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Is the Sacker–Sell type spectrum equal to the contractible set?

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For linear differential systems, the Sacker–Sell spectrum (dichotomy spectrum) and the contractible set are the same. However, we claim that this is not true for the linear difference equations. A counterexample is given. For the convenience of research, we study the relations between the dichotomy spectrum and the contractible set under the framework on time scales. In fact, by a counterexample, we show that the contractible set could be different from dichotomy spectrum on time scales established by Siegmund [J. Comput. Appl. Math., 2002]. Furthermore, we find that there is no bijection between them. In particular, for the linear difference equations, the contractible set is not equal to the dichotomy spectrum. To counter this mismatch, we propose a new notion called *generalized contractible set* and we prove that the generalized contractible set is exactly the dichotomy spectrum. Our approach is based on roughness theory and Perron's transformation. In this paper, a new method for roughness theory on time scales is provided. Moreover, we provide a time-scaled version of the Perron's transformation. However, the standard argument is invalid for Perron's transformation. Thus, some novel techniques should be employed to deal with this problem. Finally, an example is given to verify the theoretical results.

Keywords: Time scales; Dichotomy spectrum; Reducibility; Exponential dichotomy; Roughness; Contractible set; Perron's transformation; Linear nonautonomous differential equations

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1. Introduction

1.1. History

The well-known notion of exponential dichotomy introduced by Perron [**35**] extends the concept of hyperbolicity from autonomous linear systems to nonautonomous linear systems and plays a crucial role in the study of the dynamical behaviour of nonautonomous dynamical systems, such as stable and unstable invariant manifolds as well as linearization theory. Since the concept was proposed by Perron, exponential dichotomy together with its variants and extensions has been extensively studied [**3–8**, **11**, **13**, **14**, **17**, **19**, **22**, **23**, **25**, **28–30**, **38**].

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Based on the study of exponential dichotomy, the well-known Sacker–Sell spectrum was introduced by Sacker and Sell [**37**] for skew-product flows on vector bundles with compact base and it plays an important role in describing the diagonal term. In fact, Bylov [**10**] introduced the concept of almost reducibility and proved that any linear system can be almost reducibility to diagonal system. Later, Lin [**24**] improved the Bylov's work and proved that the contractible set is equal to the Sacker–Sell spectrum of the linear system. More specifically, Lin proved that the diagonal terms are contained in the Sacker–Sell spectrum and this spectrum is the minimal compact set where the diagonal coefficients belong to. The main ideas of Lin's work are based on the roughness theory of classical exponential dichotomy and Perron's transformation [**35**] by which a linear system can be reduced to an upper triangular system. Later, Catañeda and Huerta [**12**] considered Lin's work in a nonuniform framework. Catañeda and Robledo [**15**] extended Lin's work to difference systems. We mention that the contractible set is a powerful tool to study the linearization with unbounded perturbations (see [**16**, **21**]).

Recently, Pötzsche [**32**, **33**] introduced the concept of the exponential dichotomy on time scales. The theory of time scale or measure chain can be traced back to Hilger [**20**], which allows a unified treatment of continuous systems, discrete systems and hybrid systems. With such a framework, many properties and applications of exponential dichotomies on time scales can be studied in a certain range. For instance, Aulbach and Pötzsche [**2**] studied the reducibility of linear dynamic equation on measure chains. Pötzsche [**34**], Xia *et al.* [**40**] studied the linearization of dynamic equations on measure chains and time scales, respectively. Siegmund [**39**] considered the exponential dichotomy which is a specialized version of the one studied in [**32**, **33**] and introduced a new notion of spectrum for this specific exponential dichotomy. Another important property of exponential dichotomy is its roughness under perturbations. Roughness of exponential dichotomy can be traced back to Massera and Schäffer [**27**] and then it has been widely studied for continuous or discrete systems [**1**, **18**, **19**, **25**, **31**, **36**] and the systems on general time scales [**41**–**43**].

1.2. Motivations and novelties

Motivated by the works of Lin [**24**], Huerta [**21**], Castañeda and Robledo [**16**], and Siegmund [**39**], in this paper, we consider the relationship between the contractible set and the dichotomy spectrum studied in [**39**] on time scales. We show, by a counterexample, that the contractible set may not only be different from the dichotomy spectrum established by Siegmund [**39**], but also there is no bijection between them. In particular, for the linear difference equations, the contractible set is not equal to the dichotomy spectrum, which contradicts the results in [**15**]. Thus, the counterexample also shows that the result of [**15**] is questionable. This is contrary to the expectation that the spectrum should be equal to the contractible set. To counter this mismatch in expectation, we propose a new definition of *generalized contractible set* and we prove that the generalized contractible set is exactly the dichotomy spectrum. In particular, if $\mathbb{T} = \mathbb{R}$, the generalized contractible set is the contractible set in [**24**] and the dichotomy spectrum is the Sacker–Sell spectrum. In

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other words, our result is consistent with the one studied in [**24**] when dynamical system is reduced to the continuous case.

Our approach is based on roughness theorem and Perron's transformation. In this paper, a new simple method for roughness theory on time scales is provided which is different from the method of using Lyapunov function or generalized Gröwall inequality [**41**–**43**]. The advantages of this method are that the range of disturbance is determined and the coefficient matrix function can be unbounded. The main steps of this new method are listed as follows.

(i) Firstly, we prove that the roughness theorem holds for the system which admits exponential dichotomy with the projection I or O;

(ii) Secondly, we show that the unperturbed system $x^{\Delta} = A(t)x$ is kinematically similar to a diagonal block system $y^{\Delta} = \mathrm{diag}(A_1(t), A_2(t))y$, where its corresponding subsystem $y_1^{\Delta} = A_1(t)y_1$ (resp. $y_2^{\Delta} = A_2(t)y_2$) admits exponential dichotomy with the projection I (resp. O);

(iii) Lastly, we construct a Lyapunov transformation $x = R(t)y$ by which the perturbed system $x^{\Delta} = (A(t) + B(t))x$ can be transformed into the system $y^{\Delta} = \mathrm{diag}(A_1(t) + B_1(t), A_2(t) + B_2(t))$. Moreover, $\sup\limits_{t\in\mathbb{T}} \|B_i(t)\| \to 0$ as $\sup\limits_{t\in\mathbb{T}} \|B\| \to 0$.

However, the standard techniques to construct the Lyapunov transformation $x = R(t)y$ for the continuous cases are not valid for the dynamic equations on time scales. In fact, it is more difficult to construct the matrix-valued function $R(t)$ than that the continuous case. Because the relation of kinematical similarity is more complex than that in the continuous case and it is difficult to deal with the term $R(\sigma(t))$ occurred in the relation, where σ is the forward jump operator. To see how to overcome the difficulty, one can refer to (4.13) and (4.14).

Furthermore, in this paper, we provide a time-scaled version of the Perron's transformation. However, on time scales, the difficulty mentioned above still exists in the discussion of Perron's transformation. The standard arguments for Perron's transformation on $\mathbb{R}$ are not valid for the systems on time scales. In fact, if we use the standard arguments, the transformation $x = U(t)y$ transforms system $x^{\Delta} = A(t)x$ into $y^{\Delta} = B(t)y$, then it leads us to an inequality that $\|B(t) + B^T(t)U^T(\sigma(t))U(t)\| \leqslant \|A(t) + A^T(t)U^T(\sigma(t))U(t)\|$, where σ is the forward jump operator and $A(t)$ is bounded. In particular, for the continuous case $\mathbb{T} = \mathbb{R}$, $\|B(t) + B^T(t)\| \leqslant \|A(t) + A^T(t)\|$. However, in order to prove the boundedness of $B(t)$ on time scales, we have to overcome the troublesome term $U^T(\sigma(t))U(t)$. Therefore, we employ some novel techniques to deal with this problem (see theorem 4.1). Finally, we include an example to illustrate the effectiveness of our main result.

1.3. Organization of the paper

The rest of this paper is organized as follows. In §2, we introduce some notations and definitions. Section 3 gives a counter example to state that the contractible set may not be equal to dichotomy spectrum and introduces the new definitions of Δ-contractibility and generalized contractible set. In § 4, the main results of this paper and an example are provided.

2. Preliminaries

For completeness, we briefly introduce some basic terminology and notations of the calculus on time scales. More details can be found in the books [**9**, **20**]. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. Throughout this paper, we always assume that a time scale $\mathbb{T}$ is unbounded to the right and left (*two-sided time*). The closed interval on time scales is denoted by $[\cdot,\cdot]_{\mathbb{T}}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. A set $\mathbb{T}^\kappa$ is defined as $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a left-scattered maximum, $\mathbb{T}^\kappa = \mathbb{T}$ otherwise. A function is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is denoted by $\mathcal{C}_{rd}$. The graininess function μ is defined by $\mu(t) := \sigma(t) - t$. A function $p : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R}$. If $p \in \mathcal{R}$, we define the cylinder operator $\xi_\mu : \mathcal{R} \to \mathcal{C}_{rd}$ and the exponential function $e_p : \mathbb{T} \to \mathbb{R}$ by

$$\xi_\mu(p)(t) := \lim_{s\searrow\mu(t)} \frac{\log(1 + p(t)s)}{s}, \quad e_p(t,s) = \exp\left(\int_s^t \xi_\mu(p)(\tau)\Delta\tau\right),$$

where $t, s \in \mathbb{T}$. The function f is said to be positively regressive if $f \in \mathcal{C}_{rd}$ and $1 + \mu(t)f(t) > 0$ holds for all $t \in \mathbb{T}$. The set of all positively regressive functions is denoted by $\mathcal{R}^+$. The range of function f is denoted by $\mathrm{Im}f$. The real part of a complex number z is denoted by $\mathrm{Re}(z)$. The function $\overline{\xi}_\mu : \mathcal{R} \to \mathcal{C}_{rd}$ is defined by

$$\overline{\xi}_\mu(p)(t) := \mathrm{Re}(\xi_\mu(p)(t)) = \lim_{s\searrow\mu(t)} \frac{\log|1 + p(t)s|}{s}. \tag{2.1}$$

Now we consider the n-dimensional linear system

$$x^\Delta = A(t)x \tag{2.2}$$

on a time scale $\mathbb{T}$, where $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $\mu(t)$ is bounded. Let $\Phi_A(t,s)$ denote the evolution operator of (2.2), i.e., $\Phi_A(\cdot,\tau)\xi$ solve the initial value problem (2.2), $x(\tau) = \xi$, for $\tau \in \mathbb{T}$ and $\xi \in \mathbb{R}^n$. Since A is regressive, $\Phi_A(t,s)$ is invertible for any $t, s \in \mathbb{T}$ with $\Phi_A^{-1}(t,s) = \Phi_A(s,t)$. Another linear dynamic equation

$$y^\Delta = B(t)y \tag{2.3}$$

with (not necessarily regressive) $B \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ is said to be *kinematically similar* to (2.2) on an interval $J \subset \mathbb{T}$ if there exists a function $\Lambda \in \mathcal{C}^1_{rd}(J, \mathbb{R}^{n\times n})$ with the following properties:

(i) $\Lambda(\cdot)$ and $\Lambda^{-1}(\cdot)$ are bounded as functions from J to $\mathbb{R}^{n\times n}$;

(ii) the identity $\Lambda^\Delta(t) = A(t)\Lambda(t) - \Lambda(\sigma(t))B(t)$ holds on $\mathbb{J}^\kappa$.

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A function $\Lambda : J \to \mathbb{R}^{n\times n}$ with these properties is called *Lyapunov transformation function* and the transformation $x = \Lambda y$ is called *Lyapunov transformation*. It is known that the corresponding linear change of variables $x = \Lambda(t)y$ transforms (2.2) into (2.3).

