 2.2, $EE = E$, i.e., E is a subgroupoid. It remains to prove that given $a, b \in E$, then the equations $xa = b$ and $ay = b$ have solutions for x and y in E. For the first equation we obtain, with the aid of R.I.P. and (2) that

$$a, b \in E \Rightarrow (\exists\, x \in G)(xa = b) \Rightarrow x = (xa)a^{-1} = ba^{-1} \in EE^{-1} \subset EE = E \Rightarrow x \in E.$$

It also follows that

$$E \subset Ea. \tag{4}$$

The disjointness of E results from (4), (1) and the minimal property as follows:

$$A \in \mathscr{F},\ a \in A \cap E \Rightarrow E \subset Ea \subset EA = A \Rightarrow E = A. \tag{5}$$

Now assume $ay = b$ with $a, b \in E$. Let $y \in Y \in \mathscr{F}$, hence $b = ay \in EY \cap E$, i.e., $EY \cap E \neq \emptyset$. On the other hand, by (1), $EY = Y$, therefore $Y \cap E \neq \emptyset$, whence, by the disjointness of E, $Y = E$, and so $y \in E$.

We recall that a groupoid H is a *left* (*right*) *division system* (see [**3**, p. 91]), if to each pair of elements $a, b \in H$ there corresponds a unique $x \in H$ such that $xa = b$ $(ax = b)$.

Let G be a loop and E a subloop of G. Recall that the family $\mathscr{F}$ of the right cosets with respect to E, i.e., $\mathscr{F} = \{Ex | x \in G\}$, is a *right coset expansion of G modulo E*, if $\mathscr{F}$ is a partition of G (see [**3**. p. 92]).

THEOREM 3.1. *If an R.I.P. loop G is covered by a family $\mathscr{F}$ of complexes, then the two assumptions:*

(1) *the members of $\mathscr{F}$ are minimal sets,*

(2) *$\mathscr{F}$ satisfies (α),*

combined, form a necessary and sufficient condition for $\mathscr{F}$ to have the following properties: it is a right coset expansion of G modulo a subloop E of G; it is a left division system by the usual multiplication of complexes with E as identity element and with two-sided unique inverses; it satisfies the R.I.P.

Proof. Necessity: The members of $\mathscr{F}$ are mutually disjoint and therefore are minimal sets. Clearly (α) holds, even in the sense of "$=$" instead of "$\subset$".

Sufficiency: Let E be that member of $\mathscr{F}$ such that $1 \in E$. Since $\mathscr{F}$ satisfies (α), then clearly (α^*) holds with this E. Hence, by Lemma 3.1, E is a subloop of G and is disjoint from the other members of $\mathscr{F}$. The remainder of the proof is now effected in four steps.

(i) $$A \in \mathscr{F} \Rightarrow A^{-1} \in \mathscr{F}.$$

Proof. Let $a \in A \in \mathscr{F}$. Then $a^{-1} \in B \in \mathscr{F}$. Therefore $a^{-1}A \subset BA \subset C \in \mathscr{F}$ (by (α)) and $a^{-1}A \cap E \neq \emptyset$ (because $a^{-1}a = 1 \in E$). From this we get $C \cap E \neq \emptyset$, hence, by Lemma 3.1, $C = E$. That is, $a^{-1}A \subset E$ for all $a \in A$. Consequently,

$$\bigcup_{a \in A} A^{-1}a = A^{-1}A = \bigcup_{a^{-1} \in A^{-1}} a^{-1}A \subset E,$$

and hence $A^{-1}a \subset E$. Therefore, by the R.I.P. and Lemma 2.2

(6) $$A^{-1} = (A^{-1}a)a^{-1} \subset Ea^{-1} \subset EB = B \quad (a^{-1} \in B \in \mathscr{F}),$$

hence $A^{-1} \subset B$. Now, since "λ" and "ρ" can be interpreted as mutually-inverse mappings of G onto itself, we can use Lemma 2.3 substituting $a\tau = a^{-1} = a\tau^{-1}$ for all $a \in G$. Consequently, $A^{-1} \in \mathscr{F}$. (6) implies also by the minimal property that

(7) $$a \in A \in \mathscr{F} \Rightarrow A^{-1} = Ea^{-1}.$$

(ii) The members of $\mathscr{F}$ are right cosets with respect to E; they are disjoint, and have the same order. $\mathscr{F}$ is a groupoid by the generalized multiplication of complexes.

