Ergod. Th. & Dynam. Sys., (2023), **43**, 2841–2862 © The Author(s), 2022. Published by Cambridge University Press.
doi:10.1017/etds.2022.66

Periodic expansion of one by Salem numbers

SHIGEKI AKIYAMA† and HACHEM HICHRI‡

† *Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan*
(e-mail: akiyama@math.tsukuba.ac.jp)
‡ *Département de Mathématiques (UR18ES15), Faculté des sciences de Monastir, Université de Monastir, Monastir 5019, Tunisie*
(e-mail: hichemhichri@yahoo.fr)

(*Received* 23 *June* 2022 *and accepted in revised form* 19 *August* 2022)

Abstract. We show that for a Salem number β of degree d, there exists a positive constant $c(d)$ where β^m is a Parry number for integers m of natural density $\geq c(d)$. Further, we show $c(6) > 1/2$ and discuss a relation to the discretized rotation in dimension 4.

Key words: beta expansion, periodicity, Salem number, discretized rotation
2020 Mathematics Subject Classification: 37E05, 37B10, 11K16 (Primary)

1. *Introduction*

Let $\beta > 1$. Rényi [25] introduced the beta transformation on $[0, 1)$ by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor.$$

This map has long been applied in many branches of mathematics, such as number theory, dynamical system, coding theory, and computer sciences. The dynamical system $([0, 1), T_\beta)$ admits the 'Parry measure' μ_β: a unique invariant measure equivalent to the Lebesgue measure [23]. The system is ergodic with respect to μ_β and gives an important class of systems with explicit invariant density. The transformation T_β gives a representation of real numbers in a non-integer base (cf. [14]). Defining $x_n = \lfloor \beta T_\beta^{n-1}(x) \rfloor$, we obtain the 'greedy' expansion:

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \cdots,$$

which is a generalization of decimal or binary ($\beta = 10, 2$) representations by an arbitrary base $\beta > 1$. The word $d_\beta(x) := x_1 x_2 \ldots$ corresponding to x is an infinite word over $\mathcal{A} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$. An infinite word $x_1 x_2 \ldots \in \mathcal{A}^{\mathbb{N}}$ is eventually periodic if it is written as $x_1 x_2 \ldots x_m (x_{m+1} \ldots x_{m+p})^\infty$. We choose the minimum m and p when such m and p exist and call it (m, p)-periodic. The dynamical properties of a piecewise linear map are governed by the orbit of discontinuities. For T_β, there exists essentially only one discontinuity corresponding to the right end point 1. The expansion of 1 is the word $d_\beta(1) := \lim_{\epsilon \downarrow 0} d_\beta(1-\epsilon) = c_1 c_2 \ldots$. For a fixed β, an element $x_1 x_2 \ldots \in \mathcal{A}^{\mathbb{N}}$ is realized as $d_\beta(x)$ with some $x \in [0, 1)$ if and only if

$$\sigma^n(x_1 x_2 \ldots) <_{\text{lex}} c_1 c_2 \ldots$$

for any $n \in \mathbb{N}$. Here the shift is defined as $\sigma((x_i)) := (x_{i+1})$ for a (one-sided or two-sided) infinite word $(x_i) \in \mathcal{A}^{\mathbb{N}} \cup \mathcal{A}^{\mathbb{Z}}$ and $<_{\text{lex}}$ is the lexicographic order. We say that $x_1 x_2 \ldots \in \mathcal{A}^{\mathbb{N}}$ is admissible if this condition holds. A finite word $x_1 \ldots x_n \in \mathcal{A}^*$ is admissible if $x_1 \ldots x_n 0^\infty$ is admissible. An element of $c_1 c_2 \ldots \in \mathcal{A}^{\mathbb{N}}$ is realized as $d_\beta(1)$ with some $\beta > 1$ if and only if

$$\sigma^n(c_1 c_2 \ldots) <_{\text{lex}} c_1 c_2 \ldots \tag{1}$$

for any $n \geq 1$, see [18, 23] for details. Given $\beta > 1$, the beta shift (X_β, σ) is a subshift consisting of the set of bi-infinite words $(a_i)_{i \in \mathbb{Z}}$ over $\mathcal{A}$ such that every subword $a_n a_{n+1} \ldots a_m$ is admissible. Beta shift X_β is sofic if and only if $d_\beta(1)$ is eventually periodic, and X_β is a subshift of finite type if and only if $d_\beta(1)$ is purely periodic, that is, $d_\beta(1)$ is $(0, p)$-periodic, see [2, 9]. (In [23], expansion of one is defined formally by $(\lfloor \beta T_\beta^n(1) \rfloor)_{n \in \mathbb{N}}$ and the purely periodic expansion $d_\beta(1) = (c_1 c_2 \ldots c_{p-1} c_p)^\infty$ is expressed as a finite expansion $c_1 c_2 \ldots c_{p-1}(1 + c_p)0^\infty$.) The β is called a Parry number in the former case, and a simple Parry number in the latter case. When the topological dynamics (X_β, σ) is sofic, T_β belongs to an important class of interval maps; Markov maps by finite partition (see [10, 24]).

A Pisot number is an algebraic integer > 1 so that all of whose conjugates have modulus less than one. A Salem number is an algebraic integer > 1 so that all of whose conjugates have modulus not greater than one and at least one of the conjugates has modulus one. If β is a Pisot number, then $\{T_\beta^i(x) \mid i \in \mathbb{N}\}$ is finite for $x \in \mathbb{Q}(\beta)$, that is, the word $d_\beta(x)$ is eventually periodic, see [8, 26]. Consequently, a Pisot number is a Parry number. Schmidt [26] proved that if $d_\beta(x)$ is eventually periodic for every $x \in \mathbb{Q} \cap [0, 1)$, then β is a Pisot or Salem number. Determining periodicity/non-periodicity of $d_\beta(x)$ by a Salem number β and $x \in \mathbb{Q}(\beta)$ remains a difficult problem. The main obstacle is that we do not have any idea until present to show that $d_\beta(x)$ is *not* eventually periodic when β is a Salem number. Boyd [11] showed that a Salem number of degree 4 is a Parry number by classifying all shapes of $d_\beta(1)$. Since then, apart from a computational or heuristic discussion like Boyd [12], Hichri [15–17], we have very few results on the beta expansion by Salem numbers. In this paper, we make some additions to this direction.

THEOREM 1. *For a Salem number β of degree d, there exist infinitely many positive integers m where β^m is a Parry number. More precisely, there exists a positive constant $c(d)$ depending only on d where*

$$\liminf_{M \to \infty} \frac{\operatorname{Card}\{m \in [1, M] \cap \mathbb{Z} \mid d_{\beta^m}(1) \textit{ is } (1, p)\textit{-periodic with some } p \in \mathbb{N}\}}{M} \geq c(d).$$

Note that $c(4) = 1$ was shown in [11]. Our method gives a rather small bound $c(d) = (3d)^{-d}$, see Remark 6. We can give a good lower bound when $d = 6$.

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

THEOREM 2. *Given a sextic Salem number β, for more than half of the positive integers m, β^m is a $(1, p)$-periodic Parry number for some $p \in \mathbb{N}$.*

Finally, we discuss an interesting connection to four-dimensional discretized rotation.

2. *Preliminary*

We review the basic results on Salem numbers (cf. [7, 28]). It is easy to show that a Salem number has even degree $2d \geq 4$ and its minimal polynomial is self-reciprocal. Thus $d_\beta(1)$ cannot be purely periodic for a Salem number β, see [1]. Let $P(x) \in \mathbb{Z}[x]$ be a monic irreducible self-reciprocal polynomial of even degree $2d$. Putting $Q(x) := P(x)/x^{2d}$, we have $Q(x) \in \mathbb{R}[x + x^{-1}]$. We write $Q(x) = G(y)$ with $y = x + x^{-1}$. Then $P(x)$ is a minimum polynomial of a Salem number if and only if $G(2) < 0$ and $G(y)$ has $d - 1$ distinct roots in $(-2, 2)$. Here, G is coined a trace polynomial of P in [12]. The factorization

$$G(y) = (y - \gamma)(y + \alpha_1) \ldots (y + \alpha_{d-1})$$

with $\gamma > 2$ and $\alpha_i \in (-2, 2)$ corresponds to the factorization of $P(x)$ in $\mathbb{R}[x]$:

$$P(x) = (x - \beta)\left(x - \frac{1}{\beta}\right) \prod_{i=1}^{d-1} (x^2 + \alpha_i x + 1), \tag{2}$$

where $\gamma = \beta + 1/\beta$ and $x^2 + \alpha_i x + 1$ gives a root $\exp(\theta_i\sqrt{-1}) = \cos(\theta_i) \pm \sin(\theta_i)\sqrt{-1}$ of $P(x)$ with $\alpha_i = -2\cos(\theta_i)$ and $\theta_i \in (0, \pi)$. It is well known that $1, \theta_1/\pi, \theta_2/\pi, \ldots, \theta_{d-1}/\pi$ are linearly independent over $\mathbb{Q}$, that is, $\exp(\theta_i\sqrt{-1})$ $(i = 1, \ldots, d-1)$ are multiplicatively independent. This is shown by applying a conjugate map to the possible multiplicative relation among them, cf. [13]. Note that this fact guarantees that β^m $(m = 1, 2, \ldots)$ are Salem numbers of the same degree $2d$. Applying this linear independence, we see that

$$\left(\frac{m\theta_1}{2\pi}, \ldots, \frac{m\theta_n}{2\pi}\right) \bmod \mathbb{Z}^n \tag{3}$$

is uniformly distributed in $(\mathbb{R}/\mathbb{Z})^n$, that is, for any parallelepiped

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

we have

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \chi_I\left(\left(\frac{m\theta_1}{2\pi}, \ldots, \frac{m\theta_n}{2\pi}\right) \bmod \mathbb{Z}^n\right) = \prod_{i=1}^{n} (b_i - a_i),$$

where χ_I is the characteristic function of I. Indeed, this is shown by the higher dimensional Weyl criterion [20]. Note that it is understood as unique ergodicity of the action

$$x \mapsto x + \left(\frac{\theta_1}{2\pi}, \ldots, \frac{\theta_n}{2\pi}\right)$$

on $(\mathbb{R}/\mathbb{Z})^n$. Since $(\mathbb{R}/\mathbb{Z})^n$ is a compact metric group, minimality and unique ergodicity are equivalent [30], and minimality is a little easier to show.

