Multiple Periodic Variable Stars
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δ SCUTI AND RRs TYPE VARIABLES : THEORETICAL ASPECTS

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Abstract

Theoretical problems in connection with δ Sct and RRs type variables are reviewed. All evidence shows that δ Sct stars are normal main-sequence or early post main-sequence stars. Results of linear stability analyses of δ Sct models agree well with observations, the presence of non-variables in the strip being understandable in terms of helium diffusion. Excitation of non-radial modes may also occur; comparison of observed multiple periodicities with the theoretically derived period patterns for radial and non-radial modes is discussed as a means to distinguish between these possibilities.

It is not known whether RRs variables are of the same nature as δ Sct stars or in a late evolutionary state with low mass. RRs variables pulsate in radial modes, and recent linear stability analyses seem to agree well with observations for both normal and low masses. The reason for the difference in amplitude between RRs and δ Sct stars is not known. Non-linear investigations may provide important information in the near future. The observed period ratios of RRs variables indicate for most of the double mode pulsators a slight deficiency in heavy elements and a normal mass; but more detailed investigations of this problem are needed.

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1. Introduction

Each group of variable stars raises questions of two types to theoreticians: (i) Can we understand the observed light- and velocity-curves by means of pulsation theory? (ii) Where do these stars fit into stellar evolution theory, and is there agreement on their properties inferred from pulsation and from evolution theory? This comparison provides very important tests of stellar evolution theory.

The δ Scuti stars are variables (i) situated in the Cepheid instability strip in the HR diagram above the zero-age line and below the RR Lyrae region (ii) which have relatively small pulsation amplitudes ($\Delta m_V < 0.3$), and (iii) periods less than 0.25 days. All evidence, observational as well as theoretical, seems to confirm the simplest possible assumption: δ Sct stars are Population I A or F stars undergoing normal evolution in the main-sequence stage or in the shell hydrogen burning phase immediately following the main-sequence stage.

RRs variables are distinguished from the δ Sct group by their larger amplitudes ($0.3 < \Delta m_V < 0.8$), but in most other respects these two types of variables have very similar properties. Parallax determinations of the RRs variables AI Vel and SX Phe indicate that these stars are placed below the Population I zero-age main-sequence (See e.g. Bessell, 1969b; Fitch, 1970); and RRs stars have often been suggested to be in a late evolutionary phase (Kippenhahn, 1965; McNamara, 1965; Bessell, 1969b; Petersen and Jørgensen, 1972; Dziembowski and Kozlowski, 1974; Percy, 1975). However the evidence, both observational and theoretical, for an intrinsic, evolutionary difference between δ Sct and RRs stars is rather weak (e.g. Baglin et al., 1973; McNamara and Langford, 1974). At present it seems possible that these two groups of variables are intrinsically similar, the only difference being that a few of the RRs stars belong to the old disk population or even the halo population of metal deficient stars. We return to these problems in more detail later.

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Also detailed analysis of the observed complicated light- and velocity curves seems to raise quite different problems for δ Sct and RRs variables. Since δ Sct and RRs stars populate the lower part of the Cepheid instability strip, we expect them to oscillate in simple radial modes excited primarily in the second ionization zone of helium, as other well-behaved Cepheids. Already Kippenhahn and Hoffmeister (Kippenhahn, 1965) found a weak instability in main-sequence models corresponding to δ Sct stars and a more pronounced one for low mass models which could be interpreted as RRs variables. More detailed investigations by Castor (1970), Chevalier (1971), Cox et al. (1973), Dziembowski and Kozlowski (1974), and Percy (1975) have confirmed these theoretical results, which will be discussed in more detail in the following sections.

However, the light- and velocity-curves of δ Sct and RRs variables manifestly do not look like those of well-behaved classical Cepheids or RR Lyrae variables. It has been particularly difficult to interpret detailed observations of δ Sct stars. At present it is not clear whether δ Sct pulsations are basically simple radial oscillations or not. Dziembowski (1974) has shown that also non-radial modes are excited in δ Sct models, and perhaps the light-curves are more easily understood in terms of non-radial oscillations. Also other complicating effects may play a role for δ Sct variables (Fitch, 1970; Le Contel et al., 1974; Warner, 1974; Fitch, this volume).

Besides, the theoretician has to face the problem of the presence of non-variables in the δ Sct instability strip. Why are about 70 per cent of the stars in the observed instability strip constant to within 0.01 mag? Baglin (1972) showed that the correlation observed in main-sequence stars between pulsation, metallicity, and rotation can be understood in terms of a competition between diffusion processes, that tend to separate the elements in radiative regions of stars, and mixing due to meridional circulation. In very slowly rotating stars with negligible circulation, downwards diffusion in the outermost layers removes helium from the driving He^+ ionization zone, while some "metals" tend to move to the surface (radiative force). Since the main excitation mechanism can not operate in such Am stars they will not pulsate. In fast rotators, on the other hand, meridional circulation provides efficient mixing. Thus fast rotators are able to pulsate, but not to develop metallicism. Breger (1972) suggested a more elaborate scheme for the properties of stars in the δ Scuti region. The extensive non-adiabatic calculations by Cox et al. (1973) have confirmed that a helium content of less than about 10 per cent in the driving He^+ ionization zone will result in pulsational stability; and Vauclair et al. (1974) have studied the change in time of the helium profile of two typical δ Sct envelope models by detailed computation of the diffusion process. The separation takes less than 1 per cent of the main-sequence life time, so non-rotating main-sequence stars in the instability strip will not pulsate.

Many RRs variables show amplitude modulations that can be understood as due to simultaneous excitation of two radial modes. For 7 RRs stars Fitch (1970) gave two accurate periods, and showed that in 6 cases the fundamental radial mode and the first overtone is excited, while 1 star oscillates in the first and second over-tone. Theoretical problems in connection with this type of model behaviour has been greatly clarified by the results of Stellingwerf (1974, 1975b) who studied models of RR Lyrae stars and double-mode Cepheids of periods 2-4 days. A similar investigation of models in the lower part of the Cepheid instability strip will probably provide valuable new information.