Proof. Let $A \in \mathscr{F}$, $a \in A$. According to (i) we have $a^{-1} \in A^{-1} \in \mathscr{F}$, hence by (7), $A = (A^{-1})^{-1} = E(a^{-1})^{-1} = Ea$; that is, $a \in A \in \mathscr{F} \Rightarrow A = Ea$. This means that each member of $\mathscr{F}$ is a right coset with respect to E. Furthermore,

(8) $$A, B \in \mathscr{F}, A \cap B \neq \emptyset \Rightarrow (\exists\ c \in G)(c \in A \cap B) \Rightarrow A = Ec = B,$$

i.e., the members of $\mathscr{F}$ are mutually disjoint. Thus we have also proved that the complex C in (α) is here determined uniquely by A and B. In other words, by virtue of Definition 2.1, the generalized multiplication of complexes holds in $\mathscr{F}$, i.e., we can write (α) here as $A \circ B = C$ or, equivalently, $AB \subset A \circ B$ for all $A, B \in \mathscr{F}$.

It is readily seen that there exists a one-to-one correspondence between any two cosets Ea and Eb by $ua \leftrightarrow ub$ $(u \in E)$. Consequently, the right cosets are of the same order.

(iii) $(\mathscr{F}, \circ)$ is a left division system with the R.I.P. and has E as identity element and A^{-1} as unique inverse of A for any $A \in \mathscr{F}$.

Proof. $\mathscr{F}$ is a partition of G and satisfies (i), i.e., $A^{-1} \in \mathscr{F}$ for all $A \in \mathscr{F}$. Therefore, using Lemma 2.1 with $\lambda = \rho = -1$, we obtain that $(\mathscr{F}, \circ)$ has E as identity element and A^{-1} as two-sided inverse of A for any $A \in \mathscr{F}$. Furthermore, by the same Lemma, since G satisfies the R.I.P., $(\mathscr{F}, \circ)$ satisfies it too, i.e., $(A \circ B) \circ B^{-1} = A$ for all $A, B \in \mathscr{F}$. Consequently (see [**3**, p. 111]), $(\mathscr{F}, \circ)$ is a left division system. Hence A^{-1} is a unique left inverse of any $A \in \mathscr{F}$, and since A^{-1} is a two-sided inverse, it is also a unique right inverse.
(iv) The generalized multiplication of complexes in $(\mathscr{F}, \circ)$ is identical with the usual multiplication of complexes: $A \circ B = AB$ for all $A, B \in \mathscr{F}$.

Proof. Let

$$A \circ B = C, \tag{9}$$

or, equivalently, $AB \subset C$. Suppose $AB \subsetneq C$; then there exists $c \in C - AB$. We now have that

$$b \in B \Rightarrow (\exists\, a' \in A' \in \mathscr{F})(c = a'b) \Rightarrow \left.\begin{matrix} A'B \cap C \neq \emptyset \\ A'B \subset A' \circ B \end{matrix}\right\} \begin{matrix} \Rightarrow A' \circ B \cap C \neq \emptyset \Rightarrow \\ A' \circ B = C \Rightarrow A' = A \end{matrix}$$

(by (9) and (iii)). Therefore $a' \in A$ and hence $c = a'b \in AB$, in contradiction to the supposition.