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

3. *Our strategy*

Our idea is to find a nice region so that if complex conjugates of a Salem number $\beta > 1$ fall into this region and β is sufficiently large, then $d_\beta(1)$ is eventually periodic. We realize this idea in a general form which can be applied to the dominant real root of self-reciprocal polynomials, not only Salem numbers. The statement seems useless for a single β (because it is easier to compute $d_\beta(1)$ directly) but we will find a nice application in the following sections.

LEMMA 3. *Let us fix constants u, v with $0 < u < v < 1$. If a monic polynomial $f(x) \in \mathbb{Z}[x]$ of degree $2n + 2$ satisfies*

$$f(\beta) = 0, \quad f(0) = 1, \quad \beta > \max\left\{\frac{2}{u}, \frac{1}{1-v}\right\}$$

and

$$\frac{f(x)}{(x-\beta)(x-1/\beta)} = x^{2n} + 1 + \sum_{i=1}^{2n-1} g_i x^i$$

with $u < g_i < v$ for $i = 1, \ldots, 2n-1$, then $f(x)$ is self-reciprocal and $d_\beta(1)$ is $(1, 2n+1)$-periodic.

Proof. Set $g_0 = g_{2n} = 1$. Putting

$$f(x) = x^{2n+2} + 1 - \sum_{i=1}^{2n+1} c_i x^i,$$

we obtain

$$c_1 = \beta + 1/\beta - g_1, \quad c_{2n+1} = \beta + 1/\beta - g_{2n-1},$$

and

$$c_i = (\beta + 1/\beta) g_{i-1} - g_{i-2} - g_i$$

for $i = 2, \ldots, 2n$. Since $c_i \in \mathbb{Z}$, g_1 and g_{2n-1} are uniquely determined by $\beta + 1/\beta$, and thus $g_1 = g_{2n-1}$ and $c_1 = c_{2n+1}$. Moreover by induction, we see $c_i = c_{2n+2-i}$ for every $i = 1, \ldots, 2n+1$, that is, $f(x)$ is self-reciprocal. By assumption, we have

$$c_i \geq \beta u - 2 > 0, \quad c_1 - c_i \geq (\beta + 1/\beta)(1 - v) - 1 > 0$$

for $i = 2, \ldots, 2n$. Therefore, we have

$$c_i \in \mathbb{Z}, \quad c_1 > c_j > 0 \; (j = 2, \ldots, 2n).$$

One can write

$$\beta^{2n+2} + 1 - \sum_{i=1}^{2n+1} c_i \beta^i = 0$$

as a representation of zero in base β:

$$(-1), c_1, c_2, \ldots, c_n, c_{n+1}, c_n, \ldots, c_2, c_1, (-1).$$

Adding its $2n+1$ shifted form, we see

$$(-1), c_1, c_2, \dots, c_{n+1}, \dots, c_2, c_1 - 1, c_1 - 1, c_2, \dots, c_{n+1}, \dots, c_2, c_1, (-1)$$

is another representation of zero. Iterating this shifted addition, we obtain an infinite expansion of 1:

$$c_1(c_2, c_3, \dots, c_n, c_{n+1}, c_n, \dots, c_2, c_1 - 1, c_1 - 1)^\infty,$$

which satisfies the lexicographic condition in equation (1). (This implies $c_1 < \lceil\beta\rceil - 1$ as well.) Therefore, $d_\beta(1)$ is $(1, 2n+1)$- periodic. □

Logically we have to fix u, v at first. However, this is often impractical. When we apply Lemma 3 to a Salem number β of degree $2d$, we take a self-reciprocal monic polynomial $R(x) \in \mathbb{Z}[x]$ and study $f(x) = R(x)P(x)$ of degree $2n+2$, where $P(x)$ is the minimum polynomial of β. Note that one can take $R(x) = 1$ as well. We check if there exist u and v which satisfy our requirements. Since

$$\frac{f(x)}{(x-\beta)(x-1/\beta)} = \sum_{i=0}^{2n} g_i x^i = R(x) \prod_{i=1}^{d-1} (x^2 + \alpha_i x + 1)$$

with $\alpha_i \in (-2, 2)$, for a fixed $R(x)$, the problem is reduced to the set of solutions $(\alpha_1, \dots, \alpha_{d-1}) \in \mathbb{R}^{d-1}$ for the system of inequalities

$$0 < g_i < 1 \ (i = 1, 2, \dots, 2n-1), \quad -2 < \alpha_j < 2 \ (j = 1, \dots, d-1),$$

over $d-1$ variables $\alpha_1, \dots, \alpha_{d-1}$. A self-reciprocal monic polynomial $R(x)$ gives a certain choice of u, v if and only if the set of solutions contains an inner point in the space $\mathbb{R}^{d-1}$, and for every inner point, we can find u, v. Solving this set of inequalities is not an easy task in general, but it is feasible for degree 6 since all the inequalities are quadratic. In §5, we use this method to find good regions for β. It is also useful to solve a slightly wider set of inequalities:

$$0 < g_i \le 1 \ (i = 1, 2, \dots, 2n-1), \quad -2 < \alpha_j < 2 \ (j = 1, \dots, d-1). \tag{4}$$

The solution may contain the case $g_i = 1$. The proof of Lemma 3 works in the same way under a little more involved assumptions and we have the following lemma.

LEMMA 4. *Let us fix constants u, v with $0 < u < v < 1$. If a monic polynomial $f(x) \in \mathbb{Z}[x]$ of degree $2n+2$ satisfies*

$$f(\beta) = 0, \ f(0) = 1, \quad \beta > \max\left\{\frac{2}{u}, \frac{1}{1-v}\right\}$$

and

$$\frac{f(x)}{(x-\beta)(x-1/\beta)} = x^{2n} + 1 + \sum_{i=1}^{2n-1} g_i x^i$$

with

$$u < g_i < v$$

or

$$(g_i = 1 \textit{ and } g_1 < g_{i-1} + g_{i+1})$$

or

$$(g_i = 1 \textit{ and } g_1 = g_{i-1} + g_{i+1} \textit{ and } g_1 - g_{i+1} > u)$$

hold for $i = 1, \ldots, 2n - 1$, *then* $f(x)$ *is self-reciprocal and* $d_\beta(1)$ *is* $(1, 2n + 1)$*-periodic.*

Proof. If $g_i = 1$ and $g_1 < g_{i-1} + g_{i+1}$ holds, then $c_1 - c_{i+1} = g_{i-1} + g_{i+1} - g_1 > 0$. If $g_i = 1$ and $g_1 = g_{i-1} + g_{i+1}$ and $g_1 - g_{i+1} > u$ hold, then $c_1 = c_{i+1}$ and $c_2 - c_{i+2} = (\beta + 1/\beta)(g_1 - g_{i+1}) - 1 - g_2 + g_i + g_{i+2} > \beta u - 2 > 0$. Therefore, the lexicographic condition in equation (1) holds in all cases. □

We shall see later that the effect of this small extension of Lemma 3 is pretty large both in theory and in practice. Indeed, Lemma 4 works very well with a discretized rotation algorithm, see the discussion in §6.

4. *Proof of Theorem 1*

We start with the following lemma.