At present the reason for the difference in pulsation amplitudes between typical δ Sct and RRs variables is not understood. The properties of these stars seem to be very similar, except that RRs stars are slow rotators, while the mean projected rotational velocity is about 90 km/s for low amplitude δ Sct stars. Also, a small fraction of the RRs variables belong to the metal deficient old disk population, while δ Sct stars seem to belong to young Population I. It has been pointed out, however, that there is a continuous variation in properties from high amplitude ($\Delta m_V \cong 0.3$) δ Sct stars to the RRs variables of similar amplitudes (Eggen, 1970; Baglin et al., 1973). And it seems possible that the delicate balance between the complicated processes that tend to mix and those that separate the stellar matter in the very thin layers that excite or damp the oscillations, in some cases will produce a particularly strong excitation.

In the following we first discuss the δ Scuti group, emphasizing the problem of the excitation mechanism, and then the RRs type variables whose nature and evolutionary state have been widely debated.

2. δ Scuti Stars

2.1 Evolutionary stage

In order to discuss the structure and evolution of δ Sct stars several authors have performed determinations of effective temperature, T_e, gravity, g, luminosity, L or M_{bol}, mass, M, and pulsation constant, Q (cf. the review article by Baglin et al., 1973). The stellar atmosphere parameters T_e and g can be obtained from photometric and spectrophotometric measurements (Bessell, 1969a; Breger and Kuhi, 1970; Dickens and Penny, 1971).

When an independent absolute magnitude is available from trigonometric parallax or cluster membership, or the like, an empirical mass value and pulsation constant can be computed from

$$\log M/M_{\odot} = 12.502 + \log g - 0.4\, M_{bol} - 4 \log T_e \qquad (1)$$

$$\log Q_{obs} = -6.454 + \log \Pi + 0.5 \log g + 0.1\, M_{bol} + \log T_e \qquad (2)$$

where Π is the period in days. Typical observational uncertainties in favourable cases are $\Delta \log T_e \cong 0.01$, $\Delta \log g \cong 0.3$, $\Delta M_{bol} \cong 0.5$, and $\Delta \log \Pi \cong 0.04$. (1) and (2) then give error estimates $\Delta \log M/M_{\odot} \cong 0.37$ and $\Delta \log Q \cong 0.16$; i.e. present uncertainties in M and Q derived directly from photometric and spectroscopic stellar atmosphere parameters combined with luminosities from parallaxes etc. are factors of at least 2.5 and 1.4 respectively. Hence, by such investigations it is not possible to decide whether masses of δ Sct stars are equal to the evolution masses, derived by comparison of their position in the HR diagram with standard evolutionary tracks for a typical Population I composition, or perhaps a factor 2 smaller. Nor can it be decided which mode is excited, because the difference between successive modes is at most 25 per cent. However, the masses and pulsation constants determined in this way by Breger and Kuhi (1970) and Dickens and Penny (1971) are in agreement with (the evolutionary) masses $M = 1.5 - 2.5\ M_{\odot}$ (cf. Fig.1) and pulsation in the fundamental mode or an over-tone of low order.

If we rely on the assumption that δ Sct stars are Population I stars of normal type undergoing standard evolution, we can use standard calibrations of e.g. uvby, β photometry to derive M_{bol} and $\log g$ to a higher accuracy than just mentioned (Strömgren, 1966; Crawford, 1970; Petersen and Jørgensen, 1972; Breger, 1972, 1974, 1975a). Estimating $\Delta M_{bol} \cong 0.2$, $\Delta \log g \cong 0.1$, $\Delta \log T_e \cong 0.01$, and $\Delta \log \Pi \cong 0.04$ we now find $\Delta \log M/M_{\odot} \cong 0.12$ and $\Delta \log Q \cong 0.07$. Thus in favourable cases it may be possible to distinguish pulsation in the fundamental mode from over-tone pulsations. The results by Breger (1972, 1975a) and Petersen and Jørgensen (1972) indicate that both fundamental mode and over-tone oscillations occur among δ Scuti stars. Masses found in such investigations are $M = 1.5 - 2.5\ M_{\odot}$ as expected.

Table 1. Ages for Population I stars of composition (X,Z) = (0.70,0.03) in the instability strip in the main-sequence or early post main-sequence phases. Min. (max.) refers to the left (right) endpoint of the relevant section of the evolutionary track (cf. Fig.1). The fast hydrogen exhaustion phase has been neglected.

	Main-sequence			Post main-sequence		
		Age (unit: 10^6y)			Age (unit: 10^6y)	
$M/M_{\odot}$	M_{bol}	Min.	Max.	M_{bol}	Min.	Max.
1.6	2.6	0	580			
1.8	2.0	0	1020	1.6	1030	1130
2.0	1.5	495	728	1.1	743	817
2.5				0.3	430	434

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Thus δ Sct stars seem to be Population I, young disk stars evolving through the instability strip in the main-sequence phase or in the much faster shell hydrogen burning phase immediately following the main-sequence. From age calibrations (Hejlesen, 1975) their age and the time interval they spend in the instability strip can be estimated. A few typical data for $(X,Z) = (0.70,0.03)$ are given in Table 1.

2.2 Excitation Mechanism, Multiple Periodicities

Linear stability investigations of δ Scuti models by Kippenhahn and Hoffmeister (Kippenhahn, 1965), Castor (1970), Chevalier (1971), Cox et al. (1973), Pamjatnikh (1974), Dziembowski (1974, 1975), and Stellingwerf (1975c) have shown that inside the observed instability strip models that have the standard helium content $Y \cong 0.30$ are unstable in the fundamental mode and the first few over-tones. Cox et al. investigated the position of the fundamental and the first over-tone blue edges for several compositions, varying Y from 0.37 to 0.00. Their results are shown in Fig.1. For $Y = 0.28$ the calculated blue edge for the second over-tone agrees well with the observed high temperature boundary of the instability strip. For higher Y the calculated blue edges become hotter, while for $Y = 0$ no instability is found in the observed strip. Of course, this type of variation with Y is to be expected, since the second ionization zone of helium provides the driving of the oscillations.