Remark 3.1. A loop satisfying the identity

$$[(xy)z]y = x[(yz)y] \tag{10}$$

for all x, y, z of the loop is called a *Bol loop* (see [**7**]); we shall call (10) the *Bol identity*. A Bol loop, as is known, has the R.I.P. (see [**7**, Theorem 2.1]). This yields a corollary to Theorem 3.1 with G denoting a Bol loop. In this case, by Lemma 2.1, $(\mathscr{F}, \cdot)$ also satisfies the Bol identity.

THEOREM 3.2. *If an R.I.P. loop G is covered by a family $\mathscr{F}$ of complexes, where $1 \in E \in \mathscr{F}$ with E finite, then the two assumptions:*
(1) *the members of $\mathscr{F}$ are minimal sets,*
(2) *$\mathscr{F}$ satisfies (α),*
combined, form a necessary and sufficient condition for $\mathscr{F}$ to be an R.I.P. quotient loop of G modulo a normal subloop E of G.

Proof. Necessity: This follows in the same manner as for Theorem 3.1.
Sufficiency: By Theorem 3.1, $\mathscr{F}$ is a right coset expansion of G modulo a subloop E and is an R.I.P. groupoid by the "$\cdot$"-multiplication with identity element E. According to Lemma 2.1 a homomorphism of G onto $(\mathscr{F}, \cdot)$ exists such that E is the kernel of the homomorphism. Since E is finite, $\mathscr{F}$ is a loop (see [**2**, Theorem 4]), whence E is a normal subloop of G (see [**3**, p. 60]). So $\mathscr{F}$ is a quotient loop of G with respect to E.

In view of Remark 3.1, we obtain

COROLLARY 3.1. *If a Bol loop G is covered by a family $\mathscr{F}$ of complexes, where $1 \in E \in \mathscr{F}$ with E finite, then the two assumptions:*

(1) *the members of $\mathscr{F}$ are minimal sets,*

(2) *$\mathscr{F}$ satisfies* (α),

combined, form a necessary and sufficient condition for $\mathscr{F}$ to be a Bol quotient loop.

4. Families of complexes of weak inverse property loops as quotient loops.

Definition 4.1 (see [**6**]). Let G be a groupoid with an identity element 1. G satisfies the *weak inverse property* (W.I.P. for short), if and only if $(ab)c = 1$ implies $a(bc) = 1$ $(a, b, c \in G)$.

LEMMA 4.1 (see also [**6**, § 1] and [**5**, Theorem 1.1.]). *If G is a groupoid with an identity element* 1 *and with unique a^λ, a^ρ for all $a \in G$, then the following properties are equivalent:*

(a) *The W.I.P.*

(b) *$b(ab)^\rho = a^\rho$ for all $a, b \in G$.*

(c) *$(ab)^\lambda a = b^\lambda$ for all $a, b \in G$.*

(d) *$a(bc) = 1$ implies $(ab)c = 1$* $(a, b, c \in G)$.

Proof. (1) (a) implies (b) and (c) exactly in the same way as for (1) and (3) in [**6**, § 1].

(2) (b) implies (c): (b) yields (a) because $(ab)c = 1 \Rightarrow c = (ab)^\rho \Rightarrow bc = b(ab)^\rho = a^\rho$ (by (b)) $\Rightarrow a(bc) = aa^\rho = 1$. This, by (1), yields (c).

(3) (c) implies (d): $a(bc) = 1 \Rightarrow a = (bc)^\lambda \Rightarrow ab = (bc)^\lambda b = c^\lambda$ (by (c)) $\Rightarrow (ab)c = c^\lambda c = 1$.

(4) (d) implies (a): (d) yields (c), since for arbitrary $a, b \in G$ there exists $c \in G$ such that $c(ab) = 1$; therefore $c = (ab)^\lambda$ and $(ca)b = 1$, hence $ca = b^\lambda$, and therefore $(ab)^\lambda a = ca = b^\lambda$. (c) implies (b) by (d) in a similar way as (b) implies (c) by (a) (see (2)). Now, as we have shown in (2), (b) implies (a).