LEMMA 5. *For a positive even integer* $2n$, *all roots of the self-reciprocal polynomial*

$$P(x) = x^{2n} + d_{2n-1}x^{2n-1} + \cdots + d_1 x + 1 \quad (d_{2n-i} = d_i)$$

with $d_i \in \mathbb{R}$ *and* $|d_i| < 1/(2n - 2)$ *are on the unit circle.*

Proof. Let $G_0(y) := (x^{2n} + 1)/x^n$ and $G_1(y) := P(x)/x^n$ with $y = x + x^{-1}$. Since

$$G_0(2\cos(\pi k/n)) = 2(-1)^k$$

for $k = 0, 1, 2, \ldots, n$, $G_0(y)$ has n real roots ψ_i $(i = 1, 2, \ldots, n)$ with

$$2 > \psi_1 > 2\cos\left(\frac{\pi}{n}\right) > \psi_2 > 2\cos\left(\frac{2\pi}{n}\right) > \psi_3 > \cdots$$

$$\cdots > 2\cos\left(\frac{(n-2)\pi}{n}\right) > \psi_{n-1} > 2\cos\left(\frac{(n-1)\pi}{n}\right) > \psi_n > -2.$$

From $|d_i| < 1/(2n - 2)$, we have $G_1(2\cos(\pi k/n)) < 0$ for odd k and $G_1(2\cos(\pi k/n)) > 0$ for even k. By intermediate value theorem, $G_1(y)$ has n real roots ψ_i' $(i = 1, \ldots, n)$ in $(-2, 2)$ satisfying the same inequality as ψ_i. Therefore, we have

$$G_1(y) = \prod_{i=1}^{n} (y - \psi_i').$$

Coming back to $P(x)$, we get the assertion. □

For degree 4, we have nothing to do since Boyd [11] showed that every Salem number of degree 4 is a $(1, p)$-periodic Parry number for some $p \in \mathbb{N}$. Consider a polynomial

$$h(x) = x^{2n} + \frac{1}{4(n-1)} \sum_{i=1}^{2n-1} x^i + 1.$$

By Lemma 5, we have

$$h(x) = \prod_{i=1}^{n}(x - \exp(\eta_i\sqrt{-1}))(x - \exp(-\eta_i\sqrt{-1}))$$

with

$$2 > 2\cos(\eta_1) > 2\cos\left(\frac{\pi}{n}\right) > 2\cos(\eta_2) > 2\cos\left(\frac{2\pi}{n}\right) > 2\cos(\eta_3) > \cdots$$
$$\cdots > 2\cos\left(\frac{(n-2)\pi}{n}\right) > 2\cos(\eta_{n-1}) > 2\cos\left(\frac{(n-1)\pi}{n}\right) > 2\cos(\eta_n) > -2. \tag{5}$$

This type of discussion is called 'interlacing' and efficiently used in the construction of Salem numbers having desired properties, see [6, 22, 27] and its references.

Considering coefficients as a continuous function of roots, there exists a constant $\varepsilon > 0$ that if

$$\psi_i \in [\eta_i - \varepsilon, \eta_i + \varepsilon]$$

for $i = 1, 2, \ldots, n$, then we have

$$\prod_{i=1}^{n}(x - \exp(\psi_i\sqrt{-1}))(x - \exp(-\psi_i\sqrt{-1})) = x^{2n} + 1 + \sum_{i=1}^{2n-1} g_i x^i \tag{6}$$

with $1/6(n-1) < g_i < 1/3(n-1)$. See Remark 6 for the choice of ε.

Let β be a Salem number of degree $2n+2$ with $n \geq 2$ and let $\theta_i \in (0, \pi)$ $(i = 1, \ldots, n)$ be the arguments of the conjugates of β on the unit circle determined as in equation (2). Since $1, \theta_1/\pi, \ldots, \theta_n/\pi$ are linearly independent over $\mathbb{Q}$, by Kronecker's approximation theorem, we find infinitely many positive integers m such that

$$\frac{m\theta_i}{2\pi} (\text{mod } \mathbb{Z}) \in [\eta_i - \varepsilon, \eta_i + \varepsilon] \tag{7}$$

hold for $i = 1, 2, \ldots, n$. For an integer m with this property, the minimum polynomial of β^m has the form

$$(x - \beta^m)\left(x - \frac{1}{\beta^m}\right)\left(x^{2n} + 1 + \sum_{i=1}^{2n-1} g_i^{(m)} x^i\right)$$

and $1/6(n-1) < g_i^{(m)} < 1/3(n-1)$ for $i = 1, 2, \ldots, 2n-1$. By Lemma 3, we see that $d_{\beta^m}(1)$ is $(1, 2n+1)$-periodic for sufficiently large m. Finally, we show that

$$\{m \in \mathbb{N} \mid d_{\beta^m}(1) \text{ is } (1, 2n+1)\text{-periodic}\}$$

has positive lower natural density. Using the fact that equation (3) is uniformly distributed in $(\mathbb{R}/\mathbb{Z})^n$, we have

$$\liminf_{M\to\infty} \frac{1}{M} \text{Card}\{m \in [1, M] \cap \mathbb{N} \mid d_{\beta^m}(1) \text{ is } (1, 2n+1)\text{-periodic}\} \geq (2\varepsilon)^n$$

in view of equation (7), giving the lower bound of the natural density. □

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

Remark 6. We give a lower bound of $c(d)$ from the above proof. First, we compute the asymptotic expansion of η_i with respect to n, when η_i is close to $\pi/2$ (cf. [3]). If n is odd, then we have

$$\eta_{\lfloor n/2\rfloor} = \frac{\pi}{2} + \frac{1}{8n^2} + \frac{9}{64n^3} + O\left(\frac{1}{n^4}\right),$$

which leads to $\tan(\eta_{\lfloor n/2\rfloor}) = -8n^2 + O(n)$. Further, we have

$$\eta_{\lfloor n/2\rfloor\pm 1} = \frac{\pi}{2} \pm \frac{\pi}{n} + \frac{1}{8n^2} + \frac{9 \mp 8\pi}{64n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$\tan(\eta_i) = O(n)$$

for $|i - \lfloor n/2\rfloor| > 1$ in light of equation (5). If n is even, then we have

$$\eta_{n/2\pm 1} = \frac{\pi}{2} \pm \frac{\pi}{2n} + \frac{1}{8n^2} + \frac{9 \mp 4\pi}{64n^3} + O\left(\frac{1}{n^4}\right).$$

Thus, $\tan(\eta_i) = O(n)$ is valid for all i by equation (5).

Second, comparing the coefficients of $\prod_{i=1}^{n}(x^2 + 2\cos(\eta_i + \nu)x + 1)$ and those of $\prod_{i=1}^{n}(x^2 + 2\cos(\eta_i)x + 1)$, we see that the desired inequality of equation (6) holds if

$$\frac{2}{3} < \prod_{i=1}^{n} \left|\frac{\cos(\eta_i + \nu)}{\cos(\eta_i)}\right| < \frac{4}{3} \tag{8}$$

for $|\nu| < \varepsilon$. Using an easy inequality

$$1 - |\tan(x)y| - \frac{y^2}{2} < \left|\frac{\cos(x+y)}{\cos(x)}\right| < 1 + |\tan(x)y|$$

for $\cos(x) \neq 0$ and the above estimates on η_i, we see that there exists a positive constant κ so that if $|\varepsilon| < \kappa/n^2$, then equation (8) holds. Thus we obtain $c(d) \geq (\kappa/n^2)^n$ with $d = 2n + 2$. Making explicit the implied constants of Landau symbols, we see that $\kappa = 1/32$ suffices. Thus we have $c(d) \geq (32(d/2)^2)^{-d/2} > (3d)^{-d}$.

5. *Proof of Theorem 2*

For the rest of this paper, we deal with a Salem number β of degree 6. For simplicity of presentation, we prove that $c(6) \geq 0.458$. To show that $c(6)$ exceeds 0.5, we have to use more polynomials and the computation becomes much harder, see Appendix A.

To apply the discussion after Lemma 3, there exists an efficient practical way to find $R(x)$ which is similar to the shift radix system (cf. [4]). Pick a random $(\alpha_1, \alpha_2) \in (-2, 2)^2$ and start with a coefficient vector $(1, c_1, c_2, c_1, 1)$ of

$$(x^2 + \alpha_1 x + 1)(x^2 + \alpha_2 x + 1) = x^4 + c_1 x^3 + c_2 x^2 + c_1 x + 1,$$

that is, $c_1 = \alpha_1 + \alpha_2$, $c_2 = \alpha_1\alpha_2 + 2$. We wish to iterate the shifted addition as

$$\begin{aligned}(1, c_1, c_2, c_1, 1) &\to (1, c_1, c_2, c_1, 1, 0) + k(0, 1, c_1, c_2, c_1, 1)\\ &= (1, c_1 + k, c_2 + c_1 k, c_1 + c_2 k, 1 + c_1 k, k),\end{aligned}$$

