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ON THE FRACTIONAL PARTS OF A POLYNOMIAL

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1. Introduction. Heilbronn [**6**] proved that for any $\epsilon > 0$ there exists $C(\epsilon)$ such that for any real θ and $N \geqq 1$ there is an integer x satisfying

$$(1) \qquad 1 \leqq x \leqq N \quad \text{and} \quad ||\theta x^2|| < C(\epsilon)N^{-1/2+\epsilon},$$

where $||\alpha||$ denotes the difference between α and the nearest integer, taken positively. Danicic [**2**] obtained an analogous result for the fractional parts of θx^k and in 1967 Davenport [**4**] generalized Heilbronn's result to polynomials of degree k with no constant term. The last condition is essential, for if there is a constant term then no analogous result can hold (see Koksma [**7**, Kap. 6 Satz 10]).

More recently, Ming-Chit Liu [**8**] proved that for any real θ and any positive integer N there is an integer x satisfying

$$(2) \qquad 1 \leqq x \leqq N \quad \text{and} \quad ||\theta x^2|| < CN^{-1/2+\epsilon(N)},$$

where C is an absolute constant and $\epsilon(N) = 1/\log \log N$. The purpose of this note is to prove that the results of Danicic and Davenport may be improved to give results analogous to Liu's.

THEOREM 1. *Let k be an integer, $k \geqq 2$, and put $K = 2^{k-1}$. For every real θ and every positive integer N, there is an integer x satisfying*

$$(3) \qquad 1 \leqq x \leqq N \quad \text{and} \quad ||\theta x^k|| < C_1 N^{-1/K+\epsilon(N)},$$

where $C_1 = C_1(k)$ depends only on k and $\epsilon(N) = 1/\log \log N$.

THEOREM 2. *Let k be an integer, $k \geqq 2$, and put $R = 2^k - 1$. For every positive integer N and every real polynomial $f(x)$, with no constant term, of degree k, there is an integer x satisfying*

$$(4) \qquad 1 \leqq x \leqq N \quad \text{and} \quad ||f(x)|| < C_2 N^{-1/R+\epsilon(N)},$$

where $C_2 = C_2(k)$ depends only on k and $\epsilon(N) = 1/\log \log N$.

For large values of k these results can be improved by using Vinogradov's estimates for trigonometric sums, in place of Weyl's (see [**1**]).

2. Notation and preliminary lemmas. By $F \ll G$ we mean that $|F| < CG$ where C depends at most on k. We write $e(z)$ for exp $(2\pi iz)$, K for 2^{k-1}, R for $2^k - 1$ and $\epsilon(N)$ for $1/\log \log N$. We may suppose that $N > N_0(k)$.

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LEMMA 1. *Let Δ satisfy $0 < \Delta < \frac{1}{2}$ and let a be a positive integer. Then there exists a function $\psi(z)$, periodic with period 1, which satisfies*

(5) $\quad \psi(z) = 0 \quad \text{for} \quad ||z|| \geqq \Delta,$

and

(6) $\quad \psi(z) = \sum_{v=-\infty}^{\infty} a_v e(vz)$

where the coefficients a_v are real numbers, $a_0 = \Delta$, $a_{-v} = a_v$ and

(7) $\quad |a_v| \ll \min \left(\Delta, \left(\frac{a}{\pi}\right)^a \Delta^{-a}|v|^{-a-1}\right).$

This is a particular case of Lemma 12 of Chapter 1 of Vinogradov [**9**].

LEMMA 2. *Let $d(n)$ denote the number of divisors of the positive integer n. For any $\epsilon > 0$ we have*

(8) $\quad d(n) \leqq 2^{(1+\epsilon)\log n/\log\log n}$

for all $n > n_0(\epsilon)$.

This is Theorem 317 of Hardy and Wright [**5**].

We apply Lemma 2 with ϵ chosen so small that $2^{1+\epsilon} < e^{3/4}$. Then for some n_0 we have

(9) $\quad d(n) < n^{(3/4)\epsilon(n)}$

for all $n \geqq n_0$.