From the results just mentioned one should expect all stars of standard Population I composition in the instability strip to oscillate. Therefore the main problem has been how to explain the fact that about 70 per cent of the stars in the strip are observed to be constant to within 0.01 mag. In Section 1 we outlined the theory of Baglin (1972), who showed that the observed correlation between pulsation, rotation, and metallicism for main-sequence stars can be explained very satisfactorily as the result of a competition between diffusion and mixing processes. Stars with well-mixed outer layers are pulsationally unstable, while stars containing outer zones where the separation processes have succeeded in removing practically all helium from the main driving layers become stable and tend to develop metallicism.

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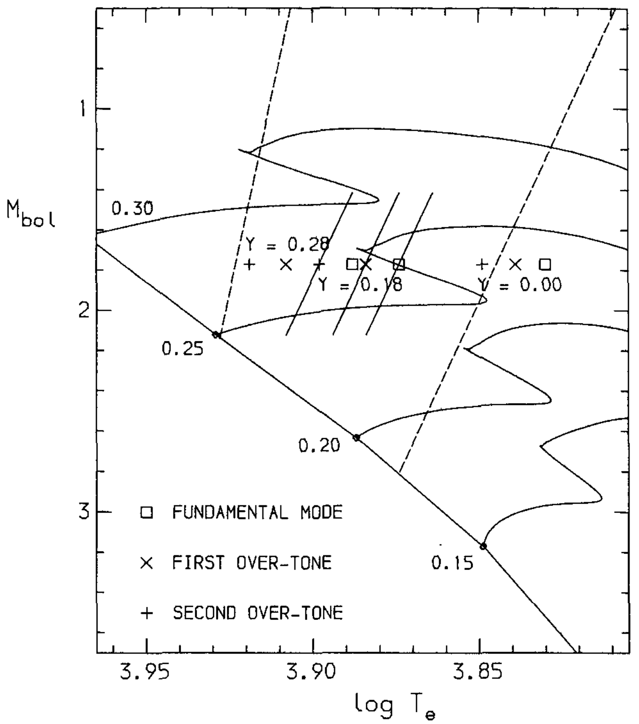

Fig.1. Theoretical blue edges for Population I ($Z = 0.02$) models of $M = 1.5$ and $2.0\,M_\odot$ as given by Cox et al. (1973). Full lines mark the edges for helium content $Y = 0.18$. The observed boundaries of the δ Sct instability strip are indicated (dashed lines). Full curves show ZAMS and standard evolutionary tracks for $(X,Z) = (0.70,0.03)$ and $\log M/M_\odot = 0.15, 0.20, 0.25, 0.30$, given by Hejlesen (1975).

Recently, Vauclair et al. (1974) have performed detailed computations of the diffusion of helium in the outermost layers of Population I models of 1.5 and $2.0\,M_\odot$. They took the changes in the structure of the models due to the helium diffusion itself into account, assuming that the two convective zones are linked by overshooting. Their result for the $1.5\,M_\odot$ model is that the helium content in the He^+ ionisation zone is reduced from $Y = 0.28$ to 0.10 in a time scale of 1.5×10^6 years. Since this is less than 1% of the nuclear time scale for such main-sequence stars, the diffusion processes in a real star will have proceeded much further. Thus such stars will appear stable, provided that no mixing occurs.

Although this simple theory for the correlation of pulsation, metallicity, and rotation correctly describes the properties of most main-sequence stars, a detailed discussion raises several difficult problems (Breger, 1972). The relations between rotation, mixing, and variability are discussed in more detail by A. Baglin in the next paper of the present volume.

Further it is not known if the mixing/separation hypothesis can explain the presence in the δ Scuti region above the main-sequence of relatively high amplitude variables, that all show slow rotation, together with non-variables that are often fast rotators. Clearly further theoretical studies of these complicated effects, also including rotation, are urgent. But it is well known that it is at present almost impossible to include rotation with meridional circulation in standard computations of stellar evolution. Mixing and separation processes are also of great interest in connection with two outstanding and until now unsolved problems in the standard theory of stellar evolution: The solar neutrino problem and the problem of the isochrones above the gap in the upper main-sequence in galactic clusters containing evolved stars of mass 1.5 $M_{\odot}$ or more (Maeder, 1974). Perhaps the assumption of totally unmixed evolution in radiative regions also in long time-scales is doubtful in several cases, and clearly much more detailed studies of mixing and separation processes should be included in many calculations of stellar evolution.

Compared with other types of variables typical δ Sct stars have very small amplitudes. This seems reasonable in view of the weak excitation (Chevalier, 1971; Dziembowski, 1974). However, the growth-times, although long compared to those of other Cepheid variables, are short compared to the main-sequence life-time. Non-linear investigations are needed to settle the question of the expected limiting amplitudes. But the initial value method (e.g. Christy, 1966) can not be applied successfully to δ Sct

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models. At present it is not known whether the small amplitudes are in agreement with the assumption of excitation of radial oscillations by the κ-mechanism. We return to these problems in more detail in the discussion of RRs stars.

Oscillations in non-radial modes would probably always result in small amplitudes, and are in this respect an attractive explanation for the group of low amplitude variables. Dziembowski (1974) showed that main-sequence stars that are unstable against radial modes are also unstable against many non-radial modes with similar growth rates. Dziembowski suggested simultaneous excitation of many non-radial modes in δ Sct stars.

Other explanations of the observed complicated light- and velocity-curves have also been offered. Fitch (1970) suggested that the complex variations observed in individual cases are due to one or more of the mechanisms: (a) tidal modulation of the basic pulsation by a faint companion in a binary orbit, (b) modulation of the primary pulsation by magnetic coupling with a companion, (c) excitation of one or more non-radial modes by rotational coupling with the intrinsic radial pulsation, (d) composite variation due to a binary pair of intrinsic variables, (e) simultaneous excitation of two successive radial pulsation modes in the same star. Le Contel et al. (1974) suggested that non-linear atmospheric effects play a role.