Remark 2.1. Kinematical similarity defines an equivalence relation on the set of all linear homogeneous dynamic equation in $\mathbb{R}^n$. Moreover, the regressivity is preserved under kinematic similarity on any time scale [**2**].

An *invariant projector* of (2.2) is defined to be a function $P : \mathbb{T} \to \mathbb{R}^{n\times n}$ of projections $P(t)$, $t \in \mathbb{T}$ such that

$$P(t)\Phi_A(t,s) = \Phi_A(t,s)P(s) \quad \text{for } t,s \in \mathbb{T}.$$

Definition 2.2 [**39**]. *For $\gamma \in \mathbb{R}$ we shall say that (2.2) admits an exponential dichotomy with growth rate γ (γ-ED) if there exists an invariant projector $P : \mathbb{T} \to \mathbb{R}^{n\times n}$ and constants $K \geqslant 1$ and $\alpha > 0$ such that for $t, s \in \mathbb{T}$, the dichotomy estimates*

$$\begin{aligned} \|\Phi_A(t,s)P(s)\| &\leqslant K\,\mathrm{e}^{(\gamma-\alpha)(t-s)}, \quad t \geqslant s,\\ \|\Phi_A(t,s)(I-P(s))\| &\leqslant K\,\mathrm{e}^{(\gamma+\alpha)(t-s)}, \quad t \leqslant s, \end{aligned}$$

hold, where I is the identity matrix.

Obviously, one can see that if system (2.2) admits γ-ED and kinematical similar to system (2.3), then system (2.3) also admits γ-ED.

Definition 2.3. *The dichotomy spectrum of system (2.2) is the set*

$$\Sigma(A) = \{\gamma \in \mathbb{R} : x^{\Delta} = A(t)x \text{ admits no } \gamma\text{-ED}\}.$$

Remark 2.4. If two systems are kinematically similar, then they have the same dichotomy spectrum.

Definition 2.5. *System (2.2) is contracted to the compact set $F \subseteq \mathbb{R}$ if for any $\delta > 0$, there exist functions $C_i(t) \in \mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$ $(i = 1, 2, \ldots, n)$ and $B(t) \in \mathcal{C}_{rd}(\mathbb{T},\mathbb{R}^{n\times n})$ satisfying*

$$\|B\| \leqslant \delta, \quad \bigcup_{i=1}^{n} \operatorname{Im} C_i(t) \subseteq F,$$

such that system (2.2) is kinematically similar to the system

$$x^{\Delta}(t) = [\operatorname{diag}(C_1(t),\ldots,C_n(t)) + B(t)]x(t).$$

Definition 2.6. *A set F is called the contractible set of system (2.2) if $F \subseteq \mathbb{R}$ satisfying*

(i) *system (2.2) is contracted to F;*

(ii) *if system (2.2) is contracted to F_1, then $F \subseteq F_1$.*

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3. A counterexample and the new definition of contractibility

From the results in [24], we know that for linear differential systems, the Sacker–Sell spectrum and the contractible set are the same. However, we claim that this is questionable for the linear difference equations. A counterexample is given. In what follows, we show, by a counterexample, that the contractible set may not be equal to dichotomy spectrum for the difference equations.

Counterexample 1: Consider the 1-dimensional discrete system (seen as time scale $\mathbb{T}=\mathbb{Z}$)

$$\Delta x = a(t)x \tag{3.1}$$

where

$$a(t)=\begin{cases} e-1, & t\geqslant 0\\ e^{-1}-1, & t\leqslant 0. \end{cases} \tag{3.2}$$

A straightforward calculation leads to

$$\Phi_a(t,s)=\mathrm{e}^{\mathrm{sgn}(t-s)\cdot(\mathrm{sgn}(t)\cdot t-\mathrm{sgn}(s)\cdot s)},$$

where

$$\mathrm{sgn}(t)=\begin{cases} 1, & t>0\\ 0, & t=0\\ -1, & t<0. \end{cases}$$

Then for any $\gamma>1$, there exists a constant α satisfying $\gamma-1>\alpha>0$, such that

$$|\Phi_a(t,s)|=\mathrm{e}^{\mathrm{sgn}(t)\cdot t-\mathrm{sgn}(s)\cdot s}\leqslant \mathrm{e}^{(\gamma-\alpha)(t-s)},\quad \text{for } t\in[s,+\infty)_{\mathbb{T}},$$

which implies system (3.1) admits γ-ED if $\gamma>1$. In a similar way, we can prove that system (3.1) admits γ-ED if $\gamma<-1$. For any $\gamma\in[-1,1]$, it can be easily verified that there are no $K\geqslant 1,\alpha>0$, such that

$$\begin{aligned}|\Phi_a(t,0)|=\mathrm{e}^t\leqslant K\,\mathrm{e}^{(\gamma-\alpha)t} \text{ or } |\Phi_a(-t,0)|&=|\Phi_a^{-1}(0,-t)|\\ &=\mathrm{e}^t\leqslant K\,\mathrm{e}^{-(\gamma+\alpha)t}\quad\text{for all } t\geqslant 0.\end{aligned}$$

Therefore, we have $\Sigma(a)=[-1,1]$. On the other hand, since (3.1) is a diagonal system, we see that the contractible set of system (3.1) is $\{e-1,\mathrm{e}^{-1}-1\}$. Therefore, we conclude, in this example, that the contractible set is not equal to the dichotomy spectrum. Furthermore, there is no bijection between the dichotomy spectrum and the contractible set of this system.

On the other hand, system (3.1) can be written as

$$x(n+1)=\begin{cases} ex(n) & n\geqslant 0\\ \mathrm{e}^{-1}x(n) & n<0. \end{cases} \tag{3.3}$$

The Sacker–Sell spectrum (definition 2.2, [15]) of system (3.3) is $[\mathrm{e}^{-1},e]$. Obviously, system (3.3) is almost reducible (definition 1.2, [15]) to itself. Therefore, system

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(3.3) is contracted (definition 1.3, [**15**]) to the compact set $\{e^{-1}, e\}$. Then the contractible set (definition 1.4, [**15**]) of system (3.3) is the subset of $\{e^{-1}, e\}$. This contradicts to the result (theorem 2.4, [**15**], saying, the Sacker–Sell spectrum of system (3.3) is the contractible set). Therefore, their assertion is questionable.

Remark 3.1. Note that proposition 5 in [**15**] plays an important role in proving theorem 2.4 in [**15**]. We now show that there is a fatal error in the proof of proposition 5 in [**15**]. For the sake of clarity, we recall proposition 5 in [**15**] and its proof:

"**proposition 5 in [15]:** If the linear system

$$x(n+1) = A(n)x(n)$$

satisfies (P1)–(P2) and can be contracted to a compact set $E \subset (0, +\infty)$, then $\Sigma(A) \subseteq E$.

Proof of proposition 5 in [15]: Let us choose $\lambda \notin E$ and notice that the compactness of E allows to define $\alpha = \inf_{x\in E}|\lambda - x| > 0$. By using definition 1.4, we have that the system $x(n+1) = A(n)x(n)$ is kinematically similar to

$$y(n+1) = Diag(C_1(n), \cdots, C_d(n))\{I + B(n)\}y(n),$$

where $C_i(n) \in E$ for any $n \in \mathbb{Z}$ and $\sup_{n\in\mathbb{Z}} \|B(n)\| < \delta/\|C\|$. Now, by lemma 3.1 we know that $x(n+1) = \lambda^{-1}A(n)x(n)$ is δ-kinematically similar to

$$y(n+1) = \frac{1}{\lambda} Diag(C_1(n), \cdots, C_d(n))\{I + B(n)\}y(n).$$

Since $C_i(n) \in E$ for any $n \in \mathbb{Z}$ and $i = 1, \cdots, d$, without loss of generality, we can assume that

$$\begin{aligned} &C_i(n) < \lambda \ if \ i = 1, \cdots, m, \\ &C_i(n) > \lambda \ if \ i = m+1, \cdots, d. \end{aligned} \tag{3.4}$$

......"

We claim that the above assumption (3.4) is false!

For continuous systems, it is true to assume that $C_i(t) < \lambda$, $i \in \{1, \cdots, m\}$ (resp. $C_i(t) > \lambda$, $i \in \{m+1, \cdots, d\}$) for all $t \in \mathbb{R}$. We illustrate this point by way of contradiction. Suppose that there exists $t_1, t_2 \in \mathbb{R}$ such that $C_i(t_1) \leqslant \lambda$ and $C_i(t_2) \geqslant \lambda$. Notice that $C_i(t)$ is continuous, $t \in \mathbb{R}, C_i(t) \in E, \lambda \notin E$. Then by intermediate value theorem, there exist $a \in [t_1, t_2]$ such that $C_i(a) = \lambda \notin E$, which contradicts $C_i(t) \in E$ for all $t \in \mathbb{R}$.

However, for discrete systems, the assumption (3.4) is false. We claim that $C_i(n) < \lambda$, $i \in \{1, \cdots, m\}$ (resp. $C_i(n) > \lambda$, $i \in \{m+1, \cdots, d\}$) does not hold for all $n \in \mathbb{Z}$. It is possible that there exist $n_1, n_2 \in \mathbb{Z}$ such that $C_i(n_1) < \lambda$ and $C_i(n_2) > \lambda$, because the value of $C_i(n)$ is discontinuous in range space. For example, we take $C_i(n) = -1$ for $n \leqslant 0$ and $C_i(n) = 1$ for $n > 0$. Let $E = \{-1, 1\}$ and $\lambda = 0 \notin E$. Obviously, E is a compact set and $C_i(n) \in E$ for $n \in \mathbb{Z}$. However, $C_i(n) < \lambda$

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for $n \leqslant 0$ and $C_i(n) > \lambda$ for $n > 0$. Therefore, the assumption (3.4) is false. This is the fatal error in the proof in [**15**].

Now we consider the 1-dimensional system

$$x^{\Delta} = a(t)x \tag{3.5}$$

on time scale $\mathbb{T} = h\mathbb{Z}$, where $h > 0, h \neq \frac{e}{e-1}$ and $a(t)$ is defined by (3.2). Then the evolution operator of (3.5) is given by

$$\Phi_a(t,s) = \begin{cases} (1+he-h)^{\frac{t-s}{h}}, & t \geqslant s \geqslant 0, \\ (1+he-h)^{\frac{t}{h}}(1+h\,\mathrm{e}^{-1}-h)^{\frac{-s}{h}}, & t \geqslant 0 > s, \\ (1+h\,\mathrm{e}^{-1}-h)^{\frac{t-s}{h}}, & 0 > t \geqslant s, \\ \Phi_a^{-1}(s,t), & t < s. \end{cases} \quad (t,s \in h\mathbb{Z})$$

It can be easily verified that for any $\gamma > \lambda_1 = h^{-1}\log(1+he-h)$, there exists α satisfying $\gamma - \lambda_1 > \alpha > 0$ such that $|\Phi_a(t,s)| \leqslant \mathrm{e}^{(\gamma-\alpha)(t-s)}$ for all $t \in [s,+\infty)_{h\mathbb{Z}}$, since $1+he-h > |1+h\,\mathrm{e}^{-1}-h|$. Similarly, we have that for any $\gamma < \lambda_2 = h^{-1}\log|1+h\,\mathrm{e}^{-1}-h|$, there exists β satisfying $\lambda_2 - \gamma > \beta > 0$ such that $|\Phi_a(t,s)| \leqslant \mathrm{e}^{(\gamma+\beta)(t-s)}$ for all $t \in (-\infty,s]_{h\mathbb{Z}}$. Therefore, $\Sigma(a) \subseteq [\lambda_2,\lambda_1]$.