LEMMA 3 (Weyl). *Let $f(x)$ be a real polynomial of degree k with leading coefficient θ:*

$$f(x) = \theta x^k + \theta_1 x^{k-1} + \dots .$$

Let B be a real number and put

$$S = \sum_{B<x<B+N} e(f(x)).$$

Then

(10) $\quad |S|^K \ll N^{K-1} + N^{K-k+(3/4)(k-1)\epsilon(N)} \sum_{m=1}^{L} \min(N, ||m\theta||^{-1}),$

where $L = k!N^{k-1}$.

This may be proved in the same way as the corresponding formula on p. 13 of Davenport [**3**] since for $m = 1, \dots, L$ we have

$$d(m) \ll L^{(3/4)\epsilon(L)} \ll N^{(3/4)(k-1)\epsilon(N)}.$$

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LEMMA 4 (Dirichlet). *Let θ be a real number and $Q \geqq 1$. Then there exist integers a, q with*

$$(11) \quad 1 \leqq q \leqq Q,\ (a, q) = 1 \quad \text{and} \quad |\theta - a/q| \leqq q^{-1}Q^{-1}.$$

See, for example, Theorem 185 of Hardy and Wright [**5**].

3. Preliminaries to Theorems 1 and 2. Let

$$(12) \quad f(x) = \theta x^k + \theta_1 x^{k-1} + \ldots + \theta_{k-1} x,$$

which contains the possibility that $f(x) = \theta x^k$. Suppose that

$$(13) \quad ||f(x)|| \geqq M^{-1} \quad \text{for} \quad 1 \leqq x \leqq N,$$

then we may also suppose that

$$(14) \quad M \leqq N^{1/K - \epsilon(N)}$$

for otherwise there is nothing to prove. We take $\Delta = M^{-1}$ in Lemma 1, then

$$0 = \sum_{x=1}^{N} \psi(f(x)) = \sum_{x=1}^{N} \sum_{v=-\infty}^{\infty} a_v e(vf(x)) = \Delta N + \sum_{v \neq 0} a_v S(v)$$

where

$$(15) \quad S(v) = \sum_{x=1}^{N} e(vf(x)).$$

Then $S(-v) = \overline{S(v)}$ so taking $M_1 = MN^{\epsilon(N)/100}$ we have

$$\Delta N \ll \sum_{0<|v|\leqq M_1} |a_v S(v)| + \sum_{|v|>M_1} |a_v S(v)| \ll \Delta \sum_{v=1}^{M_1} |S(v)| + N \sum_{|v|>M_1} |a_v|$$

and, from Lemma 1,

$$\sum_{|v|>M_1} |a_v| \ll \left(\frac{a}{\pi}\right)^a \Delta^{-a} \sum_{|v|>M_1} v^{-a-1} \ll a^a \Delta^{-a} M_1^{-a}.$$

Therefore

$$N(1 - a^a \Delta^{-a-1} M_1^{-a}) \ll \sum_{v=1}^{M_1} |S(v)|.$$

We take $a = [100/\epsilon(N)] = [100 \log \log N]$, then

$$a^a \Delta^{-a-1} M_1^{-a} \ll (100 \log \log N)^{100 \log \log N} M^{a+1} M^{-a} N^{-a\epsilon(N)/100} = o(1) \quad \text{as } N \to \infty.$$

Therefore $N \ll \sum_{v=1}^{M_1} |S(v)|$ so, by Hölder's inequality,

$$(16) \quad M_1^{1-K} N^K \ll \sum_{v=1}^{M_1} |S(v)|^K.$$

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Applying Weyl's estimate we have

$$M_1^{1-K}N^K \ll \sum_{v=1}^{M_1}\left(N^{K-1} + N^{K-k+(3/4)(k-1)\epsilon(N)}\sum_{m=1}^{L}\min(N, ||mv\theta||^{-1})\right)$$

$$\ll M_1N^{K-1} + N^{K-k+(3/4)(k-1)\epsilon(N)}\sum_{v=1}^{M_1}\sum_{m=1}^{L}\min(N, ||mv\theta||^{-1})$$