LEMMA 4.2. *A groupoid G with an identity element and with unique a^λ, a^ρ for all $a \in G$ is a loop, if G satisfies any of the properties* (a)-(d).

Proof. Assume that G satisfies (b). Arbitrary elements a, c of G determine uniquely the element $b = (c^\lambda a)^\rho$; hence, by (b), $ab = a(c^\lambda a)^\rho = (c^\lambda)^\rho = c$. That is, for given $a, c \in G$ there exists $b = (c^\lambda a)^\rho \in G$ such that $ab = c$. Conversely, this equation can only be satisfied by this b, because $ay = c$ implies $c^\lambda = (ay)^\lambda$, and by Lemma 4.1, G satisfies (c), therefore $c^\lambda a = (ay)^\lambda a = y^\lambda$, whence $y = (y^\lambda)^\rho = (c^\lambda a)^\rho$. It is shown analogously that for given $b, c \in G$ there exists a unique $a \in G$ such that $ab = c$. This proves that G is a loop. Since, by Lemma 4.1, any one of the properties (a), (c), (d) implies (b), the proof is complete.

LEMMA 4.3. *If a W.I.P. loop G is covered by a family $\mathscr{F}$ of minimal complexes* *such that* $1 \in E \in \mathscr{F}$ *and* $\mathscr{F}$ *satisfies* (α^*) *with this* E, *then* E *is a subloop of* G *and* E *is disjoint from each other member of* $\mathscr{F}$.

Proof. (1) of Lemma 2.2 implies $EE = E$, hence E is a subgroupoid. If a is in E, then by (2) of Lemma 2.2, the (unique) inverses a^λ, a^ρ are also in E. Clearly E has the W.I.P. and is consequently a loop by Lemma 4.2. The disjointness of E follows from the fact that it is a subloop (i.e., (4) holds for $a \in E$), as well as from (1) and from the minimal property—exactly in the same way as in (5).

THEOREM 4.1. *If a W.I.P. loop is covered by a family* $\mathscr{F}$ *of complexes, then the two assumptions:*

(1) *the members of* $\mathscr{F}$ *are minimal sets,*

(2) $\mathscr{F}$ *satisfies* (α),

combined, form a necessary and sufficient condition for $\mathscr{F}$ *to be a W.I.P. quotient loop of* G *modulo a normal subloop* E *of* G, *or in short,* $\mathscr{F} = G/E$.

Proof. Necessity: This follows as in Theorem 3.1.

Sufficiency: By Lemma 4.3, the member E of $\mathscr{F}$ with $1 \in E$ is a subloop of G and is disjoint from every other member of $\mathscr{F}$.

(i) $$A \in \mathscr{F} \Rightarrow A^i \in \mathscr{F} \quad (i = \lambda, \rho).$$

Proof. Beginning as for (i) in Theorem 3.1, we obtain by (α) and by Lemma 4.3:

$$a \in A \in \mathscr{F} \Rightarrow a^\rho \in B \in \mathscr{F} \Rightarrow \begin{cases} Aa^\rho \subset AB \subset C \in \mathscr{F} \\ \text{and} \\ Aa^\rho \cap E \neq \emptyset \end{cases} \Rightarrow C = E,$$

hence

(11) $$Aa^\rho \subset E.$$

Using (b) which, by Lemma 4.1, holds in G, we obtain with the aid of (11) and Lemma 2.2. that

(12) $$A^\rho = a^\rho(Aa^\rho)^\rho \subset a^\rho E^\rho \subset a^\rho E \subset BE = B.$$

Analogously, we show that $(\exists\ D \in \mathscr{F})(A^\lambda \subset D)$. We now use Lemma 2.3, substituting $a\tau = a^\lambda$ and $a\tau^{-1} = a^\rho$ for all $a \in G$. Therefore $A^i \in \mathscr{F}$ $(i = \lambda, \rho)$. From (12) it also follows, in view of the minimal property, that

