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DISPERSIVE ORDERING RESULTS

JAMES LYNCH,*
GILLIAN MIMMACK** AND
FRANK PROSCHAN,** *Florida State University*

Abstract

A distribution F is *less dispersed* than a distribution G if $F^{-1}(\beta)-F^{-1}(\alpha)\leqq G^{-1}(\beta)-G^{-1}(\alpha)$ for all $0<\alpha<\beta<1$ $(F\overset{\text{disp}}{\leqq}G)$.

We generalize a characterization of dispersive ordering of Shaked (1982) concerning sign changes of F_c-G, where F_c is a translate of F. We then use this generalization plus total positivity to develop a simple proof of a characterization of dispersive distributions due to Lewis and Thompson (1981); a distribution H is *dispersive* if $F\overset{\text{disp}}{\leqq}G\Rightarrow H*F\overset{\text{disp}}{\leqq}H*G$.

1. Introduction

Dispersive ordering is a partial ordering of distributions according to their degree of dispersion. In this note we (1) generalize a characterization of dispersive ordering by Shaked (1982) from the smooth case in which the distributions are absolutely continuous with interval supports, to the general case and (2) using this generalization, develop a simple proof of a characterization by Lewis and Thompson (1981) of distributions which preserve dispersive ordering under convolution.

More precisely, for a distribution function F, define $F^{-1}(\alpha)=\inf\{t:F(t)\geqq\alpha\}(=\sup\{t:F(t)<\alpha\})$ for $0<\alpha<1$. Then we have the following.

Definition. A distribution function F is less dispersed than is a distribution function G if $F^{-1}(\beta)-F^{-1}(\alpha)\leqq G^{-1}(\beta)-G^{-1}(\alpha)$ for all $0<\alpha<\beta<1$; we write $F\overset{\text{disp}}{\leqq}G$.

2. Extension of the Shaked characterization

Let $S(x_1,\cdots,x_n)$ denote the number of sign changes of the sequence $x_1,\cdots,x_n$, where zero terms are discarded. Let $S(f)$ denote the number of sign changes of the

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* Permanent address: Department of Statistics, Pennsylvania State University, University Park, PA 16802, U.S.A.

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† Postal address: Department of Statistics and Statistical Consulting Center, The Florida State University, Tallahassee, FL 32306, U.S.A.

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function f on $(-\infty, \infty)$; specifically,

$$S(f) = \sup S[f(t_1), \cdots, f(t_m)],$$

the supremum being taken over all $t_1 < t_2 < \cdots < t_m$, $m = 2, 3, \cdots$. Finally, let $S_c \equiv S(F_c - G)$ for distribution functions F and G.

Theorem 1. $F \overset{\text{disp}}{\leqq} G \Leftrightarrow$ for each real c, (a) $S(F_c - G) \leqq 1$ *and* (b) if $S_c = 1$, then $F_c - G$ changes sign from $-$ to $+$.

Proof. $\Rightarrow$ For fixed c, let t_0 satisfy $G(t_0) > F_c(t_0)$. Let $t_* = \inf\{t : G(s) \geqq F_c(s)$ for $t \leqq s \leqq t_0\}$. We shall show that $t_* = -\infty$. From this it will follow that $F_c - G$ can never change from $+$ to $-$. This will complete the proof.

Suppose $t_* > -\infty$. By the definition of t_* and the right continuity of F and G, $G(t_*) \geqq F_c(t_*)$ and $G(t_* - \varepsilon) < F_c(t_* - \varepsilon)$ for sufficiently small $\varepsilon > 0$. Fix such an ε; then let $\alpha = F_c(t_* - \varepsilon)$ and $\beta = G(t_0)$. It follows that

$$G^{-1}(\alpha) \geqq t_* - \varepsilon \geqq F_c^{-1}(\alpha)$$

and

$$G^{-1}(\beta) \leqq t_0 < F_c^{-1}(\beta).$$

Thus

$$G^{-1}(\beta) - G^{-1}(\alpha) < F_c^{-1}(\beta) - F_c^{-1}(\alpha),$$

contradicting $F \overset{\text{disp}}{\leqq} G$. This proves the $\Rightarrow$ part.

$\Leftarrow$ Let H_n denote the exponential distribution function with mean $\mu_n \to 0$ as $n \to \infty$. Let $F_n = H_n * F$ and $G_n = H_n * G$, where $*$ denotes convolution. By the variation diminishing theorem of Karlin (1968), the sign change properties of $F_c - G$ are inherited by $F_{nc} - G_n$. Thus $F_n \overset{\text{disp}}{\leqq} G_n$ by Theorem 2.1 of Shaked (1982) since F_n and G_n are both absolutely continuous with interval supports. Since $F_n \to F$ ($G_n \to G$) in distribution as $n \to \infty$, it follows that $F \overset{\text{disp}}{\leqq} G$.

3. Dispersive distributions

Definition. A distribution function H is said to be *dispersive* if $H * F \overset{\text{disp}}{\leqq} H * G$ whenever $F \overset{\text{disp}}{\leqq} G$.

Next we present a simpler proof of Theorem 8 of Lewis and Thompson (1981) characterizing dispersive distribution functions. The simplification derives from the use of total positivity which is usually the appropriate way to treat problems of sign change.

Theorem 2. Let H be a non-degenerate distribution function. Then H is dispersive$\Leftrightarrow$ H is absolutely continuous with a log concave density.

Proof. $\Leftarrow$ Let $F \overset{\text{disp}}{\leqq} G$. From Theorem 1, for each real c, $S_c = S(F_c - G) \leqq 1$, with a $-$ to $+$ sign change if $S_c = 1$. Since H has a log concave density, then by the variation diminishing theorem of Karlin (1968), Chapter 5, $S[(H * F)_c - (H * G)] \leqq 1$, with a $-$ to $+$ sign change if $S[(H * F)_c - (H * G)] = 1$. It follows from Theorem 1 that $H * F \overset{\text{disp}}{\leqq} H * G$

$\Rightarrow$ We use essentially Lewis and Thompson's argument, but avoid unnecessary details of their proof. Our proof is based on their elementary proof of the 'only if' part of their Theorem 7; we restate this for completeness.

Lemma. Let F be dispersive and twice continuously differentiable. Then F has a log concave density.

To complete the proof of Theorem 2, let H be dispersive and let Φ_σ denote a normal distribution with mean 0 and variance σ^2. From the first part of Theorem 2, note that Φ_σ is dispersive, and thus, $\Phi_\sigma * H$ is also dispersive. Since $\Phi_\sigma * H$ is infinitely differentiable, it follows from the lemma above that $\Phi_\sigma * H$ has a log concave density. Since $\Phi_\sigma * H \to H$ in distribution as $\sigma \to 0$, H has a log concave density. (See Ibragimov (1956); note that a distribution is 'strongly unimodal' in Ibragimov's terminology iff its density is log concave.)

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