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

with $k = -\lceil c_1 - 1 \rceil$, to find a longer coefficient vector where all entries except the first and the last ones fall in $(0, 1]$, as in Lemma 4. To make this idea into an algorithm, set $z_{-1} = (0, 0, 0, 0)$ and for $z_n = (t_n(1), t_n(2), t_n(3), t_n(4))$, we define

$$z_{n+1} := (t_n(2) + c_1 k_{n+1}, t_n(3) + c_2 k_{n+1}, t_n(4) + c_1 k_{n+1}, k_{n+1}) \tag{9}$$

with $k_{n+1} = -\lceil t_n(1) - 1 \rceil$. Thus we see $k_0 = 1$ and $z_0 = (c_1, c_2, c_1, 1)$. We stop this iteration when $0 < t_n(i) \le 1$ for $i = 1, 2, 3$ and $t_n(4) = 1$ (that is, $k_n = 1$) and obtain a candidate $R(x) = \sum_{i=0}^{n} k_i x^i$. It is natural to set $k_{n+1} = k_{n+2} = k_{n+3} = k_{n+4} = 0$ and we obtain

$$(x^4 + c_1 x^3 + c_2 x^2 + c_1 x + 1) R(x) = \sum_{i=0}^{n+4} g_i x^i$$

with $g_0 = g_{n+4} = 1$ and $g_i = t_{i-1}(1) + k_i \in (0, 1]$ $(i = 1, \ldots, n+3)$. If n became larger than a given threshold, then we restart with a different (α_1, α_2). Applying this random search, we find polynomials $R_i(x)$ $(i = 1, \ldots, 18)$ so that Lemma 3 gives relatively large regions $\mathcal{R}_i$ $(i = 1, \ldots, 18)$ defined by equation (4).

$$\begin{aligned}
R_1 &= 1,\\
R_2 &= 1 + x,\\
R_3 &= 1 + x^2,\\
R_4 &= 1 + 2x + x^2,\\
R_5 &= 1 + x + x^2 + x^3,\\
R_6 &= 1 + 2x + 2x^2 + x^3,\\
R_7 &= 1 - x + x^2 - x^3 + x^4,\\
R_8 &= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + x^5,\\
R_9 &= 1 + x - x^2 - x^3 + x^4 + x^5,\\
R_{10} &= 1 + 2x^2 + 2x^4 + x^6,\\
R_{11} &= 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6,\\
R_{12} &= 1 + x + x^2 + 2x^3 + 2x^4 + x^5 + x^6 + x^7,\\
R_{13} &= 1 + x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + x^6 + x^7,\\
R_{14} &= 1 + 2x + x^2 - x^3 - x^4 + x^5 + 2x^6 + x^7,\\
R_{15} &= 1 + x^2 - x^3 + x^4 - x^5 + x^6 + x^8,\\
R_{16} &= 1 - 2x + 2x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 2x^7 + x^8 - x^9 + 2x^{10} - 2x^{11} + x^{12},\\
R_{17} &= 1 + 3x + 4x^2 + 3x^3 + x^4,\\
R_{18} &= 1 + 3x + 4x^2 + 2x^3 - 2x^4 - 4x^5 - 2x^6 + 2x^7 + 4x^8 + 3x^9 + x^{10}.
\end{aligned}$$

For example,

$$(x^4 + c_1 x^3 + c_2 x^2 + c_1 x + 1) R_8 = x^9 + (c_1 + 2) x^8 + \cdots + (c_1 + 2) x + 1$$

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

gives the coefficient vector

$$(1, c_1+2, 2c_1+c_2+2, 3c_1+2c_2+2, 4c_1+2c_2+3, 4c_1+2c_2+3, 3c_1+2c_2+2, 2c_1+c_2+2, c_1+2, 1)$$

which gives rise to a system of linear inequalities

$$0 < c_1+2 \le 1, \quad 0 < 2c_1+c_2+2 \le 1, \quad 0 < 3c_1+2c_2+2 \le 1, \quad 0 < 4c_1+2c_2+3 \le 1.$$

Solving this system, we obtain a triangular region:

- $c_1 > \dfrac{-2c_2-3}{4}$ if $c_2 \in \left(\dfrac{1}{2}, \dfrac{5}{2}\right)$;
- $c_1 \le -1$ if $c_2 \in \left(\dfrac{1}{2}, 1\right]$;
- $c_1 \le \dfrac{-2c_2-1}{3}$ if $c_2 \in \left(1, \dfrac{5}{2}\right)$.

This collection of three sentences with • reads c_2 must be in at least one of the intervals, and we take the logical 'and' of the three. Replacing (c_1, c_2) by $(\alpha_1+\alpha_2, \alpha_1\alpha_2+2)$, we can confirm that $-2 < \alpha_i < 2$ for $i = 1, 2$ hold in this triangle. (For a general $R(x)$, we compute the intersection with the region $-2 < \alpha_i < 2$ $(i = 1, 2)$. If this intersection is empty, then we have to restart with a different (α_1, α_2).) Thus we find the two curvilinear triangles $\mathcal{R}_8$ in Figure 1 bounded by segments and hyperbola. We also found polynomials L_i $(i = 1, \ldots, 5)$ giving large regions $\mathcal{L}_i$ $(i = 1, \ldots, 5)$ where Lemma 3 does not apply and Lemma 4 is necessary.

$$\begin{aligned}
L_1 &= 1 - x^2 + x^3 + x^4 - x^5 + x^7,\\
L_2 &= 1 - x + x^3 - x^5 + x^6,\\
L_3 &= 1 - x^2 + x^3 + x^6 - x^7 + x^9,\\
L_4 &= 1 - 2x + 2x^2 - 2x^4 + 3x^5 - 2x^6 + 2x^8 - 2x^9 + x^{10},\\
L_5 &= 1 - x + x^3 - x^6 + x^7 + x^8 - x^9 + x^{12} - x^{14} + x^{15}.
\end{aligned}$$

For example, the coefficients of x^3 and x^8 in

$$\begin{aligned}
&(x^4 + c_1x^3 + c_2x^2 + c_1x + 1)L_1\\
&\quad = 1 + c_1x - x^2 + c_2x^2 + x^3 + 2x^4 + c_1x^4 - c_2x^4 - x^5 + c_2x^5 - x^6 + c_2x^6\\
&\qquad + 2x^7 + c_1x^7 - c_2x^7 + x^8 - x^9 + c_2x^9 + c_1x^{10} + x^{11}
\end{aligned}$$

are equal to 1, so we have to use Lemma 4.

We can check directly that those 23 sets are mutually disjoint. We will see later in §6 that Lemma 4 works fine for any period and this disjointness is natural. Indeed, if two period cells share an inner point, then their periods of the corresponding discretized rotations must coincide, since $d_{\beta^m}(1)$ for a Salem number β is uniquely determined by m. This implies

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

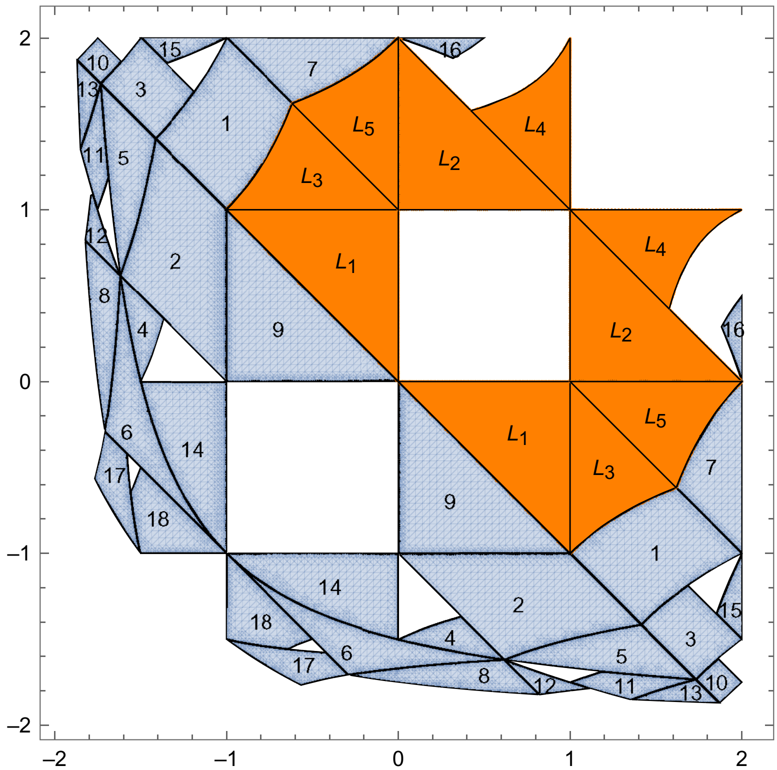

FIGURE 1. Period cells in (α_1, α_2) coordinate.

that the period cells must be identical. See Figure 1 and the explicit computation of period cells below.