For any $\gamma \in [\lambda_2,\lambda_1]$ and $\alpha > 0$, we have

$$\mathrm{e}^{(\gamma-\alpha)h} < \mathrm{e}^{\gamma h} \leqslant \mathrm{e}^{\lambda_1 h} = 1+he-h \quad \text{and} \quad \mathrm{e}^{(\gamma+\alpha)h} > \mathrm{e}^{\lambda_2 h} = |1+h\,\mathrm{e}^{-1}-h|.$$

Then we get $c = \mathrm{e}^{(\gamma-\alpha)h}(1+he-h)^{-1} < 1$ and for any $K \geqslant 1$, there exists a positive integer k such that $Kc^k < 1$, namely, $K\,\mathrm{e}^{(\gamma-\alpha)kh} < (1+he-h)^k$, which implies that

$$K\,\mathrm{e}^{(\gamma-\alpha)kh} < |\Phi(kh,0)|.$$

Similarly, we have that for any $K \geqslant 1$, there exists a negative integer $l < 0$ such that $K\,\mathrm{e}^{(\gamma+\alpha)lh} < |1+h\,\mathrm{e}^{-1}-h|^l$, which implies that

$$K\,\mathrm{e}^{(\gamma+\alpha)lh} < |\Phi(lh,0)|.$$

Therefore, we obtain that for any $\gamma \in [\lambda_2,\lambda_1]$, there are no $K \geqslant 1, \alpha > 0$ such that

$$|\Phi_a(t,s)| \leqslant K\,\mathrm{e}^{(\gamma-\alpha)(t-s)} \text{ or } |\Phi_a(s,t)| \leqslant K\,\mathrm{e}^{(\gamma+\alpha)(s-t)} \quad \text{for all } t \in [s,+\infty)_{h\mathbb{Z}}.$$

Hence, $[\lambda_2,\lambda_1] \subseteq \Sigma(a)$ and then $[\lambda_2,\lambda_1] = \Sigma(a)$. However, it can be seen that the contractible set of system (3.5) is $\{e-1, \mathrm{e}^{-1}-1\}$.

In fact, the counterexample shows that the contractible set could be different from dichotomy spectrum on time scales established by Siegmund [**39**]. Furthermore, we find that there is no bijection between them.

To counter this mismatch in expectation, we propose a new notion of contractible set, named by *generalized contractible set*. For the convenience of research, we study the relations between the dichotomy spectrum and the contractible set under the framework on time scales. Suppose that S is a subset of $\mathbb{R}$. Let $\varpi(S)$ denote the minimal closed interval which contains S. The notion of Δ-contractibility is defined as follows.

Definition 3.2 Δ-contractibility. *System (2.2) is said to be Δ-contracted to the set $F = \bigcup_{i=1}^{n} \varpi(F_i) \subseteq \mathbb{R}$ if for any $\delta > 0$, there exist functions $c_i(t) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ $(i = 1, 2, \ldots, n)$ and $B(t) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ satisfying*

$$\|B\| \leqslant \delta, \quad \operatorname{Im} \overline{\xi}_\mu(c_i)(t) \subseteq F_i,$$

such that system (2.2) is kinematically similar to the system

$$x^\Delta(t) = [\operatorname{diag}(c_1(t), \ldots, c_n(t)) + B(t)]x(t),$$

where $\overline{\xi}_\mu$ is defined by (2.1) and $\operatorname{Im}(c_i)$ denotes the range of the function $c_i(t)$.

Remark 3.3. System (2.2) is said to be almost reducible to $x^\Delta = C(t)x$ if for any $\delta > 0$, system (2.2) is kinematically similar to $x^\Delta = (C(t) + B(t))x$ with $\|B\| \leqslant \delta$. Now we use the notion of almost reducibility to explain Δ-contractibility. If there exist sets $F_1, F_2, \cdots, F_n \subseteq \mathbb{R}$ such that system (2.2) is almost reducible to a diagonal system

$$x^\Delta = \operatorname{diag}(c_1(t), \ldots, c_n(t))x$$

with $\operatorname{Im} \overline{\xi}_\mu(c_i)(t) \subseteq F_i$, then we say that system (2.2) is Δ-contracted to the set $F = \bigcup_{i=1}^{n} \varpi(F_i)$. In particular, for discrete systems, if there exist sets $F_1, F_2, \cdots, F_n \subseteq \mathbb{R}$ such that the system

$$x(k+1) = A(k)x(k) \tag{3.6}$$

is almost reducible to a diagonal system

$$x(k+1) = \operatorname{diag}(c_1(k), \ldots, c_n(k))x(k)$$

with $\log |c_i(k)| \in F_i$ for all $k \in \mathbb{Z}$, then system (3.6) is Δ-contracted to the set $F = \bigcup_{i=1}^{n} \varpi(F_i)$. In [**15**], the authors give a concept of contractibility. In their paper, system (3.6) is contracted to the compact subset $E \subseteq (0, +\infty)$ if it is almost reducible to a diagonal system

$$x(k+1) = \operatorname{diag}(c_1(k), \ldots, c_n(k))x(k)$$

with $c_i(k) \in E$ for all $k \in \mathbb{Z}$. In our paper, we study the relation between $c_i(k)$ and the dichotomy spectrum established by Siegmund [**39**] (Sacker–Sell type spectrum). However, the authors of [**15**] consider the Sacker–Sell spectrum.

Definition 3.4 Generalized contractible set. *A set F is called the generalized contractible set of system (2.2) if $F \subseteq \mathbb{R}$ satisfying*

(i) *system (2.2) is Δ-contracted to F;*

(ii) *if system (2.2) is Δ-contracted to F_1, then $F \subseteq F_1$.*

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4. Main results

The main purpose of this paper is to prove that the generalized contractible set is equal to dichotomy spectrum. Our approach is based on roughness theorem and Perron's transformation. This section is divided into three subsections. In § 4.1, we provide a time-scaled version of Perron's transformation. In § 4.2, a new simple method for roughness theory on time scales is provided which is different from the method of using Lyapunov function or generalized Gröwall inequality [**41**–**43**]. In § 4.3, we prove that the generalized contractible set is exactly the dichotomy spectrum and we provide an example to illustrate the effectiveness of our result.

4.1. Perron's transformation

THEOREM 4.1 Perron's transformation. *If $A(t)$ is bounded, then system (2.2) is kinematically similar to the system $x^{\Delta} = B(t)x$, where $B(t)$ is an upper triangular bounded matrix function and $B \in \mathcal{R}$.*

Proof. Let $X(t)$ be a fundamental matrix of system (2.2). By QR decomposition, we obtain a real orthogonal matrix $U(t)$ (i.e., $U(t)U(t)^T = U(t)^T U(t) = I$ holds for all $t \in \mathbb{T}$) and a real upper triangular matrix $Y(t)$ such that

$$X(t) = U(t)Y(t).$$

Since $X(t)$ is rd-continuously differentiable, it is easily seen that $U(t)$ and $Y(t)$ are also. The change of variables $x = U(t)y$ replaces the equation (2.2) by

$$y^{\Delta} = B(t)y,$$

where $B = (U^{\sigma})^T AU - (U^{\sigma})^T U^{\Delta}$. Then we have

$$\begin{aligned} I + \mu(t)B(t) &= I + \mu(t)[(U^{\sigma})^T AU - (U^{\sigma})^T U^{\Delta}] \\ &= (U^{\sigma})^T [I + \mu(t)A]U + I - (U^{\sigma})^T \left(U + \mu(t)U^{\Delta}\right) \\ &= (U^{\sigma})^T [I + \mu(t)A]U + I - (U^{\sigma})^T U^{\sigma} = (U^{\sigma})^T [I + \mu(t)A]U, \end{aligned}$$

which implies that $B \in \mathcal{R}$ since $A \in \mathcal{R}$. Since $Y(t)$ is a fundamental matrix of the transformed equation, $B(t) = Y^{\Delta}(t)Y^{-1}(t)$ is real upper triangular. Note that

$$U^{\sigma} BU^T = A - U^{\Delta} U^T, \quad U^{\sigma} B^T (U^{\sigma})^T = U^{\sigma} U^T A^T - U^{\sigma} (U^{\Delta})^T. \tag{4.1}$$

and

$$I^{\Delta} = (UU^T)^{\Delta} = U^{\sigma} (U^T)^{\Delta} + U^{\Delta} U^T = O. \tag{4.2}$$

From (4.1) and (4.2), we have

$$U^{\sigma}[B + B^T (U^{\sigma})^T U]U^T = U^{\sigma} BU^T + U^{\sigma} B^T (U^{\sigma})^T = A + U^{\sigma} U^T A^T. \tag{4.3}$$

We introduce the norm defined by $\|A\| := \sup_{x \in \mathbb{R}^n} |Ax|/|x|$. It is well known that $\|U\| = 1$ if U is an orthogonal matrix. Then, from (4.3), we have

$$\|B + B^T (U^{\sigma})^T U\| \leqslant \|A + U^{\sigma} U^T A^T\| \leqslant \|A\| + \|A^T\|. \tag{4.4}$$

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We claim that A^T is bounded. In fact, let $\|A\|_\infty = \sum |a_{ij}|$ and we obtain $\|A\|_\infty = \|A^T\|_\infty$. Since $M_n(\mathbb{R})$ is a finite dimensional linear space, there are positive constants c_1, c_2, such that $c_1\|\cdot\| \leqslant \|\cdot\|_\infty \leqslant c_2\|\cdot\|$. Therefore, we get $\|B + B^T(U^\sigma)^T U\| \leqslant (1+c_2)\|A\|$

Suppose that $B(t)$ is unbounded. Then there exists a sequence $\{t_m | m \in \mathbb{N}_+\}$, such that $\|B(t_m)\| \geqslant m$. Note that

$$\|B\| = \|(U^\sigma)^T AU - (U^\sigma)^T U^\Delta\| \leqslant \|A\| + \|U^\Delta\| = \|A\| + \|\frac{U^\sigma - U}{\mu(t)}\| \leqslant \|A\| + \frac{2}{\mu(t)},$$

which implies that $\mu(t_m) \to 0$ as $m \to +\infty$. Let

$$C(t) = (U^\sigma)^T U - I = -(U^\sigma)^T(U^\sigma - U).$$

Thus, $\|C(t_m)\| \leqslant \|U(t_m + \mu(t_m)) - U(t_m)\|$. Since $U(t)$ is continuous, we have $\|C(t_m)\| \to 0$ as $m \to +\infty$. For any $n \times n$ matrix $D = (d_{ij})$, we have

$$n^{-1}\sum_{i,j}|d_{ij}|^2 \leqslant \|D\|^2 \leqslant \sum_{i,j}|d_{ij}|^2.$$

This inequality can be found in the page 88 of [**19**]. Since B is upper triangular, we have

$$\begin{aligned}(1+c_2)\|A\| &\geqslant \|B + B^T(U^\sigma)^T U\| = \|B + B^T + B^T C\| \geqslant \|B + B^T\| - \|C\|\|B\| \\ &\geqslant \left(n^{-1}\sum_{j,k}|b_{jk} + b_{kj}|^2\right)^{\frac{1}{2}} - \|C\|\|B\| \\ &\geqslant \left(n^{-\frac{1}{2}} - \|C\|\right)\|B\|.\end{aligned}$$

Thus, $(1+c_2)\|A(t_m)\| \geqslant \left(n^{-\frac{1}{2}} - \|C(t_m)\|\right)\|B(t_m)\| \geqslant m\left(2n^{-\frac{1}{2}} - \|C(t_m)\|\right)$. Since $A(t)$ is bounded, let $m \to +\infty$ and we have the right side of the above inequality unbounded, which leads to a contradiction. Therefore, $B(t)$ is bounded. □

4.2. Roughness

Lemma 4.2. *Let $X(t)$ be a fundamental matrix of system (2.2). Then system (2.2) admits γ-ED if and only if there exist a projection matrix Q (i.e., $Q^2 = Q$) on $\mathbb{R}^n$ and constants $K \geqslant 1, \alpha > 0$ such that for any $t, s \in \mathbb{T}$,*

$$\begin{aligned}\|X(t)QX^{-1}(s)\| &\leqslant K\,\mathrm{e}^{(\gamma-\alpha)(t-s)}, \quad t \geqslant s, \\ \|X(t)(I-Q)X^{-1}(s)\| &\leqslant K\,\mathrm{e}^{(\gamma+\alpha)(t-s)}, \quad t \leqslant s.\end{aligned}$$

Remark 4.3. Obviously, system (2.2) has an evolution operator $\Phi_A(t,s) = X(t)X^{-1}(s)$. In the proof of sufficiency we construct the invariant projector $P(t) = X(t)QX^{-1}(t)$ and for the necessary condition we let $Q = X^{-1}(\tau)P(\tau)X(\tau)$. The proof of lemma 4.2 is simple and we omit it.

