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ON INFINITE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. Let $A = [\alpha_{ij}]$ $(i, j = 1, 2, \ldots)$ be an infinite matrix with complex entries, and let $z = (\zeta_j)$ $(j = 1, 2, \ldots)$ be a sequence of complex numbers. In this paper we wish to investigate the existence, uniqueness and asymptotic behavior of solutions to the infinite system of linear differential equations

$$(1.1) \qquad \xi_i'(t) = \sum_{j=1}^{\infty} \alpha_{ij}\xi_j(t) \quad (i = 1, 2, \ldots)$$

with the initial conditions

$$(1.2) \qquad \xi_j(0) = \zeta_j \quad (j = 1, 2, \ldots).$$

By a *solution*, we mean a sequence $x(t) = (\xi_j(t))$ of complex-valued functions on $[0, \infty)$ which satisfies (1.1) for each $t \in [0, \infty)$ and (1.2).

As far as we are aware, Arley and Borchsenius [**1**] were the first to show that if $z \in l^1$ and if

$$(1.3) \qquad N_1(A) = \sum_{i,j} |\alpha_{ij}| < \infty,$$

then the system (1.1)-(1.2) has one and only one solution $x(t)$ such that $x(t) \in l^1$ for all $t \geqq 0$. Bellman later showed in [**2**] that if $1 < p < \infty$, $z \in l^p$ and

$$(1.4) \qquad N_q(A) = \left[\sum_i \left(\sum_j |\alpha_{ij}|^q \right)^{p-1} \right]^{1/p} < \infty$$

$(1/p + 1/q = 1)$, then the system (1.1)-(1.2) has exactly one solution $x(t)$ such that $x(t) \in l^p$ for all $t \geqq 0$. In a recent paper [**9**], Shaw considered the case where

$$(1.5) \qquad \begin{aligned} \sum_j |\alpha_{ij}| &\leqq \alpha < \infty \quad (i = 1, 2, \ldots) \\ \sum_i |\alpha_{ij}| &\leqq \beta < \infty \quad (j = 1, 2, \ldots) \end{aligned}$$

for some constants α, $\beta \geqq 0$. He proved that if $z \in l^1$, then there exists a solution $x(t)$ of (1.1)-(1.2) satisfying

$$||x(t)|| \leqq ||z||e^{\beta t}.$$

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(Here $\|\cdot\|$ denotes the usual norm in l^1.) Moreover, this is the only solution with the property that $\|x(t)\|$ is continuous on $[0, \infty)$.

Each of the conditions (1.3)-(1.5) implies that the entries of A form a bounded set of numbers. Our aim in this paper is to relax this restriction; in particular, we shall allow $|\alpha_{ii}| \to \infty$ as $i \to \infty$ (cf. Bellman [**2**, p. 704]). Assuming

$$(\mathrm{A}_1) \qquad \omega = \sup\{\operatorname{Re} \alpha_{ii} : i = 1, 2, \ldots\} < \infty$$

and

$$(\mathrm{A}_2) \qquad \sum_{i=1,\, i \neq j}^{\infty} |\alpha_{ij}| \leqq M \quad (j = 1, 2, \ldots),$$

for some finite constant M, we shall show the existence of a solution $x(t)$ to (1.1)-(1.2) such that $x(t) \in l^1$ $(t \geqq 0)$, for any initial value $z \in l^1$. Note that (A_1) does allow the diagonal entries to form an unbounded set. Our solution will have the property that $x(t)$ is continuous on $[0, \infty)$, and we shall show that there is only one such solution. Finally, if A is strictly diagonally dominant in its columns (see condition (A_3)), then we shall show that the solution to (1.1)-(1.2) is the limit of solutions to the finite systems obtained by truncating A after n rows and n columns, the limit being approached uniformly on any compact subset of $[0, \infty)$. With a somewhat stronger condition (see (A_4)), the solution decays exponentially to zero as $t \to \infty$.

To simplify notation in the rest of the paper, we shall write $\sum_{i \neq j}$ in place of the summation notation in (A_2).

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2. Existence of solutions. In vector and matrix notation, (1.1)-(1.2) can be written as $x'(t) = Ax(t)$ $(t \geqq 0)$ and $x(0) = z$. For the moment these notations are purely formal, however they do suggest that an attempt be made on the problem through Banach space techniques. In fact, we shall use methods from both classical and functional analysis. For the Banach space results that we need, we shall refer to the book of Krein [**6**].