(13) $$a \in A \in \mathscr{F} \Rightarrow A^\rho = a^\rho E.$$

Analogously, we obtain

(14) $$a \in A \in \mathscr{F} \Rightarrow A^\lambda = Ea^\lambda.$$

(ii) The members of $\mathscr{F}$ are mutually disjoint and are two-sided cosets modulo E, i.e.,

(15) $$aE = A = Ea \text{ for all } A \in \mathscr{F} \text{ and all } a \in A.$$

$\mathscr{F}$ is a groupoid by the generalized multiplication of complexes, i.e., $\mathscr{F} = (\mathscr{F}, \circ)$.

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Proof. For $i, \hat{\imath} = \lambda, \rho$; $i \neq \hat{\imath}$ we get according to (i), (13) and (14);

$$a \in A \in \mathscr{F} \Rightarrow a^i \in A^i \in \mathscr{F} \Rightarrow A = (A^i)^{\hat{\imath}} = \begin{cases} E(a^\rho)^\lambda = Ea \\ (a^\lambda)^\rho E = aE \end{cases} \Rightarrow aE = A = Ea.$$

Disjointness of the members of $\mathscr{F}$ is derived from (15) as in (8), therefore the condition (α) defines here the groupoid $\mathscr{F} = (\mathscr{F}, \circ)$.

(iii) $(\mathscr{F}, \circ)$ is a W.I.P. loop.

Proof. By (ii), $\mathscr{F}$ is a partition of G, and by (i), $A^i \in \mathscr{F}$ $(i = \lambda, \rho)$. Hence, in view of Lemma 2.1, E is the identity element of $(\mathscr{F}, \circ)$, A^i is an i-sided inverse of A for each $A \in \mathscr{F}$, and (b), (c) hold in $(\mathscr{F}, \circ)$ because by Lemma 4.1, they hold in G. Now we can verify that A^i is a unique i-sided inverse of A. Indeed, if $A \circ B = E$ and $C \circ A = E$, then by (b) and (c) we have $A^\rho = B \circ (A \circ B)^\rho = B \circ E^\rho$ and $A^\lambda = (C \circ A)^\lambda \circ C = E^\lambda \circ C$. Since (i), (2) and the minimal property imply $E^\lambda = E = E^\rho$, we get $A^\lambda = E \circ C = C$ and $A^\rho = B \circ E = B$. Finally, Lemmas 4.1 and 4.2 yield that $(\mathscr{F}, \circ)$ is a W.I.P. loop.

(iv) $(\mathscr{F}, \circ) = (\mathscr{F}, \cdot)$.

Proof. This follows as for (iv) in Theorem 3.1.

(v) E is a normal subloop of G.

Proof. By Lemma 2.1, $(\mathscr{F}, \cdot)$ is a homomorphic image of G with E as kernel of the homomorphism. E, therefore, is a normal subloop of G, i.e., $\mathscr{F} = G/E$.

Remark 4.1. If, in Theorem 4.1, the assumption that $\mathscr{F}$ is a covering is replaced by another, namely that $\mathscr{F}$ is a family of complexes of G where $V^i \subset V$ $(V = \bigcup_{A \in \mathscr{F}} = A;\ i = \lambda, \rho)$, with the remaining assumptions valid, then we have, by (α), $1 \in V$ and $VV \subset V$. Hence, V is a subgroupoid with an identity element, with unique i-sided inverses, and satisfying (a). Consequently, by Lemma 4.2, V is a W.I.P. (sub) loop. In other words, $\mathscr{F}$ is a covering of the W.I.P. loop V. Now application of Theorem 4.1 to V instead of G yields $\mathscr{F} = V/E$.