Choose a small $\varepsilon > 0$ and consider the subset $\mathcal{R}_i(\varepsilon)$ of solutions of the system of inequalities

$$\varepsilon < g_i < 1 - \varepsilon \ (i = 1, 2, \ldots, 2n-1), \quad -2 < \alpha_j < 2 \ (j = 1, \ldots, d-1)$$

for a polynomial R_i $(i = 1, \ldots, 18)$. We also define $\mathcal{L}_i(\varepsilon)$ by the system of inequalities

$$\varepsilon < g_i < 1 - \varepsilon \ (i \in \{j \in [1, 2n-1] \mid g_j \neq 1\}), \quad -2 < \alpha_j < 2 \ (j = 1, \ldots, d-1)$$

for a polynomial L_i $(i = 1, \ldots, 5)$. Since $g_i \in \mathbb{Z} + (\alpha_1 + \alpha_2)\mathbb{Z} + \alpha_1\alpha_2\mathbb{Z}$, g_i is a constant if and only if $g_i = 1$. Therefore, unless $g_i = 1$, the boundary equalities $g_i = \varepsilon$ and $g_i = 1 - \varepsilon$ give one-parameter families of (linear or hyperbolic) curves, which continuously move along ε. In light of Lemmas 3 and 4, there exist $m_0 = m_0(\varepsilon) \in \mathbb{N}$ where if $m \geq m_0$ and $(-2\cos(m\theta_1/2\pi), -2\cos(m\theta_2/2\pi))$ falls into

$$S(\varepsilon) := \left(\bigcup_{i=1}^{18} \mathcal{R}_i(\varepsilon)\right) \cup \left(\bigcup_{i=1}^{5} \mathcal{L}_i(\varepsilon)\right),$$

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

then $d_{\beta^m}(1)$ is $(1, p)$-periodic with some $p \in \mathbb{N}$. Since $(m\theta_1/2\pi, m\theta_2/2\pi) (\mathrm{mod}\ \mathbb{Z}^2)$ is uniformly distributed and $S(\varepsilon)$ is Jordan measurable, the induced probability measure is computed by

$$\frac{1}{\pi^2}\int\int_{S(\varepsilon)} \frac{d\alpha_1 d\alpha_2}{\sqrt{(4-\alpha_1^2)(4-\alpha_2^2)}},$$

since the normalized Lebesgue measure $1/2\pi\, d\theta$ on the unit circle is projected 2 to 1 to the interval $[-2, 2]$ by the map $\theta \mapsto -2\cos(\theta) =: \alpha$ and

$$\frac{2}{2\pi}d\theta = \frac{1}{\pi}d(\arccos(-\alpha/2)) = \frac{1}{\pi}d(\pi - \arccos(\alpha/2)) = \frac{1}{\pi}\frac{d\alpha}{\sqrt{4-\alpha^2}}.$$

Let

$$S := \left(\bigcup_{i=1}^{18} \mathcal{R}_i\right) \cup \left(\bigcup_{i=1}^{5} \mathcal{L}_i\right)$$

and μ be the two-dimensional Lebesgue measure. Since both $S(\varepsilon)$ and S have piecewise smooth boundaries, we have

$$\mu(S \setminus S(\varepsilon)) \to 0$$

as $\varepsilon \to 0$. Thus the above measure converges to

$$\frac{1}{\pi^2}\int\int_{S} \frac{d\alpha_1 d\alpha_2}{\sqrt{(4-\alpha_1^2)(4-\alpha_2^2)}} \approx 0.458895$$

which proves our theorem. The explicit forms of the regions $\mathcal{R}_i$ $(i = 1, 2, \ldots, 18)$ and $\mathcal{L}_i$ $(i = 1, 2, \ldots, 5)$ are listed below, using the coordinate $(x, y) = (\alpha_1, \alpha_2)$ with $y < x$. Each collection of sentences with • is read in the similar way as before.

$\mathcal{R}_1$:
- $y \le -\dfrac{1}{x}$ if $x \in \left[1, \dfrac{1+\sqrt{5}}{2}\right]$;
- $y \le 1 - x$ if $x \in \left[\dfrac{1+\sqrt{5}}{2}, 2\right]$;
- $y > -x$ if $x \in [1, \sqrt{2}]$;
- $y > -\dfrac{2}{x}$ if $x \in [\sqrt{2}, 2]$.

$\mathcal{R}_2$:
- $y \le -1$ if $x \in [0, 1]$;
- $y \le -x$ if $x \in [1, \sqrt{2}]$;
- $y > -1 - x$ if $x \in \left[0, \dfrac{\sqrt{5}-1}{2}\right]$;
- $y > -\dfrac{2+x}{1+x}$ if $x \in \left[\dfrac{\sqrt{5}-1}{2}, \sqrt{2}\right]$.

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

$\mathcal{R}_3$:

- $y \leq \dfrac{-2}{x}$ if $x \in \left[\sqrt{2}, \dfrac{1+\sqrt{33}}{4}\right]$;
- $y \leq \dfrac{1}{2} - x$ if $x \in \left[\dfrac{1+\sqrt{33}}{4}, 2\right]$;
- $y > -x$ if $x \in [\sqrt{2}, \sqrt{3}]$;
- $y > \dfrac{-3}{x}$ if $x \in [\sqrt{3}, 2]$.

$\mathcal{R}_4$:

- $y \leq -\dfrac{3+2x}{2+2x}$ if $x \in \left[0, \dfrac{\sqrt{3}-1}{2}\right]$;
- $y \leq -1 - x$ if $x \in \left[\dfrac{\sqrt{3}-1}{2}, \dfrac{\sqrt{5}-1}{2}\right]$;
- $y > -\dfrac{3+2x}{2+x}$ if $x \in \left[0, \dfrac{\sqrt{5}-1}{2}\right]$.

$\mathcal{R}_5$:

- $y \leq -\dfrac{2+x}{1+x}$ if $x \in \left[\dfrac{\sqrt{5}-1}{2}, \sqrt{2}\right]$;
- $y \leq -x$ if $x \in [\sqrt{2}, \sqrt{3}]$;
- $y > -\dfrac{3+2x}{2+x}$ if $x \in \left[\dfrac{\sqrt{5}-1}{2}, \sqrt{3}\right]$.

$\mathcal{R}_6$:

- $y \leq -\dfrac{3+2x}{2+x}$ if $x \in \left[-1, \dfrac{\sqrt{5}-1}{2}\right]$;
- $y > -2 - x$ if $x \in \left[-1, \dfrac{\sqrt{2}-2}{2}\right]$;
- $y > -\dfrac{5+3x}{3+2x}$ if $x \in \left[\dfrac{\sqrt{2}-2}{2}, \dfrac{\sqrt{5}-1}{2}\right]$.

$\mathcal{R}_7$:

- $1 - x < y \leq \dfrac{2-x}{2+x}$ if $x \in \left[\dfrac{\sqrt{5}+1}{2}, 2\right)$.

$\mathcal{R}_8$:

- $y \leq -\dfrac{5+3x}{3+2x}$ if $x \in \left[\dfrac{\sqrt{2}-2}{2}, \dfrac{\sqrt{5}-1}{2}\right]$;
- $y \leq -1 - x$ if $x \in \left[\dfrac{\sqrt{5}-1}{2}, \dfrac{\sqrt{7}-1}{2}\right]$;
- $y > -\dfrac{7+4x}{4+2x}$ if $x \in \left[\dfrac{\sqrt{2}-2}{2}, \dfrac{\sqrt{7}-1}{2}\right]$.

$\mathcal{R}_9$:

- $-1 < y \leq -x$ if $x \in (0, 1]$.

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

$\mathcal{R}_{10}$:

- $y \le \frac{-3}{x}$ if $x \in \left[\sqrt{3}, \frac{1+\sqrt{193}}{8}\right]$;
- $y \le \frac{1}{4} - x$ if $x \in \left[\frac{1+\sqrt{193}}{8}, 2\right]$;
- $y > -x$ if $x \in [\sqrt{3}, \sqrt{\frac{7}{2}}]$;
- $y > \frac{-7}{2x}$ if $x \in [\sqrt{\frac{7}{2}}, 2]$.

$\mathcal{R}_{11}$:

- $y \le -\frac{5+2x}{2+2x}$ if $x \in \left[\frac{\sqrt{111}-3}{8}, \frac{\sqrt{33}-1}{4}\right]$;
- $y \le -\frac{3+2x}{2+x}$ if $x \in \left[\frac{\sqrt{33}-1}{4}, \sqrt{3}\right]$;
- $y > -\frac{4+3x}{3+x}$ if $x \in \left[\frac{\sqrt{111}-3}{8}, \frac{\sqrt{41}-1}{4}\right]$;
- $y > -\frac{3+x}{1+x}$ if $x \in \left[\frac{\sqrt{41}-1}{4}, \sqrt{3}\right]$.

$\mathcal{R}_{12}$:

- $y \le -\frac{4+3x}{3+x}$ if $x \in \left[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{19}-1}{3}\right]$;
- $y > -1 - x$ if $x \in \left[\frac{\sqrt{5}-1}{2}, \frac{\sqrt{7}-1}{2}\right]$;
- $y > -\frac{6+3x}{3+2x}$ if $x \in \left[\frac{\sqrt{7}-1}{2}, \frac{\sqrt{19}-1}{3}\right]$.

$\mathcal{R}_{13}$:

- $y \le -\frac{3+x}{1+x}$ if $x \in \left[\frac{\sqrt{41}-1}{4}, \sqrt{3}\right]$;
- $y \le -x$ if $x \in [\sqrt{3}, \sqrt{\frac{7}{2}}]$;
- $y > -\frac{7+4x}{4+2x}$ if $x \in \left[\frac{\sqrt{41}-1}{4}, \sqrt{\frac{7}{2}}\right]$.

$\mathcal{R}_{14}$:

- $-\frac{3+2x}{2+x} < y \le -1$ if $x \in [-1, 0]$.

$\mathcal{R}_{15}$:

- $\frac{1}{2} - x < y \le \frac{3-x}{1-x}$ if $x \in \left[\frac{\sqrt{41}+1}{4}, 2\right)$.