How can we decide whether the basic pulsation is radial or non-radial, or which complications are present in individual cases? The first problem can presumably be handled theoretically in a few years, while the second one seems more difficult. At present comparison of several frequencies in the observed complicated light-curves with the frequency spectrum calculated theoretically for the above mentioned mechanisms, seems to be the only potentially powerful possibility. When more frequencies are known in a star, their ratios and differences can be compared with calculated ones, and sometimes a clear decision may be possible. If, for instance, two radial modes are simultaneously excited in a Population I star, the period ratios will be $\Pi_1/\Pi_0 = 0.75$ to 0.76, $\Pi_2/\Pi_1 \cong 0.80$ etc. (Petersen and Jørgensen, 1972), while non-radial modes give other well defined period ratios (Ledoux,

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1974; Dziembowski, 1974; see below). Another example is the equal frequency split found in the analysis of 1 Mon by Shobbrook and Stobie (1974), which was interpreted by Warner (1974) as a non-radial pulsation involving a rotational perturbation. Today very few low amplitude variables have unambiguously determined multiple periodicities (Fitch and Szeidl, 1975), and theoretical investigations of complicated systems have just begun.

Cox (1974) has pointed out that Castor's results (Castor, 1970) of linearized calculations of radial pulsations of δ Sct models may pose a difficulty. Castor found that e-folding times for amplitude changes are much smaller for the third and fourth over-tone than for the low order modes, and similar results have been obtained by Dziembowski (1974) and Stellingwerf (1975c). This indicates (but does not prove - non-linear calculations are necessary to solve the problem) that δ Sct stars according to pulsation theory should oscillate with the corresponding small periods. However, the observations seem to agree best with fundamental or low order over-tone pulsations. Also the e-folding times of non-radial p-modes for two 1.75 $M_{\odot}$ main-sequence models given by Dziembowski (1974) show this unexpected dependence on the order of the p-modes.

If δ Sct stars oscillate in such high order modes their frequency spectrum is expected to be complicated. Table 2 gives theoretically calculated period ratios for radial modes alone, non-radial modes alone, and for simultaneously excited radial and non-radial p-modes. The period ratios given for radial modes are valid to an accuracy of about 1% for all Population I variables of evolutionary masses in the lower Cepheid instability strip. Radial periods depend little on the detailed internal structure of the model, because the oscillation amplitudes are very small below relative radius about 0.5. Only very near ZAMS, where few variables are situated, an increase of 2 - 4% in the Q values for radial modes occur, due to the increasing importance of the central region (Jørgensen and Petersen, 1974). Here the ratios of over-tone to fundamental mode periods are up to 3% higher than those given in the table.

Table 2. Theoretically determined period ratios for radial oscillations and non-radial p-modes (with $l = 4$) of δ Sct models.

Radial modes		Radial modes of order				
Order	$Q \times 10^4$	0	1	2	3	4
0	333	1.000				
1	252	0.757	1.000			
2	203	0.610	0.806	1.000		
3	170	0.511	0.675	0.837	1.000	
4	146	0.438	0.579	0.719	0.859	1.000
5	128	0.384	0.508	0.631	0.752	0.877

Model	Non-radial p-modes		Non-radial p-modes of order			
	Order	$Q \times 10^4$	1	2	3	4
ZAMS	1	240	1.000			
	2	194	0.807	1.000		
	3	161	0.670	0.830	1.000	
	4	137	0.573	0.710	0.856	1.000
	5	120	0.501	0.621	0.749	0.875
X_c=0.08	1	214	1.000			
	2	196	0.916	1.000		
	3	161	0.752	0.822	1.000	
	4	138	0.644	0.704	0.857	1.000

Table 2 (continued)

Model	Non-radial p-modes		Radial modes of order and Q x 10^4					
			0	1	2	3	4	5
	Order	Q x 10^4	333	252	207	174	149	130
ZAMS	1	240	0.719	0.950	1.157	1.374	1.607	1.845
	2	194	0.581	0.767	0.934	1.109	1.297	1.490
	3	161	0.482	0.637	0.775	0.920	1.076	1.236
	4	137	0.412	0.545	0.663	0.787	0.921	1.058
	5	120	0.361	0.477	0.580	0.689	0.806	0.925
X_c=0.08	1	214	0.643	0.850	1.034	1.228	1.436	1.650
	2	196	0.589	0.778	0.947	1.124	1.315	1.510
	3	161	0.484	0.639	0.778	0.924	1.080	1.241
	4	138	0.414	0.548	0.666	0.791	0.925	1.063

The frequency spectrum for non-radial modes is much more complicated. Firstly, non-radial periods depend significantly on the detailed internal structure of the models, due to the relatively high amplitude in the central region for non-radial modes. Secondly, the value of the l parameter of the chosen spherical harmonic is important. And finally the non-radial spectrum includes besides the p-modes also the f-mode and the g-modes. Dziembowski's growth rates indicate that it is reasonable to concentrate on p-modes of order less than about 4, and the observed δ Sct periods also correspond to such oscillations. If we further restrict ourselves to fairly small l ($l \leqq 4$) and to main-sequence models, rather well defined period patterns result. The Q values and period ratios of Table 2 are based upon the frequencies given by Dziembowski (1974) for two main-sequence models of 1.75 $M_\odot$, a zero-age model (ZAMS) and a model evolved

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to central hydrogen content equal to 0.08 ($X_c = 0.08$). Dziembowski (1975) reports work in progress for post main-sequence models.

3. RRs Variables

3.1 Evolutionary stage

The nature of RRs variables has been widely debated. We first review the following possibilities: RRs stars are (a) as δ Sct stars main-sequence or immediately post main-sequence stars, (b) post-red-giant stars of mass $M \cong 0.5\ M_\odot$ probably belonging to intermediate Population II, (c) stars of mass $M = 0.20 - 0.25 M_\odot$ perhaps formed in late stages of the evolution of close double star systems now contracting towards the white dwarf region, (d) blue stragglers, (e) (similar to) RR Lyrae stars of Population I; and then discuss (a), which at present seems to be most popular, in more detail.

