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THINNEST PACKING OF CUBES WITH A GIVEN NUMBER OF NEIGHBOURS

BY
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As a contribution to various investigations [1–11] about packing of convex bodies with certain conditions imposed on the number of neighbours of each body, V. Chvátal [12] recently proved the following theorem: If in a packing of translates of a square each square has at least six neighbours then the density of the packing is at least $11/15$.

Let us recall the definitions of the terms occurring in this theorem. A set of convex bodies is said to form a *packing* if no two of them have interior points in common. Two bodies are called *neighbours* if they have a boundary point in common. The *density* is defined in the usual way [13, 14] as a limiting value which can be interpreted as the total volume of the bodies divided by the total volume of the space.

The constant $11/15$ in the above theorem is best possible. An extremal packing is exhibited in Fig. 1. The proof of the theorem implies that in a certain sense this packing is unique.

The problem solved by the theorem of Chvátal can be generalized in various directions. In this paper we are concerned with the analogous problem in n-space taking into consideration besides direct neighbours also more distant neighbours.

We say that in a packing of bodies each body is a 0-neighbour of itself. A body B is a *k-neighbour* of the body A if B is a neighbour of some $(k-1)$-neighbour of A other than a j-neighbour of A with $0 \leq j < k$.

In a packing of translates of a unit n-cube all j-neighbours of a cube A with $0 \leq j \leq k$ are contained in a concentric, homothetic cube with edge-length $2k+1$, showing that the number of such neighbours is at most $(2k+1)^n$. This number is attained in a *grid* of cubes, i.e. in a packing of cubes joining along whole cells and filling the n-space completely.

Let p be the plane of an $(n-1)$-dimensional cell of a cube. Let p' be a plane parallel to p having a distance $k+1$ of p. The cubes of the grid lying between p and p' form a packing in which each cube has $(k+1)(2k+1)^{n-1}$ j-neighbours with $0 \leq j \leq k$. The density of this packing is equal to zero. On the other hand, we shall prove the following.

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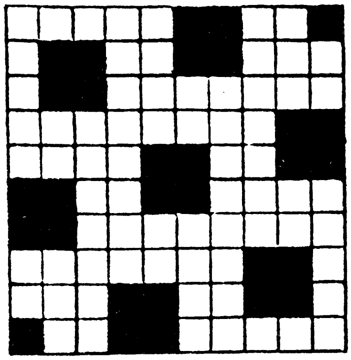

Figure 1

THEOREM. *If in a packing of translates of a cube each cube has more than* $(k+1)(2k+1)^{n-1}$ *j-neighbours with* $0 \leq j \leq k$ *then the cubes have a positive density.*

This theorem can be considered as the first approach to the problem of getting more information about the density $d_k^n(m)$ of the thinnest packing of translates of n-cubes each having at least m j-neighbours with $0 \leq j \leq k$. As we have seen, for $0 \leq m \leq (k+1)(2k+1)^{n-1}$ we have $d_k^n(m) = 0$. According to our theorem, for $(k+1)(2k+1)^{n-1} < m \leq (2k+1)^n$ we have $d_k^n(m) > 0$. For $m = (2k+1)^n$ we obviously have $d_k^n(m) = 1$ with equality only for the cubical grid. It is not difficult to show that for $m = (2k+1)^n - 1$ we have $d_k^n(m) = 1 - (k+1)^{-n}$. Going from $m = (2k+1)^n - 1$ successively down to $m = (k+1)(2k+1)^{n-1} + 1$ the problem of determining the value of $d_k^n(m)$ becomes more and more difficult. The theorem of Chvâtal claims that $d_1^2(7) = 11/15$.