Let l_0 be the vector space of sequences with only finitely many non-zero coordinates; clearly l_0 is dense in l^1. For any matrix $A = [\alpha_{ij}]$, $\eta_i = \sum_j \alpha_{ij}\zeta_j$ exists $(i = 1, 2, \ldots)$ whenever $z = (\zeta_i) \in l_0$, and we have $(\eta_i) \in l^1$ for every $z \in l_0$ if and only if $\sum_i |\alpha_{ij}| < \infty$ for all j.

Assume that this is the case, and let

$$\mathscr{D}(A) = \{(\zeta_j) \in l^1 : \eta_i = \textstyle\sum_j \alpha_{ij}\zeta_j \text{ exists for all } i \text{ and } (\eta_i) \in l^1\}.$$

Then $\mathscr{D}(A)$ is a dense linear subspace in l^1, and A may be regarded as a linear operator on $\mathscr{D}(A)$ into l^1.

By the strong derivative at t_0 of a function $x(t)$ with values in l^1, we mean the limit

$$\lim_{h\to 0} \frac{1}{h} [x(t_0 + h) - x(t_0)],$$

the limit to be taken with respect to the norm in l^1. Left and right strong derivatives at t_0 are defined similarly. The problem

$$(2.1) \qquad x'(t) = Ax(t) \quad x(0) = z, \quad (t \geqq 0)$$

where differentiation is to be in the strong sense and only the right strong derivative is required at $t = 0$, is known as the *Cauchy problem*. It is clear that a solution to (2.1) will yield a solution to (1.1)-(1.2), but in general the converse is not true.

Now we decompose A into its diagonal and off-diagonal parts: we let $D = \text{diag}[\alpha_{ii}]$, and $B = A - D$. Condition (A_2) implies that B defines a bounded linear operator on l^1. (In fact, a matrix $C = [\gamma_{ij}]$ defines a bounded operator on l^1 if and only if $\sup_j \{\sum_i |\gamma_{ij}|\} < \infty$, and in that case the operator bound of C is this supremum.) The diagonal matrix D defines a closed linear operator with domain (clearly containing l_0) $\{(\zeta_j) \in l^1 : (\alpha_{ii}\zeta_i) \in l^1\}$. It is easy to see that this set is $\mathscr{D}(A)$, and hence A is a closed operator with dense domain in l^1. Denote by $\exp(tD)$ the diagonal matrix $\text{diag}[\exp(\alpha_{ii}t)]$ $(t \geqq 0)$. If A satisfies (A_1), then these matrices form a semigroup of bounded operators on l^1. In fact, the operators are uniformly bounded on compact subsets of $[0, \infty)$:

$$(2.2) \qquad ||\exp(tD)|| \leqq e^{\omega t} \quad (t \geqq 0).$$

(When we write $||U||$ with U an operator, we mean the usual operator bound $\sup\{||Uz||: ||z|| \leqq 1\}$.)

LEMMA 1. *Assume that condition* (A_1) *holds. Then*

(i) *The semigroup* $\{\exp(tD) : t \geqq 0\}$ *is of class* C_0 [**6**, Definition 2.1, p. 43], *and*

(ii) *The generating operator of the semigroup is* D.

Proof. Let us prove the assertion in (ii) first. Clearly, if

$$\lim_{t\to +0} \frac{1}{t} [\exp(tD)y - y]$$

exists, its value will be Dy, so that y will be in $\mathscr{D}(A)$. Hence we need to show that for each $y = (\eta_i) \in \mathscr{D}(A)$

$$\lim_{t\to +0} \left|\left| \frac{1}{t} [\exp(tD)y - y] - Dy \right|\right| = 0.$$

Given any $\epsilon > 0$, we may choose N so that

$$(2.3) \qquad \sum_{i=N+1}^{\infty} |\alpha_{ii}\eta_i| < \frac{\epsilon}{5}.$$

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Also, we may choose δ so that whenever $0 < t < \delta$, $e^{\omega t} + 2 < 4$ and

$$(2.4) \qquad \sum_{i=1}^{N} \left| \frac{1}{t} [\exp(\alpha_{ii}t)\eta_i - \eta_i] - \alpha_{ii}\eta_i \right| < \frac{\epsilon}{5}.$$