(a) Eggen (1970) concluded that "there appears to be no obvious argument against the view that most of the ultrashort-period variables, regardless of amplitude, are stars of $M = 1.5$ to $2.5\ M_\odot$ in the shell hydrogen-burning stage of their evolution. Their presence as late evolvers in the old disk population, like that of other blue stragglers, still needs explanation, but it seems unlikely that they represent advanced evolution of lower mass stars". Chevalier (1972) also concluded that there is no compelling need to suppose that RRs variables differ basically from δ Sct stars, emphasizing the weakness of the arguments for (1) large space motions corresponding to intermediate Population II, (2) masses about $M = 0.5\ M_\odot$ or smaller, and (3) a low metal content ($Z < 0.005$) inferred from period ratios (See (b) below). Recently Bessell (1974) found no clear distinction between RRs stars and δ Sct stars on the basis of measured abundance, gravity and temperature; and from similar information derived from intermediate band photometry for 9 RRs stars McNamara and Langford (1974) concluded that RRs stars are post main-sequence stars that have not yet entered the red-giant stage.

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(b) Kippenhahn (1965) discussed the stability of a model of 0.4 solar mass on the main-sequence and found 100 times more excitation than in models of normal mass ($\cong 1.5\,M_\odot$). He concluded that such models could correspond to RRs variables, and that these stars must be evolved stars which have lost most of their mass. McNamara (1965) investigated the position of RRs stars in the HR diagram and suggested that they are descendants of red-giant stars of Population I, evolving from the right to the left through the instability strip. Bessell (1969b) found from space motions and abundances that RRs stars are old disk stars intermediate between young disk Population I and old disk Population II. He also derived low masses and suggested mass-loss and perhaps mixing. These views were supported by Petersen and Jørgensen (1972), who found that the observed period ratios for RRs variables indicate a considerable deficiency in heavy elements ($Z < 0.005$) relative to the sun. Percy (1975) based his linear stability analysis for RRs stars on models of mass $M = 0.5\,M_\odot$. The derived blue edges of the instability strip (for a helium content $Y = 0.28$; see below) agree well with the observed edge. Percy determined the position of the variables in the HR diagram from a $T_e - \langle B-V \rangle$ calibration and the formula

$$0.3\,M_{bol} = \log Q/\Pi - 0.5\log M/M_\odot - 3\log T_e + 12.70 \qquad (3)$$

using the observed periods, the theoretically well determined pulsation constants, and the assumed mass $M = 0.5\,M_\odot$. Comparing Percy's results with the observed red boundary of the instability strip as given by Breger (1972), it is seen that about half of the variables are situated to the right of the strip (by up to $\Delta \log T_e = 0.03$). This may be due to (1) that the red boundary of the instability strip has lower T_e for lower masses, (2) systematic errors in the temperature calibration, or (3) that the assumption of a low mass is wrong. It is not possible today to decide whether (1) or (2) or (3) is correct. (1) can not be investigated successfully, because it is not yet possible to take convection into account properly in pulsation analysis. An uncertainty $\Delta \log T_e = 0.03$ is not much more than expected, but we note that an assumed mass of $2\,M_\odot$ gives a higher luminosity

($\Delta M_{bol} = -1$), bringing the variables inside the instability strip defined by δ Sct stars.

(c) Comparing results of linear non-adiabatic pulsation analysis of full stellar models of mass $M = 0.20 - 0.25\ M_{\odot}$ with observational data for RRs variables, Dziembowski and Kozlowski (1974) concluded that such very low mass models are plausible for RRs stars. The models have a degenerate helium core and a hydrogen-burning shell, and represent pre-white-dwarf evolution. Such objects may originate by mass exchange in close binary systems or possibly by mass loss from single stars. Since it seems rather unlikely that all RRs variables are members of close binaries, Dziembowski and Kozlowski prefer the last possibility. The blue edges for these very low masses agree well with those derived by Percy for $M = 0.5\ M_{\odot}$ and those based on evolutionary masses (see below). But the position in the HR diagram based on $M = 0.2\ M_{\odot}$ gives rise to the same problems as those just described for models of $M = 0.5\ M_{\odot}$. The approach of Dziembowski and Kozlowski seems to be the only alternative to the simple possibility (a) which can produce RRs models that are not in obvious conflict with the presently accepted theory of stellar evolution.

(d) Eggen (1970, 1971) suggested that some ultrashort-period Cepheids (RRs) are blue stragglers of the old disk population, while SX Phe, having extreme space motion and metal deficiency, could be similar to the blue stragglers of the halo population. The evolutionary status of the blue stragglers is not known. Such stars may be formed in binary systems with mass exchange (McCrea, 1964; Refsdal and Weigert, 1969; Refsdal et al., 1974) or by mixing (Rood, 1970; Petersen, 1972).

(e) RRs stars are separated from the c-type RR Lyrae variables by their (1) lower periods, (2) smaller amplitudes, and (3) often more irregular light-curves. With very few exceptions all Cepheids having $\Pi < 0.25$ days are found to be RRs stars. Although the RRc and RRs groups are well separated, it seems that the RRs stars of $\Pi = 0.20 - 0.24$ days have luminosities $M_{bol} \cong 1$ and also other properties very similar to the RR Lyrae group. It is therefore natural to suspect a close relationship between these groups. Kippenhahn (1965) noted that RRs stars may be some kind of RR

Lyrae stars for Population I, and Breger (1969) suggested that RRs stars correspond to the RR Lyrae stars in the sense that they are similar evolved low-mass stars. McNamara (1974) suspected the strong line RR Lyrae stars to be an extension of the RRs group.

From the point of view of stellar evolution theory the interesting point here is whether (some) RRs stars, as the RR Lyrae group, are in an evolutionary phase following the red-giant phase and the helium flash, or not. Such RRs stars would be basically different from δ Sct stars. However, present knowledge of helium-burning stellar models seems to require that all helium-burning stars must have a helium core of mass $M_c \lesssim 0.5\,M_\odot$ and an absolute magnitude $M_{bol} \lesssim 1$. RRs stars with $M_{bol} = 1 - 4$ probably can not have the same structure as RR Lyrae stars. We conclude that the similarity in properties of RRs and RRc variables seems to be fortuitous from this point of view; stellar evolution theory does not predict a continuation of the RRc group to lower luminosities and smaller periods. But the similarity can probably be understood from pulsation theory.