Figures 2–9 show thin packings of translates of a square having altogether at least $m = 16$ or 17 or ... 23 0-, 1- and 2-neighbours. These configurations show that $d_2^2(16) \leq 69/121 = 0.570\ldots$, $d_2^2(17) \leq 2/3 = 0.666\ldots$, $d_2^2(18) \leq 7/10 = 0.7$, $d_2^2(19) \leq 3/4 = 0.75$, $d_2^2(20) \leq 19/24 = 0.791\ldots$, $d_2^2(21) \leq 5/6 = 0.833\ldots$, $d_2^2(22) \leq 7/8 = 0.875$, $d_2^2(23) \leq 11/12 = 0.916\ldots$. We conjecture that at least some of these packings are extremal. For $m = 17$ and 20 two different equally good packings are exhibited, showing that in general unicity cannot be expected.

Let us now turn to the proof of our theorem.

Let P be a packing of unit cubes considered in the theorem. We introduce a rectangular coordinate-system whose axes are parallel to the edges of a cube. We translate each cube in such a way that its center $(x_1, \ldots, x_n)$ becomes $([x_1], \ldots, [x_n])$. Let us record some properties of the set S of the translated cubes. 1. S is a packing. For two cubes of S with centers $([x_1], \ldots, [x_n])$ and $([y_1], \ldots, [y_n])$ cannot overlap without coinciding. But the assumption that $([x_1], \ldots, [x_n]) = ([y_1], \ldots, [y_n])$ would imply that $|x_1 - y_1| < 1, \ldots, |x_n - y_n| < 1$,

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in contradiction to the fact that the original cubes in P do not overlap. 2. Since the distance between the centers of a cube of P and its image in S has a uniform upper bound, namely $\sqrt{n}$, S has the same density as P. 3. If in P two cubes are neighbours, so are their images in S. It follows that two j-neighbours in P become i-neighbours in S for some $i \le j$.

Because of these properties it is enough to prove the theorem for the set S, i.e. in the case when the cubes are elements of a grid.

Now we associate with the cube c its Dirichlet cell (or Voronoi polyhedron) D defined as the set of those points of the space whose distance from the center of c is less than their distance from the center of any other cube of S. The Dirichlet cells associated with the various cubes of S fill the space without overlapping and without (n-dimensional) gaps.

Our aim is to prove that the Dirichlet cell D of c is contained in some polyhedron. We do this by showing that not all of the j-neighbours of c can be on one side only of some plane containing the center of c.

Let c be a cube of S. Let h be a closed half-space bounded by a plane p containing the center of c. We claim that the number N of the centers of the j-neighbours of c with $0 \le j \le k$ contained in h satisfies the inequality

$$(*) \qquad N \le (k+1)(2k+1)^{n-1}.$$

Obviously, the j-neighbours of c with $0 \le j \le k$ are contained in a cube C of edge-length $2k+1$ concentric with and homothetic to c. Therefore it suffices to show that the number of the centers of the cubes in the grid, in short the number of grid-points, contained in the intersection $h \cap C$ cannot exceed $(k+1)(2k+1)^{n-1}$.

Let e be an edge of C not parallel to p. A plane through the center of C perpendicular to e intersects C in an $(n-1)$-dimensional cube q. Thus q contains $(2k+1)^{n-1}$ grid-points. Since any grid-point in $p \cap C$ is a projection of a grid-point in g by a line parallel to e, we have for the number x of the grid-points contained in $p \cap C$ the inequality $x \le (2k+1)^{n-1}$. On the other hand, denoting the number of the grid-points contained in the interior of $h \cap C$ by X, we have $2X + x = (2k+1)^n$. Hence we obtain for the number $X + x$ of the grid-points contained in $h \cap C$ the inequality $X + x = \frac{1}{2}(2k+1)^2 - \frac{1}{2}x + x = \frac{1}{2}(2k+1)^2 + \frac{1}{2}x \le \frac{1}{2}\{(2k+1)^n + (2k+1)^{n-1}\} = (k+1)(2k+1)^{n-1}$, as stated.