Using the inequality $|e^z - 1| \leqq |z|[e^{\mathrm{Re}\, z} + 1]$, we obtain from (2.3) and (2.4)

$$\left|\left| \frac{1}{t} [\exp(tD)y - y] - Dy \right|\right|$$

$$(2.5) \qquad = \sum_{i=1}^{N} \left| \frac{1}{t} [\exp(\alpha_{ii}t)\eta_i - \eta_i] - \alpha_{ii}\eta_i \right| + \sum_{i=N+1}^{\infty} \left| \frac{1}{t} [\exp(\alpha_{ii}t)\eta_i - \eta_i] - \alpha_{ii}\eta_i \right|$$

$$< \frac{\epsilon}{5} + \sum_{i=N+1}^{\infty} |\alpha_{ii}\eta_i| (\exp(\mathrm{Re}\, \alpha_{ii}t) + 2),$$

where $0 < t < \delta$. Since $\exp(\mathrm{Re}\, \alpha_{ii}t) \leqq \exp(\omega t)$ for all i by condition (A_1), (2.5) implies that

$$\left|\left| \frac{1}{t} [\exp(tD)y - y] - Dy \right|\right| < \epsilon$$

for $0 < t < \delta$. We have thus proved assertion (ii).

To prove assertion (i), it is sufficient (because of (2.2)) to show

$$(2.6) \qquad \lim_{t \to +0} || \exp(tD)y - y|| = 0$$

for every $y \in l^1$. We omit the verification of (2.6), since it is very similar to that of (ii).

The following theorem is now an immediate consequence of Theorem 2.8, p. 47 and Theorem 7.5, p. 151 in [**6**].

THEOREM 1. *Suppose that A satisfies conditions* (A_1) *and* (A_2). *Then for any* $z \in \mathscr{D}(A)$, *the Cauchy problem* (2.1) *has a unique solution. Furthermore, this solution depends continuously on the initial value in the sense that if* $z_n \to 0$ $(z_n \in \mathscr{D}(A))$, *then* $x_n(t) \to 0$ *uniformly on compact subset of* $[0, \infty)$, *where* $x_n(t)$ *is the solution corresponding to the initial value* z_n $(n = 1, 2, \ldots)$.

Remark 1. If we write $\exp(tA)$ $(t \geqq 0)$ for the C_0-semigroup generated by A, the solution of (2.1) for $z \in \mathscr{D}(A)$ is given by

$$(2.7) \qquad x(t) = \exp(tA)z.$$

By (2.2) and [**6**, Remark 7.4, p. 152], we have

$$(2.8) \qquad || \exp(tA)|| \leqq e^{(\omega + ||B||)t}, \quad t \geqq 0.$$

Theorem 1 gives a unique solution to the problem (2.1), and hence provides

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a solution to the system (1.1)-(1.2) when $z \in \mathscr{D}(A)$. Our next theorem will show that (1.1)-(1.2) in fact has a solution for any $z \in l^1$.

THEOREM 2. *Suppose that A satisfies* (A_1) *and* (A_2). *Then for any* $z \in l^1$, *the system* (1.1)-(1.2) *has a solution* $x(t)$ *such that* $x(t) \in l^1$ *for all* $t \geqq 0$ *and* $x(t)$ *is continuous on* $[0, \infty)$.

Throughout the paper, continuity of an l^1-valued function means strong continuity, i.e. continuity with respect to the topology on l^1 determined by the norm.

Proof. Fix $z = (\zeta_i) \in l^1$. Since $\mathscr{D}(A)$ is dense in l^1, we may choose $z_n = (\zeta_{nj}) \in \mathscr{D}(A)$ $(n = 1, 2, \ldots)$ so that $z_n \to z$ in l^1 as $n \to \infty$. By Theorem 1, there are (unique) functions $x_n(t) = (\xi_{nj}(t))$ on $[0, \infty)$ such that

$$(2.9) \qquad x_n'(t) = Ax_n(t), \quad x_n(0) = z_n \quad (t \geqq 0)\ (n = 1, 2, \ldots).$$

In view of Remark 1, we may write $x_n(t) = \exp(tA)z_n$, and hence obtain from (2.8)

$$||x_n(t) - x_m(t)|| \leqq e^{(\omega+||B||)t}||z_n - z_m|| \quad (n, m = 1, 2, \ldots).$$

Therefore the sequence $x_n(t)$ converges to a limit function $x(t) = (\xi_j(t))$, and the convergence is uniform on compact subsets of $[0, \infty)$, so that $x(t)$ is continuous on $[0, \infty)$. Clearly $x(0) = z$, so it only remains to show that $x(t)$ satisfies (1.1) on $[0, \infty)$.