$\mathcal{R}_{16}$:

- $\frac{6-3x}{3-x} < y \le \frac{7-4x}{4-3x}$ if $x \in \left[\frac{11+\sqrt{61}}{10}, 2\right)$.

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

$\mathcal{R}_{17}$:

- $y \le -\dfrac{8+5x}{5+3x}$ if $x \in \left[-1, \dfrac{\sqrt{2}-3}{3}\right]$;
- $y \le -2-x$ if $x \in \left[\dfrac{\sqrt{2}-3}{3}, \dfrac{\sqrt{2}-2}{2}\right]$;
- $y > -\dfrac{6+3x}{3+x}$ if $x \in \left[-1, \dfrac{\sqrt{13}-7}{6}\right]$;
- $y > -\dfrac{5+3x}{3+2x}$ if $x \in \left[\dfrac{\sqrt{13}-7}{6}, \dfrac{\sqrt{2}-2}{2}\right]$.

$\mathcal{R}_{18}$:

- $y \le -2-x$ if $x \in \left[-1, \dfrac{-1}{2}\right]$;
- $y > -\dfrac{8+5x}{5+3x}$ if $x \in \left[-1, \dfrac{\sqrt{21}-11}{10}\right]$;
- $y > -\dfrac{7+5x}{5+4x}$ if $x \in \left[\dfrac{\sqrt{21}-11}{10}, \dfrac{-1}{2}\right]$.

$\mathcal{L}_1$:

- $-x < y \le 1$ if $x \in (0, 1]$.

$\mathcal{L}_2$:

- $0 < y \le 2-x$ if $x \in (1, 2)$.

$\mathcal{L}_3$:

- $-\dfrac{1}{x} < y \le 1-x$ if $x \in \left(1, \dfrac{1+\sqrt{5}}{2}\right]$.

$\mathcal{L}_4$:

- $y \le 1$ if $x \in [1, 2]$;
- $y > 2-x$ if $x \in \left[1, \dfrac{3+\sqrt{2}}{3}\right]$;
- $y > \dfrac{6-4x}{4-3x}$ if $x \in \left[\dfrac{3+\sqrt{2}}{3}, 2\right]$.

$\mathcal{L}_5$:

- $y \le 0$ if $x \in [1, 2]$;
- $y > 1-x$ if $x \in \left[1, \dfrac{1+\sqrt{5}}{2}\right]$;
- $y > \dfrac{2-x}{1-x}$ if $x \in \left[\dfrac{1+\sqrt{5}}{2}, 2\right]$.

Remark 7. By examining the beta expansion of 23 899 Salem numbers of degree 6 and trace at most 19, there are 18 250 (approximately 76%) Salem numbers that satisfy

$$-2 < \alpha_1 < 0 < \alpha_2 < 2.$$

They are Parry numbers with relatively small orbit size ($\max(m, p) < 1000$).

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

A heavy computational effort may be required to substantially improve Theorem 2 (or Proposition B.2 in Appendix B. For example, for $-r < \alpha_1 < 0 < \alpha_2 < r < 2 < \gamma$ with $r > 1$ quite close to 1, the orbit size starts taking many different values. For instance, the following polynomials:

$x^6 - 7x^5 - 3x^4 - 11x^3 - 3x^2 - 7x + 1$, $x^6 - 9x^5 - x^4 - 11x^3 - x^2 - 9x + 1$ and $x^6 - 8x^5 + 10x^4 - 15x^3 + 10x^2 - 8x + 1$ satisfy respectively: $\alpha_1 \approx -1.08$, $\alpha_1 \approx -1.05$ and $\alpha_2 \approx 1.1$, but (m, p) equals to $(6, 23)$, $(6, 35)$ and $(1, 119)$, respectively.

6. *Four-dimensional discretized rotation*

Substituting the variables of the algorithm (9) in §5 by

$$\begin{pmatrix} t_n(1) \\ t_n(2) \\ t_n(3) \\ t_n(4) \end{pmatrix} = \begin{pmatrix} 1 & c_1 & c_2 & c_1 \\ 0 & 1 & c_1 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{n-3} \\ k_{n-2} \\ k_{n-1} \\ k_n \end{pmatrix},$$

we obtain an integer sequence $(k_n)_{n\geq -4}$ which satisfies

$$0 < k_{n+4} + c_1 k_{n+3} + c_2 k_{n+2} + c_1 k_{n+1} + k_n \leq 1, \tag{10}$$

where $c_1 = \alpha_1 + \alpha_2$ and $c_2 = \alpha_1\alpha_2 + 2$. Here $\alpha_i \in (-2, 2)$ are arbitrary chosen constants. The bijective map T on $\mathbb{Z}^4$:

$$(k_n, k_{n+1}, k_{n+2}, k_{n+3}) \mapsto (k_{n+1}, k_{n+2}, k_{n+3}, -\lceil c_1 k_{n+3} + c_2 k_{n+2} + c_1 k_{n+1} + k_n - 1\rceil)$$

is conjugate to equation (9). By bijectivity, the orbit is purely periodic if and only if it is eventually periodic. Moreover, the periodicity is equivalent to the boundedness of the orbit. We are interested in the recurrence of the orbit of $(k_{-4}, k_{-3}, k_{-2}, k_{-1}) = (0, 0, 0, 0)$ by T. This map approximates a linear map Φ defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -c_1 & -c_2 & -c_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

for $(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$. The map Φ has four eigenvalues $\exp(\pm\theta_i\sqrt{-1})$ with $\alpha_i = -2\cos(\theta_i)$ for $i = 1, 2$. Therefore, the map $T : \mathbb{Z}^4 \to \mathbb{Z}^4$ is understood as a discretized version of rotation. A simpler case: the discretized rotation in $\mathbb{Z}^2$ is extensively studied in the literature. It is defined similarly by a recurrence

$$0 \leq a_{n+2} + \lambda a_{n+1} + a_n < 1, \quad a_n \in \mathbb{Z}, \tag{11}$$

with a fixed $\lambda \in (-2, 2)$. A notorious conjecture states that any sequence produced by this recursion is periodic for any initial vector $(a_0, a_1) \in \mathbb{Z}^2$. The validity is known only for 11 values of λ, see [**5**, **19**, **21**].

Note that if $\alpha_1 = \alpha_2$, then there is an unbounded real sequence $(b_n)_{n\in\mathbb{N}}$ which satisfies

$$b_{n+4} + c_1 b_{n+3} + c_2 b_{n+2} + c_1 b_{n+1} + b_n = 0$$

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

because of the shape of general terms of this recurrence. In particular, if $\alpha_1 = \alpha_2 \in \{-1, 0, 1\}$, then $\{T^n(0, 0, 0, 0)| \ n \in \mathbb{N}\}$ is unbounded. Thus we can not expect the boundedness of the orbits of T when $\alpha_1 = \alpha_2$. (However, there are points with $\alpha_1 = \alpha_2$ where the T-orbits of $(0, 0, 0, 0)$ are periodic, e.g., (α, α) with $\alpha \in (1 - \sqrt{3}, -2/3] \cup [-1/4, -2/9]$.) Excluding these cases, a natural generalization of the above conjecture for equation (11) would be the following conjecture.

Conjecture 8. If $c_2 > 2|c_1| - 2$ and $c_2 - 2 < c_1^2/4 < 4$, then for any initial vector $(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4$, the sequence satisfying equation (10) is periodic.

This is because $c_2 > 2|c_1| - 2$ and $c_2 - 2 < c_1^2/4 < 4$ is equivalent to $\alpha_i \in (-2, 2)$ for $i = 1, 2$ and $\alpha_1 \neq \alpha_2$.

We are pessimistic about its validity, due to the existence of very large orbits. However, even if Conjecture 8 may not hold, it could be true for almost all cases.

Let us restrict ourselves to the orbit of the origin. Since periodic orbits are often dominant in zero entropy systems, unbounded orbits may not give a contribution of positive measure in §5 and period cells would exhaust the total square $(-2, 2)^2$ in measure. We propose a weaker conjecture.

Conjecture 9. Letting $(k_{-4}, k_{-3}, k_{-2}, k_{-1}) = (0, 0, 0, 0)$, the sequence satisfying equation (10) is periodic for all most all $(\alpha_1, \alpha_2) \in (-2, 2)^2$ in measure.

Here our measure is equivalent to the two-dimensional Lebesgue measure. See Figure 2 for period cells. Black dots are the points where the orbit of the origin might be unbounded.