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LEMMA 4.4 Theorem 1, [26]. *Let* $-p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ *and* $y \in \mathcal{R}(\mathbb{T}, \mathbb{R})$. *Suppose that* $p(t) \geqslant 0, y(t) \geqslant 0$ *and* $\alpha > 0$. *Then*

$$y(t) \leqslant \alpha + \left| \int_s^t y(\tau)p(\tau)\Delta\tau \right|, \quad \forall t \in \mathbb{T}$$

implies

$$y(t) \leqslant \begin{cases} \alpha e_p(t,s), & \text{for } t \geqslant s, \\ \alpha e_{-p}(t,s), & \text{for } t \leqslant s. \end{cases}$$

LEMMA 4.5 Corollary 4.12, [2]. *Let equation* (2.2) *admit a* γ*-ED with* K, α, *and projection* Q *with rank* $m \leqslant n$. *Then system* (2.2) *is kinematically similar to the block diagonal system*

$$y^\Delta = \begin{pmatrix} B_1(t) & \\ & B_2(t) \end{pmatrix} y \tag{4.5}$$

which has the following properties:

(a) $B_1(t) \in \mathbb{R}^{m\times m}$ *and* $B_2(t) \in \mathbb{R}^{(n-m)\times(n-m)}$ *for all* $t \in \mathbb{T}$;

(b) *there exists a constant* $\overline{K} \geqslant 1$ *such that the estimates*

$$\|\Phi_{B_1}(t,s)\| \leqslant \overline{K}\,\mathrm{e}^{(\gamma-\alpha)(t-s)}, \quad t \geqslant s,$$
$$\|\Phi_{B_2}(t,s)\| \leqslant \overline{K}\,\mathrm{e}^{(\gamma+\alpha)(t-s)}, \quad t \leqslant s,$$

hold for $t, s \in \mathbb{T}$.

LEMMA 4.6. *Let* $\mu(t)$ *be bounded and the upper bound of* $\mu(t)$ *is denoted by* μ^*. *Suppose* $B(t) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ *and equation* (2.2) *admits* γ*-ED with the constants* K, α *and projection* $Q = O$ *(reps.* $Q = I$*). If*

$$\delta = \|B(t)\| \leqslant \delta_1 \ (\text{resp. } \delta_2),$$

then system

$$y^\Delta = [A(t) + B(t)]y \tag{4.6}$$

also admits γ*-ED with the projection* $Q_1 = O$ *(resp.* $Q_1 = I$*), where*

$$\delta_1 = \min\{(\mu^* K\,\mathrm{e}^{|\gamma+\alpha|\mu^*})^{-1}, \alpha(K\,\mathrm{e}^{|\gamma+\alpha|\mu^*})^{-1}\},$$
$$\delta_2 = \min\{(\mu^* K\,\mathrm{e}^{|\gamma-\alpha|\mu^*})^{-1}, \alpha(K\,\mathrm{e}^{|\gamma-\alpha|\mu^*})^{-1}\}.$$

Proof. We only prove the case $Q = O$ and the case $Q = I$ can be proved in a similar way. Let $\Phi(t,s)$ and $\Psi(t,s)$ denote the evolution operators of systems

(2.2) and (4.6), respectively. It can be seen that

$$\|\Phi(t,s)\| \leqslant K\,\mathrm{e}^{(\gamma+\alpha)(t-s)}, \quad t \leqslant s.$$

For any $y \in \mathbb{R}^n$, we have

$$\Psi(t,s)y = \Phi(t,s)y + \int_s^t \Phi(t,\sigma(\tau))B(\tau)\Psi(\tau,s)y\Delta\tau,$$

hence for $t \leqslant s$,

$$\begin{aligned}\|\Psi(t,s)y\| &\leqslant \|\Phi(t,s)y\| + \left|\int_s^t \|\Phi(t,\sigma(\tau))\|\,\|B(\tau)\|\,\|\Psi(\tau,s)y\|\Delta\tau\right| \\ &\leqslant K\,\mathrm{e}^{(\gamma+\alpha)(t-s)}\|y\| + K\delta\left|\int_s^t \|\Psi(\tau,s)y\|\mathrm{e}^{(\gamma+\alpha)(t-\sigma(\tau))}\Delta\tau\right|.\end{aligned}$$

Mutiplying both sides by $\mathrm{e}^{-(\gamma+\alpha)t}$, we get

$$\begin{aligned}\mathrm{e}^{-(\gamma+\alpha)t}\|\Psi(t,s)y\| &\leqslant K\,\mathrm{e}^{-(\gamma+\alpha)s}\|y\| + K\delta\left|\int_s^t \|\Psi(\tau,s)y\|\mathrm{e}^{-(\gamma+\alpha)\tau}\,\mathrm{e}^{-(\gamma+\alpha)\mu(\tau)}\Delta\tau\right| \\ &\leqslant K\,\mathrm{e}^{-(\gamma+\alpha)s}\|y\| + K\delta\left|\int_s^t \|\Psi(\tau,s)y\|\,\mathrm{e}^{-(\gamma+\alpha)\tau}\,\mathrm{e}^{|\gamma+\alpha|\mu^*}\Delta\tau\right|.\end{aligned}$$

It can be seen that $-K\delta\,\mathrm{e}^{|\gamma+\alpha|\mu^*}$ is positively regressive if $\delta < [\mu^* K\,\mathrm{e}^{|\gamma+\alpha|\mu^*}]^{-1}$. By lemma 4.4, for $\delta < [\mu^* K\,\mathrm{e}^{|\gamma+\alpha|\mu^*}]^{-1}$ we have

$$\mathrm{e}^{-(\gamma+\alpha)t}\|\Psi(t,s)y\| \leqslant K\,\mathrm{e}^{-(\gamma+\alpha)s}\|y\|e_{-K\delta\mathrm{e}^{|\gamma+\alpha|M}}(t,s).$$

It can be seen that the function $\xi(v) = \log(1 - vK\delta\,\mathrm{e}^{|\gamma+\alpha|\mu^*})/v$ is decreasing with respect to v. Thus,

$$\mathrm{e}^{-(\gamma+\alpha)t}\|\Psi(t,s)y\| \leqslant K\,\mathrm{e}^{-(\gamma+\alpha)s}\|y\|\exp\{-K\delta\,\mathrm{e}^{|\gamma+\alpha|\mu^*}(t-s)\}.$$

Mutiplyng both sides by $\mathrm{e}^{(\gamma+\alpha)t}$, we get

$$\begin{aligned}\|\Psi(t,s)y\| &\leqslant K\,\mathrm{e}^{(\gamma+\alpha)(t-s)}\|y\|\exp\{-K\delta\,\mathrm{e}^{|\gamma+\alpha|\mu^*}(t-s)\} \\ &\leqslant K\|y\|\exp\{(\gamma+\alpha-K\delta\,\mathrm{e}^{|\gamma+\alpha|\mu^*})(t-s)\}, \quad t \leqslant s.\end{aligned}$$

Note that $\alpha_1 = \alpha - K\delta\,\mathrm{e}^{|\gamma+\alpha|\mu^*} > 0$ for $\delta < \delta_1 = \min\{[\mu^* K\,\mathrm{e}^{|\gamma+\alpha|\mu^*}]^{-1}, \alpha[K\,\mathrm{e}^{|\gamma+\alpha|\mu^*}]^{-1}\}$, therefore system (4.6) admits γ-ED with projection $Q_1 = O$ if $\delta < \delta_1$. The other assertion can be proved in a similar way. □

Remark 4.7. If $\mathbb{T} = \mathbb{R}$, then $\mu^* = 0$, $\delta_1 = \delta_2 = \alpha/K$, which are consistent with lemma 4.6 in [**25**]. If $\mathbb{T} = \mathbb{Z}, \gamma = 0$, then $\mu^* = 1$, $\delta_1 = \delta_2 = \min\{K^{-1}\,\mathrm{e}^{-\alpha}, \alpha K^{-1}\,\mathrm{e}^{-\alpha}\}$

Theorem 4.8 Roughness theorem. *Assume that $\mu(t)$ is bounded, $B(t) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ and system (2.2) admits γ-ED with the constants K, α and projection Q. Then there exists $\delta > 0$ such that (4.6) admits γ-ED with projection Q_1 similar to Q when $\|B(t)\| \leqslant \delta$.*

Proof. Let k be the rank of Q. By lemma 4.5, system (2.2) is kinematically similar to the block diagonal system (4.5) by a Lyapunov transformation $x = J(t)y$, where $B_1(t) \in \mathbb{R}^{m\times m}$, $B_2(t) \in \mathbb{R}^{(n-m)\times(n-m)}$ for all $t \in \mathbb{T}$, and there exists a constant $K_0 \geqslant 1$ such that the estimates

$$\begin{aligned} \|\Phi_{B_1}(t,s)\| &\leqslant K_0\,\mathrm{e}^{(\gamma-\alpha)(t-s)}, \quad t \geqslant s, \\ \|\Phi_{B_2}(t,s)\| &\leqslant K_0\,\mathrm{e}^{(\gamma+\alpha)(t-s)}, \quad t \leqslant s, \end{aligned} \tag{4.7}$$

hold for $t, s \in \mathbb{T}$. Let

$$B_0(t) = \begin{pmatrix} B_1(t) & \\ & B_2(t) \end{pmatrix}.$$

Note that

$$\begin{aligned} J^{\Delta}(t) &= A(t)J(t) - J(\sigma(t))B_0(t) \\ &= [A(t) + B(t)]J(t) - J(\sigma(t))[B_0(t) + J^{-1}(\sigma(t))B(t)J(t)], \end{aligned}$$

which implies that system (4.6) is kinematically similar to the system

$$z^{\Delta}(t) = [B_0(t) + J^{-1}(\sigma(t))B(t)J(t)]z. \tag{4.8}$$

Let

$$D(t) = J^{-1}(\sigma(t))B(t)J(t)$$

and

$$\mathcal{X} = \{H \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n}) : \|H\| \leqslant \infty\},$$

where $\|H\| := \sup_{t\in\mathbb{T}} \|H(t)\|$. It can be seen that $\mathcal{X}$ is a Banach space with the norm $\|\cdot\|$. Let $E_k = \mathrm{diag}(I_k, O)$, where I_k is the identity matrix of order k. Consider a matrix function $H \in \mathcal{X}$, and the mapping T defined by

$$\begin{aligned} TH(t) = &\int_{-\infty}^{t} \Phi_{B_0}(t,\sigma(s))E_k(I - H(\sigma(s)))D(s)(I + H(s))(I - E_k)\Phi_{B_0}(s,t)\Delta s \\ &- \int_{t}^{\infty} \Phi_{B_0}(t,\sigma(s))(I - E_k)(I - H(\sigma(s)))D(s)(I + H(s))E_k\Phi_{B_0}(s,t)\Delta s. \end{aligned}$$