We continue to prove that the diameters of the Dirichlet cells have a uniform upper bound depending only on n and k. Let 0 be the center of c and $o_1, \ldots, o_m$ the centers of its j-neighbours with $0 < j \le k$. D is contained in the intersection P of the half-spaces containing 0 which are bounded by the orthogonal bisectors of the segments $oo_1, \ldots, oo_m$. Because of the inequality (*) and the assumption that $m > (k+1)(2k+1)^{n-1}$ there is no half-space having o as boundary point which contains the points $o_1, \ldots, o_m$. Therefore P is a polyhedron. Since $o_1, \ldots, o_m$ are grid-points contained in C, there is only a

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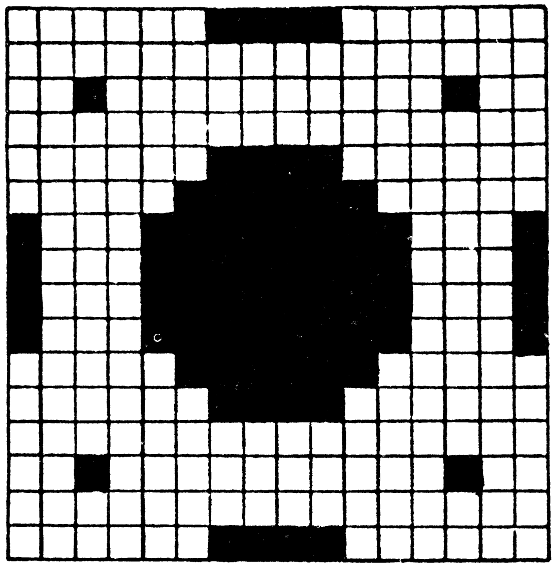

Figure 2

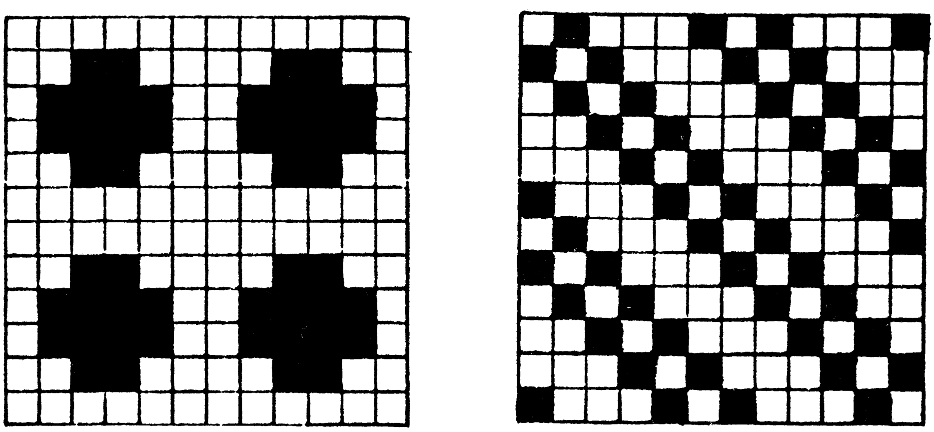

Figure 3

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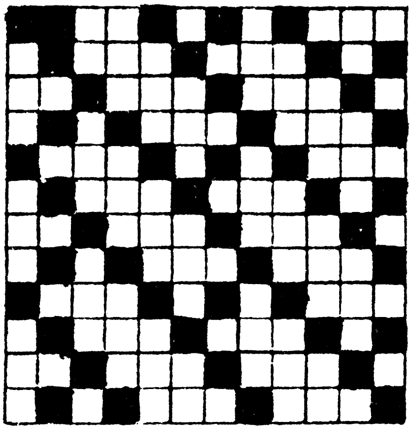

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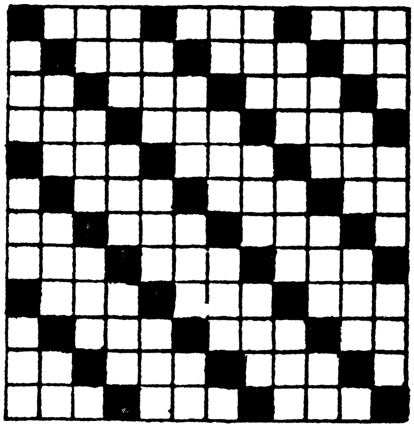