Note that once we find a period of T starting from the origin, Lemmas 3 and 4 give us a (possibly degenerated) period cell. This fact is clear when g_i does not visit 1 and we can apply Lemma 3 with

$$u = \frac{1}{2} \min_i g_i, \quad v = \frac{1}{2}(1 + \max_i g_i).$$

For Lemma 4, it looks like we have additional constraints. We shall show that this is not the case. Taking the period $p \in \mathbb{N}$ with $T^p((0, 0, 0, 0)) = (0, 0, 0, 0) = (k_{-4}, k_{-3}, k_{-2}, k_{-1})$, we have

$$g_n = t_{n-1}(1) + k_n = k_{n-4} + c_1 k_{n-3} + c_2 k_{n-2} + c_1 k_{n-1} + k_n$$

for $n = 0, \ldots, p$ and

$$k_0 = 1, \quad k_1 = -\lceil c_1 - 1 \rceil.$$

If $g_i = 1$ occurs as a polynomial of $\mathbb{Z}[\alpha_1, \alpha_2]$ with $1 \le i \le p - 1$, then $k_{i-2} = k_{i-3} + k_{i-1} = 0$ and $k_{i-4} + k_i = 1$. From $g_{i\pm 1} \in (0, 1]$, we have

$$0 < g_{i-1} = k_{i-5} + c_1 k_{i-4} + (c_2 - 1) k_{i-3} \le 1$$

and

$$0 < g_{i+1} = (1 - c_2) k_{i-3} + c_1 (1 - k_{i-4}) + k_{i+1} \le 1.$$

These imply

$$k_{i-5} = -\lceil c_1 k_{i-4} + (c_2 - 1) k_{i-3} - 1 \rceil$$

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

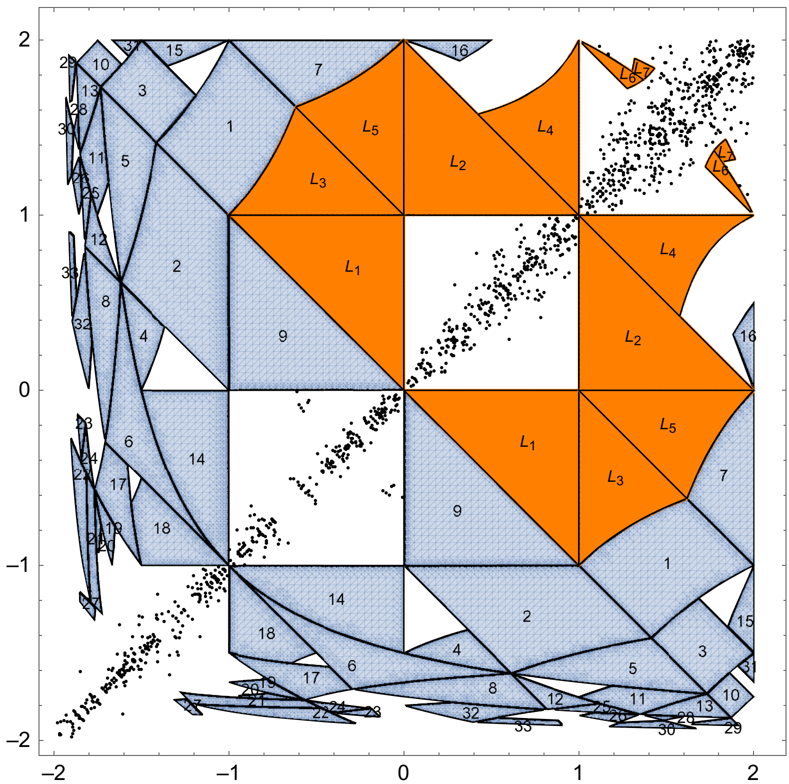

FIGURE 2. Period cells occupy more than half in measure. The orbit of the origin does not form a period until 50 000 iterations at the black dots.

and

$$k_{i+1} = -\lceil (1 - c_2)k_{i-3} + c_1(1 - k_{i-4}) - 1\rceil.$$

We see

$$k_1 \le k_{i-5} + k_{i+1}$$

from an easy fact

$$\lceil x\rceil + \lceil y\rceil - \lceil x + y + 1\rceil \in \{-1, 0\}$$

for any $x, y \in \mathbb{R}$. Therefore, when $g_i = 1$, we have

$$g_{i-1} + g_{i+1} = k_{i-5} + c_1 + k_{i+1} \ge c_1 + k_1 = g_1.$$

Since $g_1 = g_{i+1} + g_{i-1}$ implies $g_1 > g_{i+1}$, we can apply Lemma 4 in any case with

$$u = \frac{1}{2} \min\bigg(\min_i g_i, \min_{\substack{g_i=1 \\ g_1 = g_{i-1}+g_{i+1}}} (g_1 - g_{i+1})\bigg), \quad v = \frac{1}{2}\Big(1 + \max_{g_i \neq 1} g_i\Big).$$

Therefore, we can apply Lemma 4 for every period starting from the origin. Since we expect such period cells to cover $(-2, 2)^2$ in measure, as in Conjecture 9, for any $\varepsilon > 0$,

https://doi.org/10.1017/etds.2022.66 Published online by Cambridge University Press

we will find a finite union of period cells whose measure is not less than $1 - \varepsilon$. Following the same proof as Theorem 2, we arrive at a plausible

Conjecture 10. $c(6) = 1$.

For general Salem numbers β, we numerically observe many (m, p)-periodic $d_\beta(1)$ with $m > 1$. Interestingly, we find no role of (m, p)-periods with $m > 1$ in the above discussion. Every period of T gives rise to $(1, p)$-periodic $d_{\beta^m}(1)$ and other orbits of T are aperiodic if they exist. Of course this does not cause any contradiction, since we are studying sufficiently large β with respect to the location of the conjugates.

Acknowledgments. We would like to thank the anonymous referee for the careful reading of the manuscript. This research was partially supported by JSPS grants (20K03528, 17K05159, 21H00989).

A. *Appendix*

Additional polynomials to improve $c(6)$:

$$
\begin{aligned}
R_{19} &= 1 + 3x + 5x^2 + 5x^3 + 3x^4 + x^5,\\
R_{20} &= 1 + 3x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6,\\
R_{21} &= 1 + 3x + 5x^2 + 6x^3 + 6x^4 + 5x^5 + 3x^6 + x^7,\\
R_{22} &= 1 + 3x + 5x^2 + 6x^3 + 6x^4 + 6x^5 + 6x^6 + 5x^7 + 3x^8 + x^9,\\
R_{23} &= 1 + 3x + 4x^2 + 4x^3 + 4x^4 + 4x^5 + 3x^6 + x^7,\\
R_{24} &= 1 + 3x + 4x^2 + 3x^3 + 2x^4 + 3x^5 + 4x^6 + 3x^7 + x^8,\\
R_{25} &= 1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6 + x^7 + x^8 + x^9,\\
R_{26} &= 1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + 3x^6 + 2x^7 + 2x^8 + 2x^9 + x^{10} + x^{11} + x^{12},\\
R_{27} &= 1 + 4x + 8x^2 + 11x^3 + 11x^4 + 8x^5 + 4x^6 + x^7,\\
R_{28} &= 1 + x + 2x^2 + 2x^3 + 2x^4 + 3x^5 + 2x^6 + 2x^7 + 2x^8 + x^9 + x^{10},\\
R_{29} &= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 3x^6 + 3x^7 + 2x^8 + 2x^9 + x^{10} + x^{11},\\
R_{30} &= 1 + x + 2x^2 + 3x^3 + 3x^4 + 4x^5 + 4x^6 + 4x^7 + 4x^8 + 3x^9 + 3x^{10}\\
&\quad + 2x^{11} + x^{12} + x^{13},\\
R_{31} &= 1 + 2x^2 - x^3 + 2x^4 - 2x^5 + 2x^6 - 2x^7 + 2x^8 - x^9 + 2x^{10} + x^{12},\\
R_{32} &= 1 + 2x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 2x^6 + 2x^7 + 2x^8 + x^9,\\
R_{33} &= 1 + 2x + 2x^2 + 3x^3 + 4x^4 + 4x^5 + 4x^6 + 4x^7 + 3x^8 + 2x^9 + 2x^{10} + x^{11},\\
L_6 &= 1 - 3x + 5x^2 - 5x^3 + 3x^4 - 2x^6 + 2x^7 - 2x^9 + 3x^{10} - 2x^{11} + 2x^{13} - 2x^{14}\\
&\quad + 3x^{16} - 5x^{17} + 5x^{18} - 3x^{19} + x^{20},\\
L_7 &= 1 - 3x + 6x^2 - 8x^3 + 8x^4 - 5x^5 + 5x^7 - 7x^8 + 5x^9 - 5x^{11} + 8x^{12} - 8x^{13}\\
&\quad + 6x^{14} - 3x^{15} + x^{16}.
\end{aligned}
$$

One can check that these 40 regions are mutually disjoint by symbolic computation. This gives the estimate $c(6) \geq 0.505254$, see Figure 2.

In the list of the regions in §5, all the inequalities defining the period cells in $y < x$ are of the form $A < y$ or $y \leq B$ with some A and B. This is no longer true for the polynomials L_6 and L_7. The cell for L_6 is an open set, and the common boundary of the two cells belongs to the one for L_7. Thus, an inequality of the form $A \leq y$ is required for the cell for L_7 in $y < x$. The point $(\alpha_1, \alpha_2) = ((11+\sqrt{2})/7, (11-\sqrt{2})/2)$ is the end point of the common boundary but belongs to neither of them. Applying our algorithm at this point, we obtain a polynomial of degree 412 and the corresponding cell in $y < x$ degenerates to a singleton (α_1, α_2).