Now we show that $TH \in \mathcal{X}$. It follows from (4.7) that

$$\begin{aligned} \|\Phi_{B_0}(t,s)E_k\| = \|\Phi_{B_1}(t,s)\| &\leqslant K_0\,\mathrm{e}^{(\gamma-\alpha)(t-s)}, \quad t \geqslant s, \\ \|\Phi_{B_0}(t,s)(I - E_k)\| = \|\Phi_{B_2}(t,s)\| &\leqslant K_0\,\mathrm{e}^{(\gamma+\alpha)(t-s)}, \quad t \leqslant s. \end{aligned}$$

Thus we have

$$\begin{aligned} \|TH(t)\| \leqslant &\int_{-\infty}^{t} K_0\,\mathrm{e}^{(\gamma-\alpha)(t-s)}\,\mathrm{e}^{-(\gamma-\alpha)\mu(s)}(1 + \|H\|)^2\|D\|K_0\,\mathrm{e}^{(\gamma+\alpha)(s-t)}\Delta s \\ &+ \int_{t}^{\infty} K_0\,\mathrm{e}^{(\gamma+\alpha)(t-s)}\,\mathrm{e}^{-(\gamma+\alpha)\mu(s)}(1 + \|H\|)^2\|D\|K_0\,\mathrm{e}^{(\gamma-\alpha)(s-t)}\Delta s \end{aligned}$$

$$= K_0^2(1+\|H\|)^2\|D\| \times \left(\mathrm{e}^{|\gamma-\alpha|\mu^*}\int_{-\infty}^{t}\mathrm{e}^{2\alpha(s-t)}\Delta s + \mathrm{e}^{|\gamma+\alpha|\mu^*}\int_{t}^{\infty}\mathrm{e}^{-2\alpha(s-t)}\Delta s\right). \tag{4.9}$$

Let us define the map $\varphi : \mathbb{R} \to \mathcal{R}^+$:

$$\varphi(\gamma) := \lim_{s\searrow\mu(t)}\frac{\mathrm{e}^{\gamma s}-1}{s}.$$

It can be verified that $\gamma \leqslant \varphi(\gamma) \leqslant \frac{\mathrm{e}^{\gamma\mu^*}-1}{\mu^*}$. Note that

$$\int_{-\infty}^{t}\mathrm{e}^{2\alpha(s-t)}\Delta s \leqslant \int_{-\infty}^{t}\mathrm{e}^{-2\alpha(t-s)}\,\mathrm{d}s = \frac{1}{2\alpha} \tag{4.10}$$

and

$$\int_{t}^{\infty}\mathrm{e}^{-2\alpha(s-t)}\Delta s = \int_{t}^{\infty}e_{\varphi(-2\alpha)}(s,t)\Delta s = \left.\frac{\mathrm{e}^{-2\alpha(s-t)}}{\varphi(-2\alpha)(s)}\right|_{s=t}^{s=\infty} \leqslant \frac{\mu^*}{1-\mathrm{e}^{-2\alpha\mu^*}}. \tag{4.11}$$

Therefore, $TH(t)$ is bounded and $TH \in \mathcal{X}$. Let

$$\mathcal{X}_0 = \left\{H : H \in \mathcal{X}, \|H\| \leqslant \frac{1}{2}\right\},$$

then for any $H_1, H_2 \in \mathcal{X}_0$, we have

$$\begin{aligned}&(I-H_1(\sigma(t)))D(t)(I+H_1(t)) - (I-H_2(\sigma(t)))D(t)(I+H_2(t))\\ &= (H_2(\sigma(t))-H_1(\sigma(t)))D(t) - D(t)(H_2(t)-H_1(t))\\ &\quad + (H_2(\sigma(t))-H_1(\sigma(t)))D(t)H_2(t) + H_1(\sigma(t))D(t)(H_2(t)-H_1(t)).\end{aligned}$$

Thus,

$$\begin{aligned}&\|TH_1(t)-TH_2(t)\|\\ &= \|\int_{-\infty}^{t}\Phi_{B_0}(t,\sigma(s))E_k[(I-H_1(\sigma(s)))D(s)(I+H_1(s))\\ &\quad - (I-H_2(\sigma(s)))D(s)(I+H_2(s))]\\ &\quad \cdot (I-E_k)\Phi_{B_0}(s,t)\Delta s - \int_{t}^{\infty}\Phi_{B_0}(t,\sigma(s))(I-E_k)\\ &\quad \times [(I-H_1(\sigma(s)))D(s)(I+H_1(s))\\ &\quad - (I-H_2(\sigma(s)))D(s)(I+H_2(s))]E_k\Phi_{B_0}(s,t)\Delta s\|\\ &= \|\int_{-\infty}^{t}\Phi_{B_0}(t,\sigma(s))E_k[(H_2(\sigma(t))-H_1(\sigma(t)))D(t) - D(t)(H_2(t)-H_1(t))\\ &\quad + (H_2(\sigma(t))-H_1(\sigma(t)))D(t)H_2(t)\\ &\quad + H_1(\sigma(t))D(t)(H_2(t)-H_1(t))](I-E_k)\Phi_{B_0}(s,t)\Delta s\end{aligned}$$

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$$
\begin{aligned}
&- \int_t^{\infty} \Phi_{B_0}(t,\sigma(s))(I-E_k)[(H_2(\sigma(t))\\
&- H_1(\sigma(t)))D(t) - D(t)(H_2(t)-H_1(t))\\
&+ (H_2(\sigma(t)) - H_1(\sigma(t)))D(t)H_2(t) + H_1(\sigma(t))D(t)(H_2(t)\\
&- H_1(t))]E_k\Phi_{B_0}(s,t)\Delta s\|\\
\leqslant{}& 3K_0^2\|D\|\,\|H_1-H_2\|\left(\mathrm{e}^{|\gamma-\alpha|\mu^*}\int_{-\infty}^{t}\mathrm{e}^{2\alpha(s-t)}\Delta s + \mathrm{e}^{|\gamma+\alpha|\mu^*}\int_t^{\infty}\mathrm{e}^{-2\alpha(s-t)}\Delta s\right).
\end{aligned}
\tag{4.12}
$$

Note that $\|D\| \leqslant \|J\|\|J^{-1}\|\|B\|$. It follows from (4.9), (4.10), (4.11) and (4.12) that there exists a constant $\delta > 0$, such that $\|B\| < \delta$ implies that

(i) $TH \in \mathcal{X}_0$, if $H \in \mathcal{X}_0$;

(ii) $\|TH_1 - TH_2\| \leqslant \frac{1}{2}\|H_1 - H_2\|$, if $H_1, H_2 \in \mathcal{X}_0$.

Therefore, T is a contraction mapping. Note that $\mathcal{X}_0$ is a closed subspace of $\mathcal{X}$, then $\mathcal{X}_0$ is a Banach space with the norm $\|\cdot\|$. By the contraction mapping principle, there exists a unique fixed point $H \in \mathcal{X}_0$ such that

$$
\begin{aligned}
H(t) ={}& \int_{-\infty}^{t} \Phi_{B_0}(t,\sigma(s))E_k(I - H(\sigma(s)))D(s)(I+H(s))(I-E_k)\Phi_{B_0}(s,t)\Delta s\\
&- \int_t^{\infty} \Phi_{B_0}(t,\sigma(s))(I-E_k)(I-H(\sigma(s)))D(s)(I+H(s))E_k\Phi_{B_0}(s,t)\Delta s.
\end{aligned}
\tag{4.13}
$$

It can be seen that $E_k\Phi_{B_0}(t,s) = \Phi_{B_0}(t,s)E_k$ and $(I-E_k)\Phi_{B_0}(t,s) = \Phi_{B_0}(t,s)(I-E_k)$ since

$$
\Phi_{B_0}(t,s) = \begin{pmatrix} \Phi_{B_1}(t,s) & \\ & \Phi_{B_2}(t,s) \end{pmatrix}.
$$

Therefore, $E_kH(t) = H(t)(I-E_k)$. Then we obtain

$$
\begin{aligned}
&H^{\Delta}(t)\\
&= B_0(t)H(t) + \int_{-\infty}^{t} \Phi_{B_0}(\sigma(t),\sigma(s))E_k(I-H(\sigma(s)))D(s)(I+H(s))(I-E_k)\\
&\quad \times (-\Phi_{B_0}(s,\sigma(t)))B_0(t)\Delta s\\
&\quad - \int_t^{\infty} \Phi_{B_0}(\sigma(t),\sigma(s))(I-E_k)(I-H(\sigma(s)))D(s)(I+H(s))\\
&\quad \times E_k(-\Phi_{B_0}(s,\sigma(t)))B_0(t)\Delta s\\
&\quad + \Phi_{B_0}(\sigma(t),\sigma(t))E_k(I-H(\sigma(t)))D(t)(I+H(t))(I-E_k)\Phi_{B_0}(t,\sigma(t))\\
&\quad + \Phi_{B_0}(\sigma(t),\sigma(t))(I-E_k)(I-H(\sigma(t)))D(t)(I+H(t))E_k\Phi_{B_0}(t,\sigma(t))
\end{aligned}
$$

$$
\begin{aligned}
&= B_0H(t) - H(\sigma(t))B_0(t) \\
&\quad + \int_t^{\sigma(t)} \Phi_{B_0}(\sigma(t),\sigma(s))E_k(I - H(\sigma(s)))D(s)(I + H(s)) \\
&\quad \times (I - E_k)\Phi_{B_0}(s,\sigma(t))B_0(t)\Delta s \\
&\quad + \int_t^{\sigma(t)} \Phi_{B_0}(\sigma(t),\sigma(s))(I - E_k)(I - H(\sigma(s)))D(s)(I + H(s)) \\
&\quad \times E_k\Phi_{B_0}(s,\sigma(t))B_0(t)\Delta s \\
&\quad + E_k(I - H(\sigma(t)))D(t)(I + H(t))(I - E_k)\Phi_{B_0}(t,\sigma(t)) \\
&\quad + (I - E_k)(I - H(\sigma(t)))D(t)(I + H(t))E_k\Phi_{B_0}(t,\sigma(t)) \\
&= B_0H(t) - H(\sigma(t))B_0(t) + [E_k(I - H(\sigma(t)))D(t)(I + H(t))(I - E_k) \\
&\quad + (I - E_k)(I - H(\sigma(t)))D(t)(I + H(t))E_k]\Phi_{B_0}(t,\sigma(t))(\sigma(t)B_0(t) + I) \\
&= B_0H(t) - H(\sigma(t))B_0(t) + E_k(I - H(\sigma(t)))D(t)(I + H(t))(I - E_k) \\
&\quad + (I - E_k)(I - H(\sigma(t)))D(t)(I + H(t))E_k \\
&= B_0(t)H(t) - H(\sigma(t))B_0(t) + E_kD(t)(I + H(t))(I - E_k) \\
&- E_kH(\sigma(t))D(t)(I + H(t))(I - E_k) \\
&\quad + (I - E_k)D(t)(I + H(t))E_k - (I - E_k)H(\sigma(t))D(t)(I + H(t))E_k \\
&= B_0(t)H(t) - H(\sigma(t))B_0(t) + E_kD(t)(I + H(t))(I - E_k) \\
&\quad - H(\sigma(t))(I - E_k)D(t)(I + H(t))(I - E_k) \\
&\quad + (I - E_k)D(t)(I + H(t))E_k - H(\sigma(t))E_kD(t)(I + H(t))E_k \\
&= B_0(t)(I + H(t)) - (I + H(\sigma(t)))B_0(t) + E_kD(t)(I + H(t))(I - E_k) \\
&\quad + (I - E_k)D(t)(I + H(t))(I - E_k) \\
&\quad - (I + H(\sigma(t)))(I - E_k)D(t)(I + H(t))(I - E_k) \\
&\quad + (I - E_k)D(t)(I + H(t))E_k + E_kD(t)(I + H(t))E_k \\
&\quad - (I + H(\sigma(t)))E_kD(t)(I + H(t))E_k \\
&= B_0(t)(I + H(t)) - (I + H(\sigma(t)))B_0(t) + D(t)(I + H(t)) \\
&\quad - (I + H(\sigma(t)))[(I - E_k)D(t)(I + H(t))(I - E_k) + E_kD(t)(I + H(t))E_k].
\end{aligned}
$$