Since the functions $x_n(t)$ are continuous on $[0, \infty)$ and converge to $x(t)$ uniformly on compact subsets of $[0, \infty)$, (A_2) implies that the functions $Bx_n(t)$ are continuous on $[0, \infty)$ and converge to $Bx(t)$ uniformly on compact subsets of $[0, \infty)$. In particular, the coordinate functions $\sum_{j\neq i}\alpha_{ij}\xi_{nj}(t)$ $(n, i = 1, 2, \ldots)$ are also continuous on $[0, \infty)$, and for each i,

$$\sum_{j\neq i} \alpha_{ij}\xi_{nj}(t) \to \sum_{j\neq i} \alpha_{ij}\xi_j(t) \quad as\ n \to \infty,$$

uniformly on compact subsets of $[0, \infty)$.

From (2.9), we have

$$\begin{aligned}\xi_{ni}'(t) &= \sum_{j=1}^{\infty} \alpha_{ij}\xi_{nj}(t)\\ &= \alpha_{ii}\xi_{ni}(t) + \sum_{j\neq i} \alpha_{ij}\xi_{nj}(t).\end{aligned}$$

Letting $n \to \infty$, the right-hand sides of these equations converge to

$$\alpha_{ii}\xi_i(t) + \sum_{j\neq i} \alpha_{ij}\xi_j(t) = \sum_{j=1}^{\infty} \alpha_{ij}\xi_j(t)$$

uniformly on compact subsets of $[0, \infty)$, in view of the remarks above. It now

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follows from a standard result in advanced calculus that

$$\xi_i'(t) = \sum_{j=1}^{\infty} \alpha_{ij}\xi_j(t) \quad (i = 1, 2, \ldots; t \geqq 0).$$

The proof is therefore complete.

3. Uniqueness of solutions. Theorem 1 contains a uniqueness assertion for solutions to the problem (2.1) when $z \in \mathscr{D}(A)$. However, as we have already pointed out, it is not always true that a solution to (1.1)-(1.2) will be a solution to (2.1), and thus there is a question about the uniqueness of solutions to (1.1)-(1.2), even when $z \in \mathscr{D}(A)$. The next theorem shows that (1.1)-(1.2) in fact has only one continuous solution for each $z \in l^1$.

THEOREM 3. *Suppose that $A = [\alpha_{ij}]$ satisfies* (A_1) *and* (A_2). *Then for any $z \in l^1$, the system* (1.1)-(1.2) *has only one solution $x(t)$ such that $x(t) \in l^1$ and $x(t)$ is continuous on* $[0, \infty)$.

Proof. Let $z = (\zeta_j) \in l^1$, and suppose that the system (1.1)-(1.2) has two continuous solutions $x(t) = (\xi_i(t))$ and $y(t) = (\eta_i(t))$. Then, for $t \geqq 0$ and $i = 1, 2, \ldots$, we have

$$\begin{aligned} &\xi_i'(t) = \alpha_{ii}\xi_i(t) + \sum_{j\neq i} \alpha_{ij}\xi_j(t) \\ (3.1) \quad &\eta_i'(t) = \alpha_{ii}\eta_i(t) + \sum_{j\neq i} \alpha_{ij}\eta_j(t) \\ &\xi_i(0) = \eta_i(0) = \zeta_i. \end{aligned}$$

Now B is continuous by condition (A_2), and the operators $\exp(tD)$ $(t \geqq 0)$ are continuous by condition (A_1). Therefore $x(t)$ continuous implies that $[\exp(t - s)D]Bx(s)$ is a continuous function of $s \in [0, t]$, for any $t \in [0, \infty)$. Hence the coordinate functions

$$\exp(\alpha_{ii}(t - s)) \sum_{j\neq i} \alpha_{ij}\xi_j(s)$$

are continuous for $s \in [0, t]$, for all i. The same argument applies to $y(t)$. Therefore we may convert equations in (3.1) into integral equations

$$(3.2) \quad \begin{aligned} \xi_i(t) &= \zeta_i \exp(\alpha_{ii}t) + \int_0^t \exp(\alpha_{ii}(t - s)) \sum_{j\neq i} \alpha_{ij}\xi_j(s)\, ds \\ \eta_i(t) &= \zeta_i \exp(\alpha_{ii}t) + \int_0^t \exp(\alpha_{ii}(t - s)) \sum_{j\neq i} \alpha_{ij}\eta_j(s)\, ds \end{aligned}$$

from which we have

$$\begin{aligned} |\xi_i(t) - \eta_i(t)| &\leqq \int_0^t \left| \exp(\alpha_{ii}(t - s)) \sum_{j\neq i} \alpha_{ij}(\xi_j(s) - \eta_j(s)) \right| ds \\ &\leqq \int_0^t e^{\omega(t-s)} \sum_{j\neq i} |\alpha_{ij}|\, |\xi_j(s) - \eta_j(s)|\, ds. \end{aligned}$$