Is the simple assumption that RRs stars are main-sequence or early post main-sequence stars evolving according to standard evolution theory in agreement with present observational data? It is well known that SX Phe presents a problem, because the luminosity derived from the reasonably reliable trigonometric parallax indicates a position in the HR diagram far below the ZAMS. The effective temperature of SX Phe has been given $\log T_e = 3.876 \pm 6$ by Bessell (1969b), $\log T_e = 3.895 \pm 5$ by Jones (1973) using a temperature calibration of his β_1 index, and $\log T_e = 3.880$ by Percy (1975). We adopt $\log T_e = 3.88 \pm 0.01$, and following the discussion of Fitch (1970) we take $M_{bol} = 4.0 \left\{ \begin{matrix} +0.7 \\ -0.9 \end{matrix} \right.$. In Fig.2 the error box for SX Phe is compared with results of stellar evolution calculations for hydrogen content $X = 0.70$, by Hejlesen (1975). It is seen that the masses (and ages) possible for SX Phe depend very much on the content of heavy elements, Z, assumed for the star. A standard Population I composition is not allowed, while a solar composition, $Z_\odot = 0.017$, and more metal deficient compositions are possible. Table 3 gives the mass and

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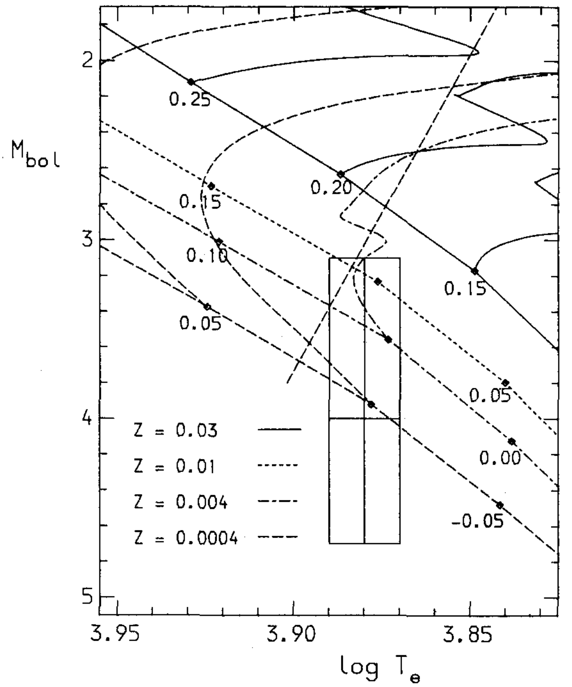

Fig.2. Comparison of the observed position of SX Phe in the HR diagram (see text) with results of stellar evolution calculations by Hejlesen (1975) for hydrogen content $X = 0.70$ and a large interval in content of heavier elements Z. Zero-age main-sequences are given for 4 Z values; numbers attached to the curves give $\log M/M_{\odot}$. A few evolutionary tracks and the cool boundary of the instability strip extended towards smaller luminosities are also shown.

Table 3. Mass and age for SX Phe

Z	Deficiency factor $Z_{\odot}/Z$	Mass (unit : $M_{\odot}$)	Maximum age (unit: 10^9y)
0.03	0.6	-	-
0.01	1.7	1.2	1.5
0.004	4	1.1	3
0.0004	40	0.9-1.0	7

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the maximum age for models in the uncertainty box for a large Z interval. We note that only models of $X = 0.70$ have been considered here, but there is no evidence for large deviations from this hydrogen content among main-sequence stars.

The observed value of Preston's ΔS indicates a very small Z, and Bessell (1969b) found the Fe abundance of SX Phe to be less than in η Lep by at least a factor of 4. Assuming η Lep to have solar abundance this gives a deficiency factor of more than 4, which agrees well with the discussion of Jones (1973). Recently Jørgensen (1975) found a metal deficiency relative to the Hyades about a factor of 10 from uvbyβ photometry. Since this agrees with Bessell's result, we trust in the following a deficiency factor 4 relative to the sun, i.e. $Z = 0.004$ for SX Phe. Table 3 then provides a mass $M = 1.1\ M_\odot$ and a maximum age 3×10^9 years. We note that the absolute magnitude for SX Phe given by Percy (1975), when corrected to $M = 1.1\ M_\odot$, becomes $M_{bol} = 2.8$, which corresponds to a position 0.3 mag above the upper boundary of the box of Fig.2 and gives an age 3×10^9 years. Eggen (1971) found $M_V = 3.1$, assuming SX Phe to be a member of Kapteyn's star group. Thus it seems possible to understand the most extreme star in the RRs group in terms of standard evolution theory.

In Table 4 the characteristics of SX Phe for the assumptions (a), (b), and (c) are given. Identifying the primary period 0.05496 days with the fundamental mode and using $Q_o = 0.032$ days, the pulsation criterion gives the mean density. Using the assumed mass, radius and gravity follow. And Eq.(3) gives M_{bol} to an accuracy better than about 0.15 mag, as $\log T_e = 3.88 \pm 0.01$.

Table 4. Three possibilities for the nature of SX Phe.

Possibility	$M/M_\odot$	$R/R_\odot$	log g	M_{bol}
(a)	1.10	1.48	4.14	2.70
(b)	0.50	1.14	4.03	3.27
(c)	0.25	0.90	3.93	3.77

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An observational distinction between these possibilities requires determinations of log g to an accuracy better than 0.1 or M_{bol} better than 0.5 mag. (a) is slightly favoured by the observed period ratio (see below), while (b) is suggested from the investigation by Jørgensen (1975) in this volume, and (c) agrees well with many observational data. A similar table for AI Vel shows that only a very small mass $\cong 0.2\,M_\odot$ can explain the observed gravities and absolute magnitudes. Then, however, it seems impossible to identify the observed period ratio, 0.773, with Π_1/Π_0.