Let

$$R(t) = I + H(t). \tag{4.14}$$

Then $\|R\| \leqslant \frac{3}{2}$ since $\|H\| \leqslant \frac{1}{2}$. For any $t \in \mathbb{T}$ and $y \in \mathbb{R}^n$, $y \neq 0$,

$$\|R(t)y\| = \|y + H(t)y\| \geqslant \|y\| - \|H(t)\|\|y\| \geqslant \|y\| - \frac{1}{2}\|y\| = \frac{1}{2}\|y\|.$$

It follows from $\|R(t)y\| \neq 0$ that $R(t)$ is invertible for any $t \in \mathbb{T}$. Then we obtain

$$\|y\| = \|R(t)R^{-1}(t)y\| \geqslant \frac{1}{2}\|R^{-1}(t)y\|,$$

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i.e., $\|R^{-1}(t)y\| \leqslant 2\|y\|$. Therefore, $\|R^{-1}(t)\| \leqslant 2$. Note that

$$\begin{aligned} R^{\Delta}(t) = H^{\Delta}(t) &= [B_0(t) + D(t)]R(t) \\ &\quad - R(\sigma(t))[B_0(t) + (I - E_k)D(t)R(t)(I - E_k) + E_k D(t)R(t)E_k], \end{aligned}$$

which implies that system (4.8) is kinematically similar to the system

$$u^{\Delta}(t) = [B_0(t) + (I - E_k)D(t)R(t)(I - E_k) + E_k D(t)R(t)E_k]u. \tag{4.15}$$

Hence, system (4.6) is kinematically similar to system (4.15). It follows from lemma 4.6 that system (4.15) admits γ-ED when $\|B\|$ is sufficiently small and the rank of projection is k. Then system (4.6) also admits γ-ED and the rank of projection is k. □

4.3. Generalized contractibility

Lemma 4.9 Theorem 11, [**39**]. *The dichotomy spectrum $\Sigma(A)$ of system (2.2) is the disjoint union of k closed intervals where $0 \leqslant k \leqslant n$, namely,*

$$\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i]$$

with $-\infty \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k \leqslant \infty$.

Lemma 4.10. *Assume that $\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i]$ with $a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k$, then there exist functions $B_i(t)$ $(i = 1, 2, \ldots, k)$ such that $\Sigma(B_i) = [a_i, b_i]$ and system (2.2) is kinematically similar to the system*

$$x^{\Delta} = \operatorname{diag}(B_1(t), \cdots, B_k(t))x. \tag{4.16}$$

Proof. Let $\gamma \in (b_{k-1}, a_k)$, then system (2.2) admits γ-ED. It follows from lemma 4.5 that system (2.2) is kinematically similar to

$$x_1^{\Delta} = A_1(t)x_1, \quad x_2^{\Delta} = A_2(t)x_2, \tag{4.17}$$

where the first equation of (4.17) admits γ-ED with the invariant projector I and the second equation of (4.17) admits γ-ED with the invariant projector O. By lemma 4.9, we have $\Sigma(A_1) = \bigcup_{i=1}^{k-1} [a_i, b_i]$, $\quad \Sigma(A_2) = [a_k, b_k]$. Let $B_0 = A_1$, $B_k = A_2$, then

$$\Sigma(B_0) = \bigcup_{i=1}^{k-1} [a_i, b_i], \quad \Sigma(B_k) = [a_k, b_k].$$

The proof is completed by repeating the above steps. □

Lemma 4.11 Theorem 5, [**26**]. *Suppose that $A(t)$ is bounded on $\mathbb{T}$ and let $L = \sup_{t\in\mathbb{T}} \|A(t)\|$. Then one has*

$$\|\Phi_A(t,t_0)x_1 - \Phi_A(t,t_0)x_2\| \leqslant \begin{cases} \|x_1 - x_2\| e_L(t,t_0), & \text{for } t \in [t_0,+\infty)_{\mathbb{T}}, \\ \|x_1 - x_2\| e_{-L}(t,t_0), & \text{for } t \in (-\infty,t_0]_{\mathbb{T}}, \end{cases}$$

where $\Phi_A(t,t_0)$ is the evolution operator of system (2.2).

Lemma 4.12. *The following statements are true.*

(a) *If $[c,d] \subseteq \mathbb{R} - \Sigma(A)$, then system (2.2) admits c-ED and d-ED with the same projector;*

(b) *If $\Sigma(A) \subseteq [a,b]$ and $\lambda > b$ (resp. $\lambda < a$), then system (2.2) admits λ-ED with the projector I (resp. O) and constants $K = 1, \alpha > 0$.*

Proof. (a). For any $\lambda \in [c,d]$, system (2.2) admits λ-ED, i.e., there exist constants $\alpha_\lambda > 0, K_\lambda \geqslant 1$ and a projection Q_λ such that

$$\|\Phi_A(t,s)P_\lambda(s)\| \leqslant K_\lambda\, \mathrm{e}^{(\lambda-\alpha_\lambda)(t-s)}, t \geqslant s,$$
$$\|\Phi_A(t,s)(I - P_\lambda(s))\| \leqslant K_\lambda\, \mathrm{e}^{(\lambda+\alpha_\lambda)(t-s)}, t \leqslant s,$$

where $\Phi_A(t,s)$ is the evolution operator of system (2.2). Obviously, system (2.2) also admits γ-ED with the projector P_λ when $\gamma \in (\lambda - \alpha_\lambda, \lambda + \alpha_\lambda)$. Since the family of open intervals $\{(\lambda - \alpha_\lambda, \lambda + \alpha_\lambda) | \lambda \in [c,d]\}$ covers the interval $[c,d]$. By Heine–Borel theorem, there are finite open intervals $(\lambda_1 - \delta_{\lambda_1}, \lambda_1 + \delta_{\lambda_1}), \cdots, (\lambda_m - \delta_{\lambda_m}, \lambda_m + \delta_{\lambda_m})$ that cover the interval $[c,d]$. Suppose $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m$. Then $(\lambda_i - \delta_{\lambda_i}, \lambda_i + \delta_{\lambda_i}) \cap (\lambda_{i+1} - \delta_{\lambda_{i+1}}, \lambda_{i+1} + \delta_{\lambda_{i+1}}) \neq \emptyset$ $(i = 1, 2 \cdots, m-1)$. Therefore, $P_{\lambda_i} = P_{\lambda_{i+1}}$ $(i = 1, 2 \cdots, m-1)$, which implies $P_c = P_d$.

(b). Let $L = \sup_{t\in\mathbb{T}} \|A(t)\|$ and let $M = \max\{L, \lambda\}$. According to lemma 4.11, for any $\lambda > b$, we have

$$\begin{aligned} \|\Phi_A(t,s)x\| &= \|\Phi_A(t,s)x - \Phi_A(t,s)\cdot 0\| \leqslant \|x\| e_L(t,s) \leqslant \mathrm{e}^{L(t-s)}\|x\| \\ &\leqslant \mathrm{e}^{M(t-s)}\|x\|, \quad t \in [s,+\infty)_{\mathbb{T}}, \end{aligned}$$

which implies that $\|\Phi_A(t,s)\| \leqslant \mathrm{e}^{M(t-s)}$ for $t \in [s,+\infty)_{\mathbb{T}}$. Thus, system (2.2) admits M-ED with the projector I. Note that $[\lambda, M] \subseteq \mathbb{R} - \Sigma(A)$. From the statement (a) in this lemma, we have system (2.2) that admits γ-ED with the projector I. The other assertion can be proved in a similar way. $\square$

Lemma 4.13. *Assume that $C(t) = (c_{ij}(t)) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ is a bounded upper triangular matrix-valued function ($c_{ij}(t) = 0$ if $i > j$) satisfying $\Sigma(C) = [a,b]$, then*

$$\Sigma(C) = \bigcup_{i=1}^{n} \Sigma(c_{ii}).$$

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Proof. Let $D(t) = \mathrm{diag}(c_{11}(t), c_{22}(t), \cdots, c_{nn}(t))$. Obviously, for any $\lambda \notin \bigcup_{i=1}^{n} \Sigma(c_{ii})$, the diagonal system

$$x^{\Delta} = D(t)x$$

admits λ-ED. From theorem 4.8, there exists $\delta > 0$, such that the system

$$y^{\Delta} = [\mathrm{diag}(c_{11}(t), c_{22}(t), \cdots, c_{nn}(t)) + B(t)]y$$

also admits λ-ED if $\sup_{t\in\mathbb{T}} \|B(t)\| < \delta$, where $B(t) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$. Let $\sup_{t\in\mathbb{T}} \|C(t)\| = L$ and $\eta = \frac{\delta}{2n^2L}$. Taking the transformation $x = \mathrm{diag}(1, \eta, \cdots, \eta^{n-1})z$, we obtain

$$\begin{aligned}
z^{\Delta} &= [\mathrm{diag}(1, \eta, \cdots, \eta^{n-1})]^{-1}x^{\Delta} \\
&= [\mathrm{diag}(1, \eta, \cdots, \eta^{n-1})]^{-1}C(t)x \\
&= [\mathrm{diag}(1, \eta, \cdots, \eta^{n-1})]^{-1}C(t)\mathrm{diag}(1, \eta, \cdots, \eta^{n-1})z \\
&= \left[\begin{pmatrix} c_{11}(t) & & & \\ & c_{22}(t) & & \\ & & \ddots & \\ & & & c_{nn}(t) \end{pmatrix}\right. \\
&\quad \left. + \begin{pmatrix} 0 & \eta c_{12}(t) & \eta^2 c_{13}(t) & \cdots & \eta^{n-1}c_{1n}(t) \\ & 0 & \eta c_{23}(t) & \cdots & \eta^{n-2}c_{2n}(t) \\ & & 0 & \cdots & \eta^{n-3}c_{3n}(t) \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}\right] z \\
&\triangleq [D(t) + B(t)]z
\end{aligned}$$

It is clear that $\|B(t)\| < \delta$. Therefore, $z^{\Delta} = [D(t) + B(t)]z$ admits λ-ED. Then the system $x^{\Delta} = C(t)x$ also admits λ-ED, which implies $\Sigma(C) \subseteq \bigcup_{i=1}^{n} \Sigma(c_{ii})$.