Summing over all i gives

$$\begin{aligned}||x(t) - y(t)|| &\leqq \sum_i \int_0^t e^{\omega(t-s)} \sum_{j\neq i} |\alpha_{ij}|\,|\xi_j(s) - \eta_j(s)|\,ds \\ &= \int_0^t e^{\omega(t-s)} \sum_j \sum_{i\neq j} |\alpha_{ij}|\,|\xi_j(s) - \eta_j(s)|\,ds \\ &\leqq M \int_0^t e^{\omega(t-s)} \sum_j |\xi_j(s) - \eta_j(s)|\,ds \\ &\leqq M \int_0^t e^{\omega(t-s)} ||x(s) - y(s)||\,ds, \quad (t \geqq 0)\end{aligned}$$

by (A_2). The interchange of limits is justified, since all terms involved are non-negative. Now, the integral equation

$$v(t) = M \int_0^t e^{\omega(t-s)} v(s)\,ds \quad (t \geqq 0)$$

has the unique solution $v(t) = 0$ on $[0, \infty)$. Hence by a comparison theorem [**7**, p. 322],

$$||x(t) - y(t)|| \leqq v(t) \quad \text{for all } t \in [0, \infty).$$

This of course implies $x(t) = y(t)$ for all $t \in [0, \infty)$. The proof is thus complete.

By making slight modifications in the above proof, we can show that for any $z \in l^1$, the system (1.1)-(1.2) has only one l^1-valued solution $x(t)$ such that $||x(t)||$ is continuous on $[0, \infty)$. Similarly, if we assume that $A = [\alpha_{ij}]$ satisfies an additional condition, namely

$$\sum_j |\alpha_{ij}| < \infty, \quad i = 1, 2, \ldots,$$

then we can prove that for any $z \in l^1$, (1.1)-(1.2) has only one l^1-valued solution $x(t)$ such that $||x(t)||$ is bounded on each bounded subset of $[0, \infty)$ (cf. [**9**, Theorem 3]). In both cases, the modifications referred to only concern the justifications of the various steps in the proof; formally, the proof remains unchanged.

4. Approximation and behavior of solutions. In this section we shall impose another condition on A, which we call strict diagonal dominance in columns. We assume that there exists a positive constant δ such that

$$(A_3) \qquad |\alpha_{jj}| \geqq \delta + \sum_{i\neq j} |\alpha_{ij}| \quad (j = 1, 2, \ldots).$$

Before proceeding to our result on approximation of solutions to the infinite system by solutions to finite systems, we prove the following two lemmas.

LEMMA 2. *If A satisfies* (A_2) *and* (A_3), *then A has a bounded inverse on* l^1.

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Proof. First we note that (A_3) implies that $|\alpha_{jj}| \geqq \delta$ for all j. Hence the α_{jj}'s do not cluster at 0, and D has the bounded inverse $D^{-1} = \operatorname{diag}[1/\alpha_{jj}]$ on l^1.

Now let $\sigma = M/(M + \delta)$. Then $0 < \sigma < 1$ and $(\sigma - 1)M + \sigma\delta = 0$. By (A_3),

$$\begin{aligned} \sigma|\alpha_{jj}| &\geqq \sigma\delta + \sigma \sum_{i \neq j} |\alpha_{ij}| \\ &= \sigma\delta + (\sigma - 1) \sum_{i \neq j} |\alpha_{ij}| + \sum_{i \neq j} |\alpha_{ij}| \\ &\geqq \sigma\delta + (\sigma - 1)M + \sum_{i \neq j} |\alpha_{ij}| \\ &= \sum_{i \neq j} |\alpha_{ij}|, \end{aligned}$$

and therefore

$$\sum_{i \neq j} |\alpha_{ij}|/|\alpha_{jj}| \leqq \sigma < 1 \quad \text{for all } j.$$

It now follows that the operator BD^{-1}, whose entry in position (i, j) is α_{ij}/α_{jj} if $i \neq j$ and 0 if $i = j$, satisfies $||BD^{-1}|| \leqq \sigma < 1$.

Since $||BD^{-1}|| < 1$, $I + BD^{-1}$ has a bounded inverse on l^1. Therefore $A = D + B = (I + BD^{-1})D$ has the bounded inverse $D^{-1}(I + BD^{-1})^{-1}$ on l^1, and the lemma is proved.