$$\ll M_1N^{K-1} + N^{K-k+(3/2)(k-1)\epsilon(N)+\epsilon(N)/K}\sum_{h=1}^{H}\min(N, ||h\theta||^{-1})$$

where $H = M_1L$ and we have put $h = mv$, since the number of representations of h in the form mv is

$$d(h) \ll H^{(3/4)\epsilon(H)} \ll (N^{k-1+1/K})^{(3/4)\epsilon(N)}.$$

From (14) we have $M_1N^{K-1} = o(M_1^{1-K}N^K)$ so, putting

$$\eta(N) = (3/2)(k-1)\epsilon(N) + \epsilon(N)/K,$$

we have

$$(17)\quad M_1^{1-K}N^{k-\eta(N)} \ll \sum_{h=1}^{H}\min(N, ||h\theta||^{-1}).$$

Let a/q be any rational number, in its lowest terms, for which

$$(18)\quad |\theta - a/q| \leqq q^{-2}.$$

We divide the sum on the right-hand side of (17) into blocks of q terms and estimate the sum of each block in the usual way (see Lemma 1 of Davenport [**3**]) to give

$$(19)\quad M_1^{1-K}N^{k-\eta(N)} \ll (q^{-1}H + 1)(N + q\log q).$$

4. Proof of Theorem 1. Now $f(x) = \theta x^k$ and we may suppose $k \geqq 3$, since Liu [**8**] has proved the result in the case $k = 2$. We take

$$M = N^{1/K-\epsilon(N)} \text{ so that } M_1 = N^{1/K-(99/100)\epsilon(N)}.$$

We choose

$$(20)\quad q \leqq M_1^{1-K}N^{k-\tau(N)},$$

where $\tau(N) = \eta(N) + (1/K)\epsilon(N)$. Then

$$q\log q \ll M_1^{1-K}N^{k-\tau(N)}\log N$$
$$= o(M_1^{1-K}N^{k-\eta(N)})$$
$$N = o(M_1^{1-K}N^{k-\eta(N)})$$

and

$$H\log q \ll M_1N^{k-1}\log N$$
$$\ll M_1^{1-K}N^{1-K(99/100)\epsilon(N)}N^{k-1}\log N$$
$$= o(M_1^{1-K}N^{k-\eta(N)})$$

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since

(21) $(99/100)K\epsilon(N) > \eta(N) + \epsilon(N)/4$ for $k \geqq 3$.

It now follows from (19) that

$$M_1^{1-K}N^{k-\eta(N)} \ll q^{-1}HN \ll q^{-1}M_1N^k$$

so that

$$q \ll M_1^K N^{\eta(N)} = N^{1-(99/100)K\epsilon(N)+\eta(N)} = o(N).$$

By Lemma 4, there exists a rational number a/q such that

(23) $q \leqq M_1^{1-K}N^{k-\tau(N)}$

and

(24) $|\theta - a/q| \leqq q^{-1}M_1^{K-1}N^{\tau(N)-k}$.

This q must also satisfy (22) and

(25) $$||\theta q^k|| \leqq |q^k\theta - aq^{k-1}| \leqq q^{k-1}M_1^{K-1}N^{\tau(N)-k}$$
$$\leqq N^{k-1}N^{1-1/K-(K-1)(99/100)\epsilon(N)}N^{\tau(N)-k} \leqq N^{-1/K+\epsilon(N)+\tau(N)-(99/100)K\epsilon(N)}$$
$$\leqq N^{-1/K+\epsilon(N)}$$

since for $k \geqq 3$, $(99/100)K\epsilon(N) \geqq \tau(N) = \eta(N) + (1/K)\epsilon(N)$, and this completes the proof of Theorem 1 since $x = q$ satisfies the theorem.

5. Proof of Theorem 2. This is proved by induction on k, we begin with the case $k = 2$. Let

(26) $f(x) = \theta x^2 + \theta_1 x$ and $M = N^{1/3-\epsilon(N)}$.