On the other hand, let $\lambda \notin \Sigma(C) = [a, b]$. If $\lambda > b$, by lemma 4.12, the system $x^{\Delta} = C(t)x$ admits λ-ED with the projector I. Then there exist constants $K_\lambda \geqslant 1$, $\alpha_\lambda > 0$ such that

$$\|\Phi_C(t, s)\| \leqslant K_\lambda \, \mathrm{e}^{(\lambda-\alpha_\lambda)(t-s)}, \quad t \in [s, +\infty).$$

It can be easily verified that

$$\Phi_C(t, s) = \begin{pmatrix} e_{c_{11}}(t, s) & * & \cdots & * \\ & e_{c_{22}}(t, s) & \cdots & * \\ & & \ddots & \vdots \\ & & & e_{c_{nn}}(t, s) \end{pmatrix}$$

and $|e_{c_{ii}}(t, s)| \leqslant \|\Phi_C(t, s)\|, i = 1, 2, \cdots, n$. Therefore,

$$|e_{c_{ii}}(t, s)| \leqslant K_\lambda \mathrm{e}^{(\lambda-\alpha_\lambda)(t-s)}, \quad t \in [s, +\infty), i = 1, 2, \cdots, n,$$

which implies that the system $x_i^{\Delta} = c_{ii}x_i$ admits λ-ED for any $i \in \{1, 2, \cdots, n\}$. Thus, $\lambda \notin \bigcup_{i=1}^{n} \Sigma(c_{ii})$. In a similar way, we can prove that $\lambda \notin \bigcup_{i=1}^{n} \Sigma(c_{ii})$ if $\lambda < a$. Therefore, $\bigcup_{i=1}^{n} \Sigma(c_{ii}) \subseteq \Sigma(C)$. In conclusion, we have $\Sigma(C) = \bigcup_{i=1}^{n} \Sigma(c_{ii})$. The proof is completed. □

LEMMA 4.14. *If system (2.2) is Δ-contracted to F, then $\Sigma(A) \subseteq F$.*

Proof. If $\lambda \notin F$, then $\alpha = \inf_{x \in F} |\lambda - x| > 0$ since $F = \bigcup_{i=1}^{n} \varpi(F_i)$, where F_i are closed intervals $(i = 1, 2, \cdots, n)$. It follows from the definition of generalized contractible set that there exist $c_1(t), c_2(t), \cdots, c_n(t) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ and $B(t) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$, such that $\sup_{t\in\mathbb{T}} \|B(t)\| \leqslant \delta, \mathrm{Im}\, \overline{\xi}_{\mu}(c_i) \subseteq F_i (i = 1, 2, \cdots, n)$ and system (2.2) is kinematically similar to

$$y^{\Delta} = [\mathrm{diag}(c_1(t), c_2(t), \cdots, c_n(t)) + B(t)]y. \tag{4.18}$$

Without loss of generalization, we assume that

$$\begin{aligned} &\varpi(F_i) < \lambda - \alpha, \quad (i = 1, 2, \cdots, k), \\ &\varpi(F_j) > \lambda + \alpha, \quad (j = k+1, \cdots, n), \end{aligned}$$

which implies

$$\begin{aligned} &\overline{\xi}_{\mu}(c_i)(t) < \lambda - \alpha, \quad \forall t \in \mathbb{T}\ (i = 1, 2, \cdots, k), \\ &\overline{\xi}_{\mu}(c_j)(t) > \lambda + \alpha, \quad \forall t \in \mathbb{T}\ (j = k+1, \cdots, n). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &|e_{c_i}(t, s)| = \mathrm{e}^{\int_s^t \overline{\xi}_{\mu}(c_i)(s)\Delta s} < \mathrm{e}^{(\lambda-\alpha)(t-s)}, \quad (t \geqslant s)(i = 1, 2, \cdots, k), \\ &|e_{c_j}(t, s)| = \mathrm{e}^{\int_s^t \overline{\xi}_{\mu}(c_j)(s)\Delta s} < \mathrm{e}^{(\lambda+\alpha)(t-s)}, \quad (t \leqslant s)(j = k+1, \cdots, n). \end{aligned}$$

Since $\sup_{t\in\mathbb{T}} \|B(t)\| \leqslant \delta$ and δ can be sufficiently small, by roughness theorem, we get system (4.18) admits λ-ED. Therefore, system (2.2) also admits λ-ED, which implies $\lambda \notin \Sigma(A)$. The proof is completed. □

THEOREM 4.15. *Assume that $A(t)$ is bounded and the generalized contractible set of system (2.2) is denoted by F. Then $F = \Sigma(A)$.*

Proof. The proof is divided into several parts. Firstly, we prove that $F \subseteq \Sigma(A)$.

Part 1: System (2.2) is kinematically similar to an upper triangular system.

Suppose $\Sigma(A) = \bigcup_{i=1}^{k} [a_i, b_i] (1 \leqslant k \leqslant n), a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k$. From lemma 4.10, system (2.2) is kinematically similar to system (4.16), where $B_i(t)(i =$

$1, 2, \cdots, k)$ are bounded rd-continuous functions and $\Sigma(B_i) = [a_i, b_i]$. Using Perron's transformation, the system

$$x_i^{\Delta} = B_i(t)x_i \tag{4.19}$$

is kinematically similar to the $n_i \times n_i$ upper triangular system

$$y_i^{\Delta} = D_i(t)y_i, \tag{4.20}$$

where $\Sigma(D_i) = [a_i, b_i]$. Let $D_i(t) = (d_{ij}^{(i)}(t))$. From lemma 4.13, $\bigcup_{r=1}^{n_i} \Sigma(d_{rr}^{(i)}) = [a_i, b_i]$. Then, by lemma 4.12, for any $\delta > 0$, the system

$$u^{\Delta} = d_{rr}^{(i)}(t)u$$

admits $(b_i + \delta)$-ED with projector I and $(a_i - \delta)$-ED with projector O. In consequence, there exist constants $K = 1, \alpha > 0$, such that

$$\begin{aligned} |e_{d_{rr}^{(i)}}(t, s)| &\leqslant \mathrm{e}^{(b_i+\delta-\alpha)(t-s)}, \quad \text{for } t \in [s, +\infty)_{\mathbb{T}}, \\ |e_{d_{rr}^{(i)}}(t, s)| &\leqslant \mathrm{e}^{(a_i-\delta+\alpha)(t-s)}, \quad \text{for } t \in (-\infty, s]_{\mathbb{T}}. \end{aligned} \tag{4.21}$$

Part 2: We are going to construct a strictly increasing and unbounded sequence $\{t_p^{(ir)}\}_{p=0}^{+\infty}$ such that for any constant $M > \exp\{(b_i + 1 - a_i + \delta)(1 + \mu^*)\}$, the estimate

$$M^{-1} \leqslant \mathrm{e}^{-\int_{t_0}^{t} h_i(\tau)+g_i(\tau)\Delta\tau} |e_{d_{rr}^{(i)}}(t, t_0)| \leqslant M \tag{4.22}$$

holds for all $t \in [t_0, +\infty)_{\mathbb{T}}$, where $h_i, g_i : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}$ is defined by:

$$h_i(t) = \begin{cases} a_i & \text{if } t \in [t_j^{(ir)}, t_{j+1}^{(ir)})_{\mathbb{T}}, \\ b_i & \text{if } t \in [t_{j+1}^{(ir)}, t_{j+2}^{(ir)})_{\mathbb{T}}, \end{cases} \quad (j = 0, 2, 4, \cdots)$$

and

$$g_i(t) = \begin{cases} -\delta & \text{if } t \in [t_j^{(ir)}, t_{j+1}^{(ir)})_{\mathbb{T}}, \\ \delta & \text{if } t \in [t_{j+1}^{(ir)}, t_{j+2}^{(ir)})_{\mathbb{T}}. \end{cases} \quad (j = 0, 2, 4, \cdots)$$

In what follows, we denote $t_p^{(ir)}$ by t_p $(p \in \mathbb{N}_0)$ if there is no ambiguity. Interchanging t by s in the second inequality of (4.21), we obtain

$$\begin{cases} \mathrm{e}^{-(b_i+\delta)(t-s)} |e_{d_{rr}^{(i)}}(t, s)| \leqslant \mathrm{e}^{-\alpha(t-s)}, & t \in [t_0, +\infty)_{\mathbb{T}}, \\ \mathrm{e}^{-(a_i-\delta)(t-s)} |e_{d_{rr}^{(i)}}(t, s)| \geqslant \mathrm{e}^{\alpha(t-s)}, & t \in [t_0, +\infty)_{\mathbb{T}}. \end{cases} \tag{4.23}$$

Let

$$U(t, t_0) = \mathrm{e}^{-(a_i-\delta)(t-t_0)} |e_{d_{rr}^{(i)}}(t, t_0)|, \quad V(t, t_0) = \mathrm{e}^{-(b_i+\delta)(t-t_0)} |e_{d_{rr}^{(i)}}(t, t_0)|.$$

Therefore, by (4.23), we have

$$\begin{cases} V(t, s) \leqslant \mathrm{e}^{-\alpha(t-s)}, & t \in [t_0, +\infty)_{\mathbb{T}}, \\ U(t, s) \geqslant \mathrm{e}^{\alpha(t-s)}, & t \in [t_0, +\infty)_{\mathbb{T}}. \end{cases} \tag{4.24}$$

It can be seen that $U(t,s) \geqslant 1$ and $V(t,s) \leqslant 1$ if $t \geqslant s$ and for any fixed $t_0 \in \mathbb{T}$, $U(t,t_0)$ is unbounded on $[t_0,+\infty)_{\mathbb{T}}$ since $\alpha > 0$. Moreover, $V(t,t_0) \to 0$ as $t \to +\infty$. In consequence, there is $t_1 \geqslant t_0$ such that

$$M^{-1} < 1 \leqslant U(t,t_0) \leqslant M \tag{4.25}$$

for any $t \in [t_0,t_1]$ and

$$U(\sigma(t_1),t_0) \geqslant M. \tag{4.26}$$

Meanwhile, we assert that $t_1 - t_0 > 1$. In fact, one can see that

$$U(t,s) \leqslant \mathrm{e}^{(b_i+1-a_i+\delta)(t-s)}, \quad V(t,s) \leqslant \mathrm{e}^{(b_i+1-b_i-\delta)(t-s)} \quad \text{for } t \geqslant s$$

since $\Sigma(d_{rr}^{(i)}) \subseteq [a_i,b_i]$ and $|e_{d_{rr}^{(i)}}(t,s)| \leqslant \mathrm{e}^{(b_i+1)(t-s)}$ for $t \geqslant s$. From (4.26), we get

$$M \leqslant \mathrm{e}^{(b_i+1-a_i+\delta)(\sigma(t_1)-t_0)},$$

which implies that

$$\sigma(t_1) - t_0 \geqslant \frac{\ln M}{b_i+1-a_i+\delta} > 1 + \mu^*$$

since $M > \exp\{(b_i+1-a_i+\delta)(1+\mu^*)\}$, and then $t_1 - t_0 > 1$.