LEMMA 3. *Let* $A = [\alpha_{ij}]$ *and define* $A_n = [\alpha_{ij}^{(n)}]$, *where* $\alpha_{ij}^{(n)} = \alpha_{ij}$ $(i, j = 1, \ldots, n)$ *and* $\alpha_{ij}^{(n)} = 0$ *otherwise. If* A *satisfies* (A_2), *then* $A_n z \to Az$ *as* $n \to \infty$ *for all* $z \in \mathscr{D}(A)$.

Proof. For any $z = (\zeta_i) \in \mathscr{D}(A)$, we define $z_n = (\zeta_1, \ldots \zeta_n, 0, 0, \ldots)$. Write $y = Az = (\eta_i)$ and $y_n = A_n z$. Then

$$\begin{aligned} ||y - y_n|| &= \sum_{i=1}^{n} \left| \sum_{j=n+1}^{\infty} \alpha_{ij}\zeta_j \right| + \sum_{i=n+1}^{\infty} \left| \sum_{j=1}^{\infty} \alpha_{ij}\zeta_j \right| \\ &= \sum\nolimits_1 + \sum\nolimits_2. \end{aligned}$$

Now $\sum_1 \leqq ||B(z - z_n)||$ and $\sum_2 = \sum_{i>n} |\eta_i|$. Choose $\epsilon > 0$. Since B is bounded and $z_n \to z$ in l^1 as $n \to \infty$, there exists N_1 such that $n \geqq N_1$ implies $\sum_1 < \epsilon/2$. Since $y \in l^1$, there exists N_2 such that $n \geqq N_2$ implies $\sum_2 < \epsilon/2$. Therefore for $n \geqq \max(N_1, N_2)$, we have $||y - y_n|| < \epsilon$. This completes the proof of Lemma 3.

THEOREM 4. *Let* A *satisfy* (A_1), (A_2) *and* (A_3), *and let* $z = (\zeta_i) \in l^1$. *Define* A_n *and* z_n *as in Lemma* 3. *Then the continuous solution* $x(t)$ *of* (1.1)-(1.2) *satisfies*

$$x(t) = \exp(tA)z = \lim_{n \to \infty} \exp(tA_n)z_n, \tag{4.1}$$

the convergence being uniform on compact subsets of $[0, \infty)$.

Proof. First we assume $z \in \mathscr{D}(A)$; the following argument is adapted from the proof of Lemma 3.1, p. 199 of [**6**]. Since A commutes with the C_0-semigroup $\exp(tA)$ on $\mathscr{D}(A)$, we have

$$x'(t) = Ax(t) = A\exp(tA)z = \exp(tA)Az.$$

Since the semigroup $\exp(tA)$ is strongly continuous, we have $x'(t)$ continuous. Now, for any n, we can write the differential equation for $x(t)$ in the form

$$x'(t) = A_n x(t) + (A - A_n)x(t).$$

The operators A_n are bounded, and $(A - A_n)x(t)$ is continuous on $[0, \infty)$, so we can convert to an integral equation, obtaining

$$\begin{aligned} x(t) &= \exp(tA_n)z + \int_0^t \exp[(t-s)A_n](A - A_n)x(s)\,ds \\ &= \exp(tA_n)z + \int_0^t \exp[(t-s)A_n](A - A_n)A^{-1}Ax(s)\,ds. \end{aligned} \tag{4.2}$$

(Note that A^{-1} exists by Lemma 2.) Fix $T \geqq 0$. Since $Ax(t)$ is continuous, the set $V_T = \{Ax(s) : 0 \leqq s \leqq T\}$ is compact. Because of Lemma 2 and Lemma 3, the operators $(A - A_n)A^{-1}$ are bounded and converge to zero strongly on l^1, and hence uniformly on the compact set V_T. Applying (2.8) to each A_n, we see that $\|\exp(tA_n)\| \leqq \exp(\omega + \|B\|)t$, and therefore that the operators $\exp(tA_n)$ are uniformly bounded with respect to n and $t \in [0, T]$. Now it follows that the integral terms in (4.2) converge to zero as $n \to \infty$, uniformly for $t \in [0, T]$, and hence we obtain for $z \in \mathscr{D}(A)$

$$\exp(tA)z = \lim_{n\to\infty} \exp(tA_n)z, \tag{4.3}$$

the convergence being uniform on compact subsets of $[0, \infty)$.