5. The property (P). We shall use in the following a property somewhat different from (α).

Definition 5.1. Let $\mathscr{F}$ be a family of complexes of a groupoid G, and let $A, B \in \mathscr{F}$. The ordered pair (A, B) is said to have the property (P), if and only if the following condition, denoted by $(A, B)_P$ holds:

If a member C of $\mathscr{F}$ exists such that $AB \cap C \neq \emptyset$, then $AB \subset C$. $\mathscr{F}$ is said to have the property (P), if and only if $(A, B)_P$ holds for all $A, B \in \mathscr{F}$.

LEMMA 5.1. *Let $\mathscr{F}$ be a family of complexes of a groupoid. If for $A, B \in \mathscr{F}$*

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there exist $A_1, A_2, B_1, B_2 \in \mathscr{F}$ *such that* $A = A_1A_2$, $B = B_1B_2$ *and* $(A_1, A_2)_P$, $(B_1, B_2)_P$ *hold, then* A *and* B *are either disjoint or identical.*

$$\textit{Proof. } A \cap B \neq \emptyset \Rightarrow \left\{\begin{matrix} A_1A_2 \cap B \neq \emptyset \Rightarrow A_1A_2 \subset B \Rightarrow A \subset B \\ \\ B_1B_2 \cap A \neq \emptyset \Rightarrow B_1B_2 \subset A \Rightarrow B \subset A \end{matrix}\right\} \Rightarrow A = B.$$

LEMMA 5.2. *Let* G *be a groupoid with an identity element* 1 *and* $\mathscr{F}$ *a family of complexes of* G *with* $1 \in E \in \mathscr{F}$. *If* $A \in \mathscr{F}$ *satisfies* $(A, E)_P$ $((E, A)_P)$, *then*

$$AE = A \quad (EA = A). \tag{16}$$

If $A_i \in \mathscr{F}$ $(i = 1, 2)$ *satisfy* $(A_i, E)_P$ $((E, A_i)_P)$, *then* A_1 *and* A_2 *are either disjoint or identical.*

Proof. For $a \in A \in \mathscr{F}$ we have $a = a1 \in AE$, whence $A \subset AE$, i.e., $AE \cap A \neq \emptyset$; therefore, by $(A, E)_P$, $AE \subset A$. Hence $AE = A$. Similarly $EA = A$ if $(E, A)_P$ holds. The second statement is obtained by Lemma 5.1 and (16), according to which $A_i = A_iE$ $(A_i = EA_i)(i = 1, 2)$.

Using this Lemma, we can replace in Lemmas 3.1, 4.3 the assumption

$$\text{The members of } \mathscr{F} \text{ are minimal complexes} \tag{17}$$

and the condition (α^*), by the assumptions

$$(A, E)_P \text{ and } (E, A)_P \text{ for all } A \in \mathscr{F}. \tag{18}$$

Indeed, by Lemma 5.2, (18) implies (16) and the disjointness of the members of $\mathscr{F}$; therefore (17) and (α^*) certainly hold.

LEMMA 5.3. *Let* $\mathscr{F}$ *be a family of complexes of a groupoid* G.

(i) *If* $\mathscr{F}$ *is a covering of* G *and has the property* (P), *then* $\mathscr{F}$ *satisfies* (α).

(ii) *If the members of* $\mathscr{F}$ *are mutually disjoint and satisfy* (α), *then* $\mathscr{F}$ *has the property* (P).

Proof. Let $A, B \in \mathscr{F}$; then clearly $AB \neq \emptyset$. In case (i) there exists $C \in \mathscr{F}$ such that $AB \cap C \neq \emptyset$, and this, by $(A, B)_P$, yields $AB \subset C$.

In case (ii), assuming $AB \cap D \neq \emptyset$ $(D \in \mathscr{F})$, we have by (α), $AB \subset C \in \mathscr{F}$, consequently $C \cap D \neq \emptyset$, whence, by the disjointness, $C = D$, i.e., $AB \subset D$.