3.2 Excitation Mechanism. Period Ratios

RRs variables clearly oscillate in radial modes as classical Cepheids and RR Lyrae stars, but they have often complicated light-curves due to simultaneous excitation of two or more modes. We have already mentioned the extensive stability analysis for models of normal masses $M = 1.5 - 2.0\,M_\odot$ (Cox et al., 1973), models of $0.5\,M_\odot$ (Percy, 1975) and models of $M = 0.20 - 0.25\,M_\odot$ (Dziembowski and Kozlowski, 1974). In Fig.3 the blue edges of the instability strip derived in these investigations for typical Population I compositions ($Z = 0.02 - 0.03$, $Y = 0.27 - 0.28$) are compared. The blue edges of the high mass models are situated at slightly higher temperatures (for same luminosity) as expected, and the differences in log T_e (at constant luminosity) for the fundamental and first harmonic blue edges are less than 0.02 - 0.03. This is scarcely more than the differences to be expected from the small differences in composition parameters, different interpolations in opacity tables etc.

It seems unlikely that linear investigations can explain why RRs variables have higher amplitudes than δ Sct stars. Even if, for certain combinations of model parameters, one found a strong excitation (in the sense that the pulsation growth times were relatively short, e.g. 1000 periods) one could not be sure that this also secures a large limiting amplitude for such models, although it is an indication.

Christy (1966) found that non-linear investigations by the initial value method are not feasible for models of a relatively

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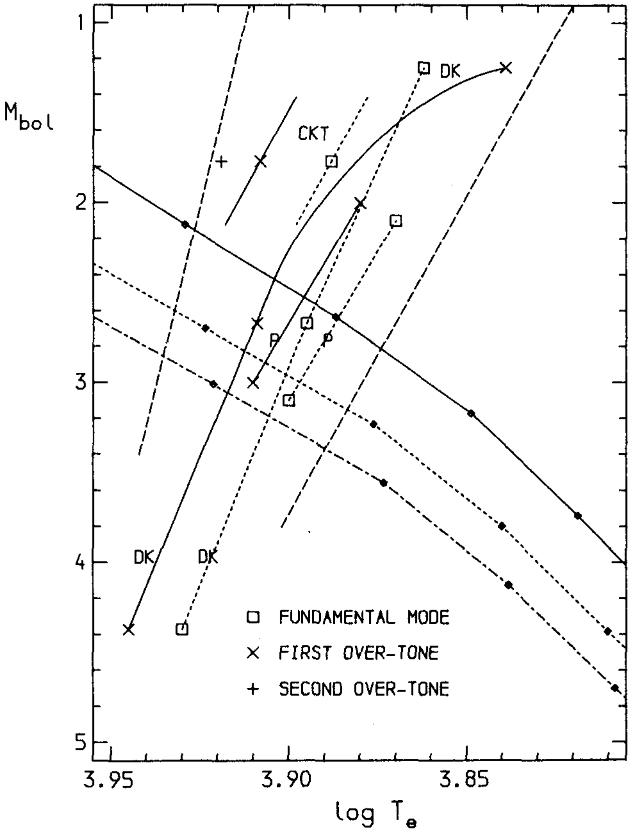

Fig.3. Blue edges of instability regions for typical Population I compositions derived by Dziembowski and Kozlowski (1974, DK) for models of $0.20-0.25\,M_\odot$, by Percy (1975, P) for models of $0.5\,M_\odot$, and by Cox et al. (1973, CKT) for models of $1.5-2.0\,M_\odot$. Fundamental mode blue edges are shown as dotted lines, and first overtone edges as full lines. For reference 3 ZAMS curves and the boundaries of the observed δ Sct instability strip are also given.

low luminosity to mass ratio, $L/M < 30\ L_\odot/M_\odot$, e.g. for δ Sct models. Such model have growth times of the order 10^6 periods; clearly excessive computer time will be used in direct simulation of the motions to limiting amplitude.

The non-linear eigenvalue approach of Baker and von Sengbusch (1969) and Stellingwerf (1974, 1975b) overcomes this difficulty, and furthermore this method provides detailed information on mode behaviour at limiting amplitude. Stellingwerf has analysed models for RR Lyrae stars and double-mode Cepheid variables of fundamental period 2 - 4 days. These stars show a complicated mode behaviour very similar to the RRs pattern. It seems likely that investigations by the non-linear eigenvalue method of models for δ Sct and RRs stars could provide valuable new information, at least for the high amplitude pulsations. In preliminary non-linear calculations for models attempted to describe AI Vel ($Z = 0.005$, $M = 0.4 M_\odot$, $T_e = 7400$ °K, $M_{bol} = 2.54$) and δ Sct ($Z = 0.02$, $M = 2 M_\odot$, $T_e = 7000$, $M_{bol} = 0.875$) Stellingwerf (1975c) finds light amplitudes of at least 1.0 mag and velocity amplitudes more than 60 km/sec, which is too much. Stellingwerf's preliminary conclusion is that some source of dissipation, perhaps convective motions, is present in actual stars, but not included in the models. Stellingwerf is continuing this investigation and has encountered no problems with the numerical technique.

Period Ratios

Accurate Q values for δ Sct and RRs models determined from linear analysis have been given by several authors (Cogan, 1970; Chevalier, 1971; Petersen and Jørgensen, 1972; Dziembowski and Kozlowski, 1974; Jørgensen and Petersen, 1974). In principle, comparison with observed pulsation constants tells in which mode a variable oscillates. However, present available observational accuracy does not allow definite answers for RRs stars (Section 2.1).