Next we show that (4.3) holds for any $z \in l^1$. Clearly

$$\begin{aligned} \|\exp(tA_n)z - \exp(tA)z\| \leqq \|\exp(tA_n) - \exp(tA)\| \cdot \|z - z_k\| \\ + \|\exp(tA_n)z_k - \exp(tA)z_k\|, \end{aligned}$$

for any positive integers n and k. Now, fix a positive number ϵ and a compact set $K \subseteq [0, \infty)$. Since the operators $\exp(tA_n)$ are uniformly bounded for $n = 1, 2, \ldots$ and $t \in K$, and $z_k \to z$ in l^1 as $k \to \infty$, the first term on the right can be made $< \epsilon/2$ uniformly in n and $t \in K$ by letting k be sufficiently large. Once k is fixed, (4.3) with z replaced by $z_k \in l_0 \subseteq \mathscr{D}(A)$ implies that the second term can also be made $< \epsilon/2$ uniformly for $t \in K$ by making n sufficiently large. Hence, for all $t \in K$ and n sufficiently large,

$$\|\exp(tA_n)z - \exp(tA)z\| < \epsilon, \tag{4.4}$$

as desired.

Now, for any $z \in l^1$ and any integer n, we have

$$\begin{aligned} \|\exp(tA_n)z_n - \exp(tA)z\| \leqq \|\exp(tA_n)\| \cdot \|z_n - z\| \\ + \|\exp(tA_n)z - \exp(tA)z\|. \end{aligned} \tag{4.5}$$

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The first term on the right approaches zero, since $z_n \to z$ as $n \to \infty$ and the operators $\exp(tA_n)$ are uniformly bounded. The second term also approaches zero, in view of (4.4). Hence the sequence $\{\exp(tA_n)z_n\}$ is uniformly convergent on compact subsets of $[0, \infty)$ to the limit $\exp(tA)z$. Thus (4.1) holds, and the theorem is proved.

Remark 2. As we have indicated, Theorem 4 shows that the solution $x(t)$ to (1.1)-(1.2) for given $z \in l^1$ can be approximated by solutions to finite systems, since $\exp(tA_n)z_n$ has for its first n entries precisely the solutions to the finite system

$$(4.6) \qquad \xi_i'(t) = \sum_{j=1}^{n} \alpha_{ij}\xi_j(t) \quad \xi_i(0) = \zeta_i, \ (t \geqq 0) \ (i = 1, \ldots, n).$$

It should also be pointed out that condition (A_3) is not necessary for Theorem 4 to hold. What is needed for the first part of the argument is that A must have a bounded inverse on l^1. Note that (A_3) gives conditions on the entries of A which imply that A has a bounded inverse, and also that it is closely related to condition (A_4), which we shall use to obtain exponential decay of solutions.

THEOREM 5. *Suppose that A satisfies condition* (A_2) *and that there exists a positive number δ such that*

$$(A_4) \qquad -\mathrm{Re}\,\alpha_{jj} \geqq \sum_{i \neq j} |\alpha_{ij}| + \delta \quad (j = 1, 2, \ldots).$$

Then for any $z \in l^1$, the continuous solution $x(t)$ of (1.1)-(1.2) *satisfies*

$$(4.7) \qquad ||x(t)|| \leqq e^{-\delta t}||z|| \quad (t \geqq 0).$$

Proof. First note that (A_4) implies (A_1) (in fact, $\omega \leqq -\delta$), so that (1.1)-(1.2) has a unique continuous solution $x(t)$. Furthermore, (A_4) implies (A_3), so that (4.1) is true. Taking norms in (4.1) gives

$$(4.8) \qquad ||x(t)|| = \lim_{n\to\infty} ||\exp(tA_n)z_n||.$$

As we have already pointed out in Remark 2, $\exp(tA_n)z_n$ is essentially the solution to the finite system (4.6), and therefore a recent result of Kahane [**5**] applies, and gives

$$(4.9) \qquad ||\exp(tA_n)z_n|| \leqq e^{-\delta t}||z_n||$$

for all n and all t in $[0, \infty)$. Combining (4.8) and (4.9) leads to (4.7), and proves the theorem.

If (A_4) is weakened to

$$(A_5) \qquad -\mathrm{Re}\,\alpha_{jj} > \sum_{i \neq j} |\alpha_{ij}| \quad (j = 1, 2, \ldots),$$

then we obtain the following boundedness theorem.