B. *Appendix*

Our strategy in this paper is to study sufficiently large β. However, for regions $\mathcal{R}_9$ and $\mathcal{L}_1$, we can remove the adjective 'sufficiently large'. Indeed, at the beginning of this study, we found Proposition B.2 below and then generalized it to our current setting. In particular, the formulation of Lemma 4 is inspired by the second case of Proposition B.2. Let P be the minimum polynomial of a sextic Salem number:

$$P(x) = x^6 - ax^5 - bx^4 - cx^3 - bx^2 - ax + 1. \tag{B.1}$$

We denote by Q its trace polynomial:

$$Q(y) = y^3 - ay^2 - (b+3)y - (c-2a) = (y-\gamma)(y-\alpha_1)(y-\alpha_2). \tag{B.2}$$

We say that β is *well-posed* if its trace polynomial $Q(y)$ has three roots $\gamma, \alpha_1, \alpha_2$ such that

$$-1 < \alpha_1 < 0 < \alpha_2 < 1 < 2 < \gamma.$$

Then we have the following lemma.

LEMMA B.1. *A real number $\beta > 1$ is a well-posed Salem number of degree 6 if and only if β is the dominant root of the polynomial in $\mathbb{Z}[x]$ of the form given by equation (B.1) satisfying the following conditions:*

(i) $2 - 2b < 2a + c$;
(ii) $c < 2a$;
(iii) $|b+2| < c - a$.

Proof. Since Q is cubic, the well-posedness is equivalent to $Q(-1) < 0$, $Q(0) > 0$, $Q(1) < 0$, $Q(2) < 0$. We see (i) $\Leftrightarrow$ $Q(2) < 0$, (ii) $\Leftrightarrow$ $Q(0) > 0$, and $Q(\pm 1) < 0 \Leftrightarrow$ (iii). □

Classifying into $b < -1$ and $b \geq -1$, we obtain the following Proposition, which gives a partial response to Problem 1 in [29].

PROPOSITION B.2. *Let β be a well-posed Salem number of minimum polynomial P in equation (B.1). Then β is Parry number and we have:*

- *if $2 \leq -b < c - a + 2$, then*

$$d_\beta(1) = a - 1(a+b+1, c-a+b+1, c-a-1, 2a-c-1, \\ 2a-c-1, c-a-1, c-a+b+1, a+b+1, a-2, a-2)^\infty; \tag{B.3}$$

- *if* $a - c + 2 < -b \leq 1$, *then*

$$d_\beta(1) = a(b+1, c-a-1, a-1, 2a+b-c+1, c-a-1, \\ c-a-1, 2a+b-c+1, a-1, c-a-1, b+1, a-1, a-1)^\infty. \quad \text{(B.4)}$$

Proof. Lemma B.1(iii) implies $c - a \geq 1$ and Lemma B.1(ii) gives $a \geq 2$ and $c \geq 3$. Starting from the representation of zero in base β:

$$-1, a, b, c, b, a, -1,$$

we can construct another representation as

-1	a	b	c	b	a	-1					
	-1	a	b	c	b	a	-1				
		1	$-a$	$-b$	$-c$	$-b$	$-a$	1			
			1	$-a$	$-b$	$-c$	$-b$	$-a$	1		
				-1	a	b	c	b	a	-1	
					-1	a	b	c	b	a	-1
-1	$a-1$	$a+b+1$	$c-a+b+1$	$c-a-1$	$2a-c-1$	$2a-c-1$	$c-a-1$	$c-a+b+1$	$a+b+1$	$a-1$	-1

Performing recursive shifted addition of this new representation in base β, we obtain the infinite representation in equation (B.3). We can check the condition in eqation (1) from Lemma B.1. Similarly, we have

-1	a	b	c	b	a	-1							
		1	$-a$	$-b$	$-c$	$-b$	$-a$	1					
			-1	a	b	c	b	a	-1				
				-1	a	b	c	b	a	-1			
					1	$-a$	$-b$	$-c$	$-b$	$-a$	1		
							-1	a	b	c	b	a	-1
-1	a	$b+1$	$c-a-1$	$a-1$	$2a+b-c+1$	$c-a-1$	$c-a-1$	$2a+b-c+1$	$a-1$	$c-a-1$	$b+1$	a	-1

By recursive shifted addition, we obtain the representation in equation (B.4). The condition in equation (1) follows from Lemma B.1. □

REFERENCES

[1] S. Akiyama. Pisot numbers and greedy algorithm. *Number Theory (Eger, 1996)*. Eds. K. Győry, A. Pethő and V. Sós. de Gruyter, Berlin, 1998, pp. 9–21.

[2] S. Akiyama. Pisot number system and its dual tiling. *Physics and Theoretical Computer Science (Amsterdam) (NATO Security through Science Series - D: Information and Communication Security, 7)*. Eds. J.-P. Gazeau, J. Nešetřil and B. Rovan. IOS Press, Amsterdam, 2007, pp. 133–154.

[3] S. Akiyama. A family of non-sofic beta expansions. *Ergod. Th. & Dynam. Sys.* **36**(2) (2016), 343–354.

[4] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. M. Thuswaldner. Generalized radix representations and dynamical systems I. *Acta Math. Hungar.* **108**(3) (2005), 207–238.

[5] S. Akiyama, H. Brunotte, A. Pethő and W. Steiner. Periodicity of certain piecewise affine planar maps. *Tsukuba J. Math.* **32**(1) (2008), 1–55.

[6] S. Akiyama and D. Y. Kwon. Constructions of Pisot and Salem numbers with flat palindromes. *Monatsh. Math.* **155**(3–4) (2008), 265–275.

[7] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse and J.-P. Schreiber. *Pisot and Salem Numbers*. Birkhäuser, Basel. 1992.

[8] A. Bertrand. Développement en base de Pisot et répartition modulo 1. *C. R. Acad. Sci. Paris Sér. I Math.* **285** (1977), 419–421.
[9] F. Blanchard. β-expansion and symbolic dynamics. *Theoret. Comput. Sci.* **65** (1989), 131–141.
[10] A. Boyarsky and P. Góra. *Laws of Chaos (Probability and Its Applications)*. Birkhäuser Boston, Inc., Boston, MA, 1997.
[11] D. W. Boyd. Salem numbers of degree four have periodic expansions. *Number Theory*. Walter de Gruyter, Berlin, 1989, pp. 57–64.
[12] D. W. Boyd. On the beta expansion for Salem numbers of degree 6. *Math. Comp.* **65** (1996), 861–875.
[13] A. Dubickas and J. Jankauskas. Linear relations with conjugates of a Salem number. *J. Théor. Nombres Bordeaux* **32**(1) (2020), 179–191.
[14] C. Frougny. Numeration systems. *Algebraic Combinatorics on Words*. Ed. M. Lothaire. Cambridge University Press, Cambridge, 1970, Ch. 7.
[15] H. Hichri. Beta expansion for some particular sequences of Salem numbers. *Int. J. Number Theory* **10**(08) (2014), 2135–2149.
[16] H. Hichri. On the beta expansion of Salem numbers of degree 8. *LMS J. Comput. Math.* **17**(01) (2014), 289–301.
[17] H. Hichri. Beta expansion of Salem numbers approaching Pisot numbers with the finiteness property. *Acta Arith.* **168** (2015), 107–119.
[18] S. Ito and Y. Takahashi. Markov subshifts and realization of β-expansions. *J. Math. Soc. Japan* **26**(1) (1974), 33–55.
[19] K. L. Kouptsov, J. H. Lowenstein and F. Vivaldi. Quadratic rational rotations of the torus and dual lattice maps. *Nonlinearity* **15**(6) (2002), 1795–1842.
[20] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences*. Wiley, New York, 1974.
[21] J. Lowenstein, S. Hatjispyros and F. Vivaldi. Quasi-periodicity, global stability and scaling in a model of Hamiltonian round-off. *Chaos* **7**(1) (1997), 49–66.
[22] J. F. McKee and C. J. Smyth. There are Salem numbers of every trace. *Bull. Lond. Math. Soc.* **37**(1) (2005), 25–36.
[23] W. Parry. On the beta-expansions of real numbers. *Acta Math. Acad. Sci. Hungary* **11** (1960), 401–416.
[24] M. Pollicott and M. Yuri. *Dynamical Systems and Ergodic Theory (London Mathematical Society Student Texts, 40)*. Cambridge University Press, Cambridge, 1998.
[25] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Hungar.* **8** (1957), 477–493.
[26] K. Schmidt. On periodic expansions of Pisot numbers and Salem numbers. *Bull. Lond. Math. Soc.* **12** (1980), 269–278.
[27] C. J. Smyth. Salem numbers of negative trace. *Math. Comp.* **69** (2000), 827–838.
[28] C. J. Smyth. Seventy years of Salem numbers. *Bull. Lond. Math. Soc.* **47**(3) (2015), 379–395.
[29] J.-L. Verger-Gaugry. On the dichotomy of Perron numbers and beta-conjugates. *Monatsh. Math.* **155**(3–4) (2008), 277–299.
[30] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer-Verlag, New York, 1982.