The last two Lemmas imply that in Theorems 3.1, 3.2, 4.1 and in Corollary 3.1 the assumptions "the members of $\mathscr{F}$ are minimal sets" and (α) may be replaced by the assumption: $\mathscr{F}$ satisfies (P). For instance, from Theorems 3.2, 4.1 we obtain

THEOREM 5.1. *Let* G *be a loop which is covered by a family* $\mathscr{F}$ *of complexes and let* $1 \in E \in \mathscr{F}$. *If* G *satisfies the R.I.P. and* E *is finite, or if* G *satisfies the W.I.P., then property* (P) *of* $\mathscr{F}$ *is a necessary and sufficient condition for* $\mathscr{F}$ *to be a quotient loop with the R.I.P. or W.I.P. respectively.*

6. Some further results. As is known, W.I.P. loops include inverse property loops and crossed-inverse loops (see [**1**]), and inverse property loops include Moufang loops and quasi-associative loops (see [**4**]). We can, therefore, suppose in Theorems 4.1, 5.1 the particular case of the loop G satisfying any of the following: the inverse property, the crossed-inverse property, the Moufang identity, or the quasi-associative property. According to Lemma 2.1, $(\mathscr{F}, \cdot)$ also has the first three properties respectively. It is readily seen that if G is quasi-associative, $(\mathscr{F}, \cdot)$ is also quasi-associative. If G is a group, then clearly E is a normal subgroup. We thus obtain

COROLLARY 6.1. *Let $\mathscr{F}$ be a covering of a loop G satisfying any of the following properties: the inverse property, the crossed-inverse property, the Moufang identity, the quasi-associative property, or the associative property. If*

(i) *the members of $\mathscr{F}$ are minimal sets and satisfy* (α), *or*

(ii) *$\mathscr{F}$ satisfies* (P),

then there exists a normal subloop (subgroup in case of a group G) E of G such that $\mathscr{F} = G/E$ and has the same property as G.

Definition 6.1. A family $\mathscr{F}$ of subsets of a set S is called a *proper family*, if and only if $\mathscr{F}$ has at least one member which is neither S nor a one-element subset of S.

COROLLARY 6.2. *Let G be a simple loop satisfying W.I.P. or R.I.P. Then there cannot exist a covering of G by a proper family $\mathscr{F}$ of complexes—finite complexes in the case of R.I.P.—such that in each case*

(i) *the members of $\mathscr{F}$ are minimal sets satisfying* (α), *or*

(ii) *$\mathscr{F}$ satisfies* (P).

Proof. Negating the assertion in each case, we have in $\mathscr{F}$ at least one member A such that

$$A \neq G \text{ and } A \neq \{g\} \text{ for any } g \in G, \tag{19}$$

and Theorems 4.1, 3.2 hold in case (i) or Theorem 5.1 holds in case (ii), i.e., in both cases $\mathscr{F} = G/E$ where E is a normal subloop of G. Consequently $A = aE$ $(a \in A)$, hence by (19) $E \neq \{1\}$ and $E \neq G$. Thus G is not simple, and this is a contradiction.

COROLLARY 6.3. *Let a groupoid H be a homomorphic image of a loop G satisfying any one of the following conditions:*

(i) *W.I.P.*,

(ii) *R.I.P. and being finite*,

(iii) *the Bol identity and being finite. Then H is a loop satisfying the corresponding condition.*

Proof. If θ is a homomorphism of G onto H, then, according to a known homomorphism theorem, H is isomorphic with the family $\mathscr{F} = \{x\theta^{-1} | x \in H\}$ which is a partition of G and a groupoid by the "o"-multiplication. This yields, using Theorem 4.1 in case (i) or Theorem 3.2 in case (ii) or Corollary 3.1 in case (iii), that $(\mathscr{F}, \circ)$ is a loop which satisfies the same condition as G does. Therefore, by the isomorphism, H is also a loop satisfying the same condition as G.

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