THEOREM 6. *If A satisfies conditions* (A_2), (A_3), *and* (A_5), *then for each* $z \in l^1$, *the continuous solution* $x(t)$ *of* (1.1)-(1.2) *satisfies*

(4.10) $||x(t)|| \leqq ||z|| \quad (t \geqq 0).$

Proof. We proceed as in the proof of Theorem 5 and arrive at equation (4.8). (A_5) implies that for each n, there is $\delta_n > 0$ such that

$$-\operatorname{Re} \alpha_{jj} \geqq \delta_n + \sum_{i=1,\, i\neq j}^{n} |\alpha_{ij}| \quad (j = 1, \ldots, n).$$

Hence we can again apply Kahane's result [**5**] and obtain

$$|| \exp(tA_n)z_n|| \leqq \exp(-\delta_n t)||z_n|| \leqq ||z_n||.$$

The desired inequality (4.10) now follows by letting $n \to \infty$.

In [**2**], Bellman has investigated the boundedness of solutions of infinite systems of the form

$$\xi_i'(t) = \sum_{j=i+1}^{\infty} \alpha_{ij}\xi_j(t) \quad (i = 1, 2, \ldots),$$

i.e., systems with an upper-triangular coefficient matrix. However, the proof of his result (Theorem 3, p. 704) appears to be incomplete.

It should also be pointed out that while the matrices in Kahane's paper [**5**] are assumed to have real entries, the result of that paper remains true for matrices with complex entries if condition (2) there is replaced by our condition (A_5).

5. Examples. Infinite systems of linear differential equations occur in various areas of science, for example, the perturbation theory for quantum mechanics [**1**], the physical chemistry of macromolecules [**8**], and in particular in the theory of stochastic processes [**1**; **4**]. However the examples in this section are intended just to illustrate the results in the present paper.

First we note that without condition (A_1), the system (1.1)-(1.2) does not have to have a solution in l^1 even when $z \in \mathscr{D}(A)$. It is easy to construct an example to illustrate this point by considering a diagonal matrix, and we shall not give details.

Example 1. Consider the system

$$(5.1) \quad \begin{cases} \xi_i'(t) = -i\xi_i(t) + \displaystyle\sum_{j=i+1}^{\infty} \xi_j(t) \\ \xi_i(0) = \zeta_i \end{cases} \quad (i = 1, 2, \ldots)$$

where $(\zeta_i) \in l^1$. In [**3**], Hille showed that (5.1) has a solution which depends on an arbitrary function, and therefore is not unique. Observe that the coefficient matrix does not satisfy (A_2). This suggests that when the diagonal entries are not bounded, it may be difficult to find general uniqueness theorems with conditions weaker than (A_2).

Example 2. (See [**9**].) Let $A = [\alpha_{ij}]$ be the symmetric matrix defined by

$$\alpha_{ij} = \begin{cases} -c, & \text{if } i = j \\ 2^{-(p+2)}, & \text{if } (i \dotplus j) = 2^p,\ p = 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

where $c > 0$ and $(i \dotplus j)$ means the modulo-two sum, without carrying, of the binary representations of i and j. Clearly, A satisfies the conditions of (1.5) with $\alpha = \beta = |c| + 1/2$, and hence, by [**9**, Theorem 3], the unique continuous solution of (1.1)-(1.2) satisfies

$$(5.2) \quad ||x(t)|| \leqq e^{(c+1/2)t}||z||.$$

In our notation, $\omega = -c$ and it is not hard to see that $||B|| = 1/2$. Since A is a bounded operator on $\mathscr{D}(A) = l^1$, (2.8) immediately gives the improved estimate

$$(5.3) \quad ||x(t)|| \leqq e^{-(c-1/2)t}||z||.$$

In particular, if $c > 1/2$, $||x(t)|| \to 0$ as $t \to \infty$. Note that if $c > 1/2$, (A_4) holds with $\delta = c - 1/2$, so (5.3) can also be obtained from Theorem 5.

Example 3. Consider the system

$$(5.4) \quad \begin{cases} \xi_i'(t) = -i\xi_i(t) + \xi_{i+1}(t) + \xi_{i+2}(t), & (i = 1, 2, \ldots) \\ \xi_i(0) = \zeta_i, \end{cases}$$

where $(\zeta_i) \in l^1$. It is easy to see that (A_1) and (A_2) hold with $\omega = -1$ and $M = ||B|| = 2$, so that (5.4) has a unique continuous solution $x(t)$. By Remark 1, we have the estimate

$$||x(t)|| \leqq e^t||z||.$$

However, (A_4) is satisfied with $\delta = 1$, so Theorem 5 gives the improved growth condition

$$||x(t)|| \leqq e^{-t}||z||.$$

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