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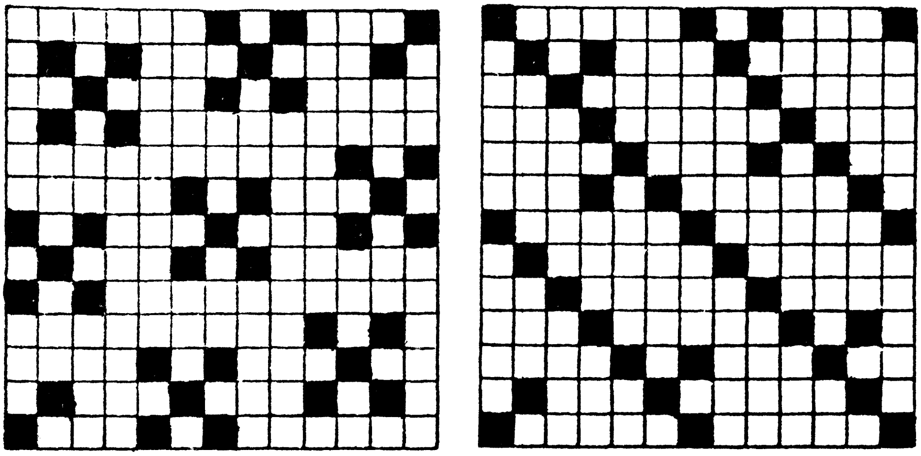

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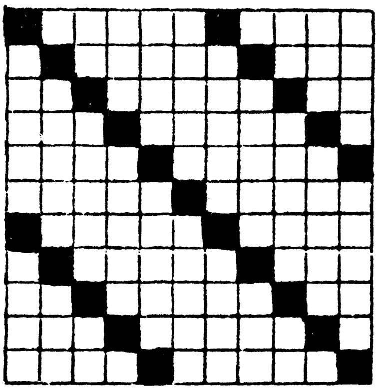

Figure 7

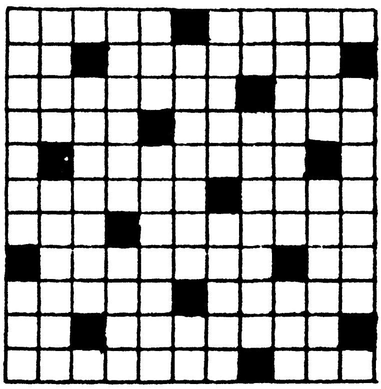

Figure 8

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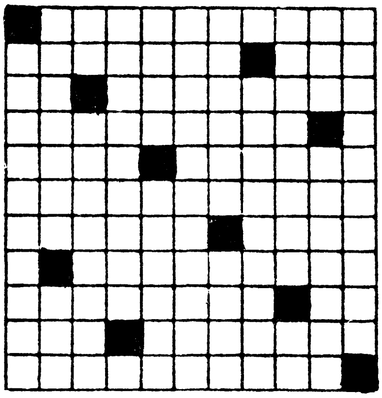

Figure 9

finite number of such polyhedra. The greatest diameter d occurring among these polyhedra is an upper bound of the diameters of the Dirichlet cells.

Let $s(R)$ be the number of the centers of the cubes of S contained in a ball of radius R centered at a fixed point of the space, $D(R)$ the total volume of the Dirichlet cells belonging to these cubes and $V(R)$ the volume of the ball. Then we have

$$V(R-d)<D(R)<V(R+d),$$

whence

$$\lim_{R\to\infty}\frac{s(R)}{V(R)}=\lim_{R\to\infty}\frac{s(R)}{D(R)}$$

which shows that the lower density of the cubes is at least $1/U$, where U is an upper bound of the volumes of the Dirichlet cells. This completes the proof of the theorem.

As to the "critical" density $d_k^n = d_k^n((k+1)(2k+1)^{n-1}+1)$ we conjecture that for any fixed k d_k^n has a positive lower bound. On the other hand, it seems to be very likely that for any n we have $\lim_{k\to\infty} d_k^n = 0$.

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