We choose an integer q satisfying

(27) $1 \leqq q \leqq M_1^{-1}N^{2-(5/2)\epsilon(N)}$, $||q\theta|| \leqq M_1N^{-2+(5/2)\epsilon(N)}$.

Then the terms N, $q \log q$ and $H \log q$ in (19) are negligible, so that

$$M_1^{-1}N^{2-\eta(N)} \ll q^{-1}HN \ll q^{-1}M_1N^2.$$

Hence

(28) $q \ll M_1^2N^{\eta(N)} = N^{2/3+(1/50)\epsilon(N)}$.

For any positive integer T we can choose an integer t satisfying

(29) $1 \leqq t \leqq T$ and $||\theta_1 qt|| \leqq T^{-1}$.

Taking $x = qt$ we have

$$||\theta x^2 + \theta_1 x|| = ||\theta q^2t^2 + \theta_1 qt|| \leqq qt^2||\theta q|| + ||\theta_1 qt||$$
$$\ll T^2N^{2/3+(1/50)\epsilon(N)}M_1N^{-2+(5/2)\epsilon(N)} + T^{-1}$$
$$\ll T^2N^{-1+(153/100)\epsilon(N)} + T^{-1}.$$

Taking $T = N^{1/3-\epsilon(N)/3}$ we have

(30) $||\theta x^2 + \theta_1 x|| \ll N^{-1/3+(259/300)\epsilon(N)} + N^{-1/3+\epsilon(N)/3} \ll N^{-1/3+\epsilon(N)}$

and

(31) $1 \leqq x = qt \ll N^{(2/3)+\epsilon(N)/50}N^{1/3-\epsilon(N)/3} = o(N),$

which completes the proof in the case $k = 2$.

For $k > 2$ let

(32) $f(x) = \theta x^k + \theta_1 x^{k-1} + \ldots + \theta_{k-1}x$ and $M = N^{1/R-\epsilon(N)}$.

We choose an integer q satisfying

(33) $1 \leqq q \leqq M_1{}^{1-K}N^{k-\tau(N)}, \ ||q\theta|| < M_1{}^{K-1}N^{\tau(N)-k},$

where $\tau(N) = \eta(N) + \epsilon(N)/K$. As before, it follows from (19) that

(34) $q \ll M_1{}^K N^{\eta(N)} = N^{(K/R)-(99/100)K\epsilon(N)+\eta(N)} = o(N^{K/R})$ for $k \geqq 3$.

By the inductive hypothesis, there exists an integer T satisfying

(35) $1 \leqq t \leqq T$ and $||\theta_1 q^{k-1}t^{k-1} + \ldots + \theta_{k-1}qt|| \ll T^{-1/(K-1)+\epsilon(T)}$,

since $2^{k-1} - 1 = K - 1$. Taking $x = qt$ we have

(36)
$$\begin{aligned} ||f(x)|| &\ll ||\theta q^k t^k|| + ||\theta_1 q^{k-1}t^{k-1} + \ldots + \theta_{k-1}qt|| \\ &\ll q^{k-1}t^k||q\theta|| + T^{-1/(K-1)+\epsilon(T)}. \end{aligned}$$

We take $T = [N^{(K-1)/R}]$, then $1 \leqq qt \leqq N$, for $N \geqq N_0(k)$, and

(37)
$$\begin{aligned} ||f(x)|| &\ll \{M_1{}^K N^{\eta(N)}\}^{k-1}N^{k(K-1)/R}M_1{}^{K-1}N^{\tau(N)-k} + T^{-1/(K-1)+\epsilon(T)} \\ &\ll M_1{}^{kK}N^{k(K-1)/R-k}M_1{}^{-1}N^{(k-1)\eta(N)+\tau(N)} + N^{-1/R+(K-1)\epsilon(T)/R} \\ &\ll M_1{}^{-1} + N^{-1/R+\epsilon(N)} \ll N^{-1/R+\epsilon(N)}, \end{aligned}$$

which completes the proof of Theorem 2.

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