On the other hand, the function $U(t_1,t_0)V(t,t_1)$ is convergent to zero as $t \to +\infty$. Then, there exists $t_2 > t_1$ such that

$$U(t_1,t_0)V(\sigma(t_2),t_1) \leqslant M^{-1}$$

and

$$U(t_1,t_0)V(t,t_1) \geqslant M^{-1} \quad \text{for any } t \in [t_1,t_2]_{\mathbb{T}}.$$

Combined with the first inequality of (4.24) and (4.25), we have

$$M^{-1} \leqslant U(t_1,t_0)V(t,t_1) \leqslant M \quad \text{for any } t \in [t_1,t_2]_{\mathbb{T}}. \tag{4.27}$$

If $t \in [t_1,t_2]_{\mathbb{T}}$, by the definition of $h_i(t)$ and $g_i(t)$, we have

$$\begin{aligned}
&\mathrm{e}^{-\int_{t_0}^{t} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t,t_0)| \\
&\quad = \mathrm{e}^{-(\int_{t_0}^{t_1}+\int_{t_1}^{t})(h_i(\tau)+g_i(\tau))\Delta\tau}|e_{d_{rr}^{(i)}}(t,t_1)||e_{d_{rr}^{(i)}}(t_1,t_0)| \\
&\quad = \mathrm{e}^{-(a_i-\delta)(t_1-t_0)}|e_{d_{rr}^{(i)}}(t_1,t_0)| \\
&\qquad \cdot \mathrm{e}^{-(b_i+\delta)(t-t_1)}|e_{d_{rr}^{(i)}}(t,t_1)| = U(t_1,t_0)V(t,t_1).
\end{aligned}$$

Combined with (4.27), equation (4.22) is verified for any $t \in [t_0,t_2]_{\mathbb{T}}$. As inductive hypothesis, we assume that there are $2m+1$ numbers $t_0 < t_1 < \cdots < t_{2m-1} < t_{2m}$

such that

$$e^{-\int_{t_0}^{\sigma(t_{2m})} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(\sigma(t_{2m}),t_0)| \leqslant M^{-1} \tag{4.28}$$

and

$$M^{-1} \leqslant e^{-\int_{t_0}^{t} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t,t_0)| \leqslant M \quad \text{for any } t \in [t_{2m-1},t_{2m}].$$

Using the second inequality of (4.24), we have that

$$e^{-\int_{t_0}^{t_{2m}} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t_{2m},t_0)|U(t,t_{2m}) \tag{4.29}$$

is unbounded on $[t_{2m},+\infty)_{\mathbb{T}}$. Then, there exists $t_{2m+1} \geqslant t_{2m}$ such that this product is less than M for any $t \in [t_{2m},t_{2m+1}]$ and

$$e^{-\int_{t_0}^{t_{2m}} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t_{2m},t_0)|U(\sigma(t_{2m+1}),t_{2m}) \geqslant M. \tag{4.30}$$

In addition, it follows from the inductive hypothesis and $U(t,t_{2m}) \geqslant 1$ $(t \geqslant t_{2m})$ that

$$M^{-1} \leqslant e^{-\int_{t_0}^{t_{2m}} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t_{2m},t_0)|U(t,t_{2m}) \leqslant M \quad \text{for any } t \in [t_{2m},t_{2m+1}]_{\mathbb{T}}.$$

Moreover, by (4.31) and (4.30), it is not difficult to verify that $t_{2m+1} - t_{2m} > 1$.

Finally, we have that

$$e^{-\int_{t_0}^{t_{2m}} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t_{2m},t_0)|U(t_{2m+1},t_{2m})V(t,t_{2m+1})$$

converges to zero as $t \to +\infty$. Then there exists t_{2m+2} such that the product above is greater than M^{-1} when $t \in [t_{2m+1},t_{2m+2}]$. Since $V(t,t_{2m+1}) \leqslant 1$ for $t \geqslant t_{2m+1}$, the product above is less than M. It can be easily verified that

$$e^{-\int_{t_0}^{t} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t,t_0)| = e^{-\int_{t_0}^{t_{2m}} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t_{2m},t_0)|U(t,t_{2m})$$

for $t \in [t_{2m},t_{2m+1}]$ and

$$\begin{aligned} &e^{-\int_{t_0}^{t} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t,t_0)| \\ &\quad = e^{-\int_{t_0}^{t_{2m}} h_i(\tau)+g_i(\tau)\Delta\tau}|e_{d_{rr}^{(i)}}(t_{2m},t_0)|U(t_{2m+1},t_{2m})V(t,t_{2m+1}) \end{aligned}$$

for $t \in [t_{2m+1},t_{2m+2}]$. This proves (4.22) and $t_p \to +\infty$ as $p \to +\infty$.

Part 3: In a similar way, we can define $h_i(t)$ and $g_i(t)$ on $(-\infty, t_0]$ satisfying

$$M^{-1} \leqslant \mathrm{e}^{-\int_{t_0}^{t} h_i(\tau)+g_i(\tau)\Delta\tau} |e_{d_{rr}^{(i)}}(t,t_0)| \leqslant M.$$

Since $h_i(t)$ and $g_i(t)$ are sectionally continuous on $\mathbb{T}$. Then there exist continuous functions $\widehat{h}_i(t)$ and $\widehat{g}_i(t)$ satisfying

$$a_i \leqslant \widehat{h}_i(t) \leqslant b_i, \quad -\delta \leqslant \widehat{g}_i(t) \leqslant \delta$$

and

$$\int_{-\infty}^{+\infty} |(\widehat{h}_i(\tau) + \widehat{g}_i(\tau)) - (h_i(\tau) + g_i(\tau))|\Delta\tau \leqslant 1.$$

Thus, we have

$$(eM)^{-1} \leqslant \mathrm{e}^{-\int_{t_0}^{t} \widehat{h}_i(\tau)+\widehat{g}_i(\tau)\Delta\tau} |e_{d_{rr}^{(i)}}(t,t_0)| \leqslant eM, \quad t \in \mathbb{T}.$$

Let $S_{ir}(t) = \mathrm{e}^{-\int_{t_0}^{t} \widehat{h}_i(\tau)+\widehat{g}_i(\tau)\Delta\tau} \cdot e_{d_{rr}^{(i)}}(t,t_0)$ and

$$L_i(t) = \mathrm{diag}\left(S_{i1}(t), S_{i2}(t), \cdots, S_{in_i}(t)\right).$$

It can be seen that $\|L_i(t)\| \leqslant eM$ and $\|L_i^{-1}(t)\| \leqslant eM$ for any $t \in \mathbb{T}$ and system (4.20) is kinematically similar to

$$z_i^{\Delta} = \Lambda_i(t) z_i \tag{4.31}$$

with $y_i = L_i(t)z_i$, where $\Lambda_i(t) = L^{-1}(\sigma(t))D_i(t)L(t) - L^{-1}(\sigma(t))L^{\Delta}(t)$ is a $n_i \times n_i$ matrix whose rj-coefficient is defined by

$$\{\Lambda_i(t)\}_{rj} = \begin{cases} d_{rr}^{(i)} S_{ir}^{-1}(\sigma(t))S_{ir}(t) - S_{ir}^{-1}(\sigma(t))S_{ir}^{\Delta}(t) & \text{if } r = j, \\ d_{rj}^{(i)} S_{ir}^{-1}(\sigma(t))S_{ij}(t) & \text{if } 1 \leqslant r < j \leqslant n_i, \\ 0 & \text{others.} \end{cases}$$

A straightforward calculation leads to

$$\begin{aligned} \{\Lambda_i(t)\}_{rj} = d_{rj}^{(i)} S_{ir}^{-1}(\sigma(t))S_{ij}(t) &= d_{rj}^{(i)} \\ &\cdot \frac{\mathrm{e}^{\mu(t)(\widehat{h}_i(t)+\widehat{g}_i(t))} e_{d_{jj}^{(i)} \ominus d_{rr}^{(i)}}(t,t_0)}{1+\mu(t)d_{rr}^{(i)}}, \quad (r \leqslant j) \end{aligned}$$

$$\begin{aligned} S_{ir}^{-1}(\sigma(t))S_{ir}^{\Delta}(t) &= S_{ir}^{-1}(\sigma(t)) \lim_{s \searrow \mu(t)} \frac{S_{ir}(t+s) - S_{ir}(t)}{s} \\ &= \lim_{s \searrow \mu(t)} \left(1 - \frac{\mathrm{e}^{s(\widehat{h}_i(t)+\widehat{g}_i(t))}}{1+sd_{rr}^{(i)}(t)}\right) \cdot s^{-1}, \end{aligned}$$

and

$$\{\Lambda_i(t)\}_{rr} = d_{rr}^{(i)} S_{ir}^{-1}(\sigma(t))S_{ir}(t) - S_{ir}^{-1}(\sigma(t))S_{ir}^{\Delta}(t) = \lim_{s \searrow \mu(t)} \frac{\mathrm{e}^{s(\widehat{h}_i(t)+\widehat{g}_i(t))} - 1}{s}. \tag{4.32}$$

Since $(eM)^{-1} \leqslant S_{ir}(t) \leqslant eM$ and $A(t)$ is bounded, we obtain $d_{rr}^{(i)}$ is bounded and $d_{rj}^{(i)} S_{ir}^{-1}(\sigma(t)) S_{ij}(t)$ is bounded, i.e., $\{\Lambda_i(t)\}_{rj}$ is bounded if $r < j$.

We define the η-transformation

$$z_i(t) = \operatorname{diag}(1, \eta, \cdots, \eta^{n_i - 1}) w_i(t).$$

It can be seen that (4.31) and (4.20) are kinematically similar to

$$w_i^{\Delta} = \Gamma_i(t) w_i,$$

where the rj-coefficient of $\Gamma(t)$ is

$$\{\Gamma_i(t)\}_{rj} = \begin{cases} \{\Lambda_i(t)\}_{rj} & \text{if } r = j, \\ \eta^{j-r}\{\Lambda_i(t)\}_{rj} & \text{if } 1 \leqslant r < j \leqslant n_i, \\ 0 & \text{others.} \end{cases}$$

Observe that $\Gamma_i(t)$ can be written as

$$\Gamma_i(t) = \operatorname{diag}(\{\Lambda_i(t)\}_{11}, \{\Lambda_i(t)\}_{22}, \cdots, \{\Lambda_i(t)\}_{n_i n_i}) + W_i(t),$$

where the rj-coefficient of $W_i(t)$ is defined by

$$\{W_i(t)\}_{rj} = \begin{cases} \eta^{j-r}\{\Lambda_i(t)\}_{rj} & \text{if } 1 \leqslant r < j \leqslant n_i, \\ 0 & \text{others.} \end{cases}$$

Since $\{\Lambda_i(t)\}_{rj}$ is bounded if $r < j$, we obtain that $\|W_i(t)\| \to 0$ as $\eta \to 0$. On the other hand, it follows from (4.32) that

$$\begin{aligned} \overline{\xi}_{\mu}(\{\Lambda_i(t)\}_{rr}) &= \operatorname{Re}\left(\lim_{s \searrow \mu(t)} \frac{\log(1 + s\{\Lambda_i(t)\}_{rr})}{s} \right) \\ &= \operatorname{Re}\left(\widehat{h}_i(t) + \widehat{g}_i(t) \right) \in (a_i - \delta, b_i + \delta). \end{aligned}$$

Therefore, it is clear that system (2.2) is Δ-contracted to the set $G_\delta = \bigcup_{i=1}^{k} (a_i - \delta, b_i + \delta)$ for any $\delta > 0$. Since the constant δ can be sufficiently small and $G_\delta \to \Sigma(A)$ as $\delta \to 0$, we have $F \subseteq \Sigma(A)$, where F is the generalized contractible set of (2.2).

On the other hand, by the definition of generalized contractible set and lemma 4.14, we have $\Sigma(A) \subseteq F$. In conclusion, we have $F = \Sigma(A)$. The proof is completed. □

Example 4.16. Let us consider the system mentioned in the counterexample. Let F denote the generalized contractible set of system (3.1). Obviously,

$$\overline{\xi}_{\mu}(e - 1) = \log|1 + e - 1| = 1, \;\; \overline{\xi}_{\mu}(\mathrm{e}^{-1} - 1) = \log|1 + \mathrm{e}^{-1} - 1| = -1, \;\; F = [-1, 1].$$

We know that the dichotomy spectrum of system (3.1) is $[-1, 1]$, which supports our result.

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Conflict of Interest

The authors declare that they have no conflict of interest.

Declarations of Authors' Contributions

All the authors have the same contributions to the paper.

Data Availability

No data was used for the research in this article. It is pure mathematics.

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