Petersen and Jørgensen (1972) studied the ($\log \Pi_0$, Π_1/Π_0) diagram, and showed that this diagram provides potential powerful means to determine mass and chemical composition for double mode pulsators having two observationally accurately known periods. They concluded that the observed period ratios for RRs variables $\Pi_1/\Pi_0 = 0.768 - 0.778$ indicate that RRs stars have $Z < 0.005$ and thus cannot belong to Population I. This result was questioned by Chevalier (1972) and Baglin et al. (1973), who emphasized

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that neither the theoretical calculations nor the observed periods were accurate enough to allow distinction between period ratios near 0.77 for RRs variables and the values 0.750 - 0.760 expected for Population I models.

The weakest point in the theoretical analysis is the application of the linear, adiabatic theory for the derivation of Q values. However, in the meantime linear non-adiabatic Q values (e.g. Cox et al., 1972; Dziembowski and Kozlowski, 1974) and Q values derived in full non-linear calculations for models corresponding to double mode Cepheids (Stellingwerf, 1975a, 1975b) have been found to agree well with the linear adiabatic pulsation constants. In the present context Stellingwerf's results are especially interesting. Analysing a mixed-mode RR Lyrae model of fundamental period 0.96 days, he found linear and non-linear periods to agree to an accuracy of 0.2% for the fundamental mode and 0.5% for the first over-tone. He also showed that the standard procedure used by observers to determine periodicities in observed light curves is correct in the sense that the periods derived by this method from the calculated light curve (with beats) are precisely the same as those corresponding to pure fundamental or first harmonic oscillations of the model. Due to the non-linear character of the problem this is not a priori evident. It now seems that the small differences in Π_1/Π_0 discussed above are significant both theoretically and observationally.

Among the RRs variables SX Phe has the highest period ratio ($\Pi_1/\Pi_0 = 0.778$) and the largest metal deficiency ($Z \cong 0.004$). From the calibration curves in Fig.4 of Petersen and Jørgensen (1972) or in Fig.4 of Dziembowski and Kozlowski (1974), or from data given by Jørgensen and Petersen (1974) it is seen that these values are in agreement with a mass $M \cong 1\ M_\odot$ for SX Phe. For very small masses, $M \cong 0.2\ M_\odot$, the calibration curves indicate a considerably smaller period ratio. However, it is not known how much reasonable changes in X and Z can change the period ratio.

If the other RRs variables are slightly metal deficient and have $M = 1.5 - 2.0\ M_\odot$, we expect period ratios near the observed ones (0.768 - 0.773). But if they have $M \cong 0.2\ M_\odot$ and a nearly normal Population I composition, the identification of the period ratio with Π_1/Π_0 seems impossible, as just mentioned for AI Vel. Clearly much work remains to be done on these problems.

4. Conclusions

(1) Comparisons of observational data for δ Sct stars with theoretically calculated evolution and pulsation properties of stellar models all agree that δ Sct variables are normal young disk stars of Population I evolving in the main-sequence or in the shell hydrogen-burning post main-sequence phases.

(2) Linear stability analyses of δ Sct models (Cox *et al.*, 1973) give fundamental and first and second over-tone blue edges of the instability strip that agree very well with the observed strip in the HR diagram, when a standard helium content is assumed. Q values determined by Breger (1975a) indicate that the high temperature δ Sct stars near the blue edge oscillate in the first and second over-tone, while the cooler variables oscillate in the fundamental mode; just as expected.

(3) The presence in the instability strip of non-variable stars, often with Am characteristics, have been explained in terms of separation processes (Baglin, 1972; Vauclair *et al.*, 1974). In the main-sequence rotation seems to control the separation and mixing processes; slowly rotating stars become non-variable Am stars, while relatively fast rotators become typical δ Sct variables.

(4) Non-radial modes are also excited in δ Sct models (Dziembowski, 1974). The observed low amplitudes can be taken as an argument for the presence of non-radial modes rather than radial modes. We suggest comparison of observed multiple periodicities with the theoretically determined period patterns of radial and non-radial oscillations as a means to distinguish between these possibilities. Period patterns may also provide information on other complicating effects, e.g. coupling of pulsation and rotation.

(5) It is not clear whether RRs variables are of the same nature as δ Sct stars or quite different, probably in a very advanced evolutionary stage. We have shown that it is possible to explain most observed properties of SX Phe from the assumption that this star, as δ Sct stars, is a main-sequence or immediately post main-sequence star.

(6) The similarity of the pulsations of RRs stars to those of other types of double mode Cepheids, and their observed period ratios, show that RRs variables pulsate in radial modes. Linear stability analyses agree well with the observed position in the HR diagram.

(7) At present the reason for the difference in amplitude between typical δ Sct and RRs variables is not known. Non-linear investigations by the initial-value method are not feasible for δ Sct and RRs models, but the new powerful eigenvalue method may provide important information on limiting amplitudes in the near future.

(8) The observed period ratio of SX Phe agrees with a mass $M \cong 1\ M_{\odot}$ for this star, and a recent investigation by Stellingwerf (1975b) of double mode behaviour of an RR Lyrae model indicates that the small differences in period ratio used in this type of analysis are significant. Clearly non-linear analysis of RRs models must be available before results derived from period ratios can be trusted.

(9) We finally mention some important developments expected in the near future : (i) Non-linear, radial survey. An investigation by Stellingwerf (1975c) is in progress. (ii) Linear, non-radial survey. Dziembowski (1975) has obtained detailed results for main-sequence and post main-sequence δ Sct models. (iii) Comparison of detailed models with data for individual objects may soon provide more reliable information on the presence of radial or/and non-radial oscillations in δ Sct stars and the nature of RRs variables.

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Discussion to the paper of PETERSEN

BAGLIN: Before asking the question concerning the different evolutionary stage of the two classes of variables, δ Scuti and RRs, one should define clearly their properties. The first classification in the Catalog of Variable Stars, made from a rough look at the light curve, cannot be retained. As far as the amplitudes are concerned, there seems to be a continuous transition from low to high amplitudes, rather than a clear break.

PETERSEN: I have used the definition based on amplitudes (limit: $\Delta V = 0.30$ mag). There is a continuous transition from relatively high amplitude δ Scuti stars with amplitudes near 0.30 mag to RRs stars of slightly higher amplitudes. But I mostly refer to typical low amplitude δ Scuti stars and RRs variables.

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