THE THIRD TERM OF THE LOWER CENTRAL SERIES OF A FREE GROUP AS A SUBGROUP OF THE SECOND

Dedicated to the memory of Hanna Neumann

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1. Introduction

In this paper the following notation will be used: for any group G, positive integer c and non-negative integer n, G_c is the cth term of the lower central series of G and $\delta^n G_c$ is the nth term of the derived series of G_c.

In a free group F, the subgroup F_2 is itself a free group which contains F_3 as a subgroup. The question arises: how nicely is F_3 situated in F_2?

In a free metabelian group G, the subgroups G_2 and G_3 are free abelian and it is easy to see that G_3 is a direct summand of G_2, since G_2/G_3 is also free abelian. Returning to the absolutely free group F, the simplest analogue of this result, that F_3 is a free factor of F_2, is just as easily seen to be false provided that the rank of F is at least 2, for then $\delta F_2 \leqq F_3$. On the other hand, it is possible to find free generating sets for F_2 and F_3 which have a large number of elements in common. The question now becomes: how close does F_3 come to being a free factor of F_2?

To answer this question one looks at members of the free generating set for F_3 that are not free generators for F_2 and, as would be expected, these all turn out to lie in δF_2. Suprisingly, they are contained in a free generating set for δF_2 and those members of this free generating set for δF_2 which are not free generators for F_3 turn out, in turn, to be contained in a free generating set for δF_3. This process continues ad infinitum through free generating sets for the subgroups $\delta^n F_2$ and $\delta^n F_3$ and results in the following theorem; the remainder of this paper will be devoted to the details of its proof.

THEOREM. *In an absolutely free group F of arbitrary rank there exist subgroups A_n and B_n $(n = 0, 1, 2, \cdots)$ such that, for each non-negative integer n, $\delta^n F_2$ is the free product of A_n and B_n and $\delta^n F_3$ is the free product of B_n and A_{n+1}.*

2. Free generating sets for certain subgroups of F

For the remainder of this paper F will be a given absolutely free group of arbitrary rank with a well-ordered free generating set G in terms of which all definitions will be made. A *generator* of F is to be understood as a member of G. The usual notation for commutators in F will be employed: $[x,y] = x^{-1}y^{-1}xy$. The element

$$[b, m_1 a_1, m_2 a_2, \cdots, m_r a_r],$$

where r and each m_i are non-negative integers and b and each a_i are members of F, is defined inductively. If $r = 0$ this is the element b, if $r \geqq 1$ and $m_r = 0$ it is the same as the element

$$[b, m_1 a_1, m_2 a_2, \cdots, m_{r-1} a_{r-1}]$$

and if $r \geqq 1$ and $m_r \geqq 1$ it is the element

$$[[b, m_1 a_1, m_2 a_2, \cdots, m_{r-1} a_{r-1}, (m_r - 1) a_r], a_r].$$

It will be necessary to refer several times to elements of the general form

$$[b, m_1 a_1^{\varepsilon_1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}]$$

where r and each m_i are *positive* integers, each $\varepsilon_i = \pm 1$, b and each a_i are members of F which have previously been ordered in some way, $b > a_1 < a_2 < \cdots < a_r$ and, if $b = [b_1, b_2]$ then $b_2 \leqq a_1$. This will be called the *standard form* and in its various manifestations further restrictions will be placed upon b and the a_i.

A free generating set for F_2 was first given by Grunberg in [1], Theorem 5.2, however it will be necessary to use here the slightly different one given in [2] together with a free generating set for F_3; these emerge as special cases of Lemma 8 of that paper upon putting $k_1 = 2$ or 3 (for free generating sets for F_2 or F_3 respectively) and $k_i = 2$ for all $i \geqq 2$ in the sequence K of the introduction. For the reader's convenience these free generating sets are described explicitly as the first lemma below. In this lemma, a *basic commutator of weight two* means as usual an element of the form $[b, a]$ where a and b are generators and $a < b$. Such elements are distinct as written and distinct from the generators. They will be considered to be ordered lexicographically: $[b_1, a_1] < [b_2, a_2]$ if and only if either $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$. The orderings of the generators and of the basic commutators of weight two individually are extended to encompass both by specifying that generators always precede commutators.

LEMMA 1. (*i*) *The elements of the standard form, where b and each a_i are generators, are distinct as written and form a free generating set for F_2.*

(ii) *The elements of the standard form, where b is a basic commutator of weight two and each a_i is either a generator or a basic commutator of weight two, are distinct as written and form a free generating set for F_3.*

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LEMMA 2. *The elements of the form*

$$[b_1, b_2, m_1 a_1^{\varepsilon_1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}] \tag{1}$$

where r is a non-negative integer, each m_i is a positive integer, each $\varepsilon_i = \pm 1$, b_1, b_2 and each a_i are generators and $b_1 > b_2 \leqq a_1 < a_2 < \cdots < a_r$, are distinct as written and form a free generating set for F_2.

Note that elements of the form (1) may alternatively be described as elements of the standard form except that r may possibly be zero, where b is a basic commutator of weight two and each a_i is a generator.

PROOF OF LEMMA 2. (I am indebted to Professor Gruenberg for suggesting this proof, which in much more satisfactory than my own.) Let S be the free generating set for F_2 given in Lemma 1 (i) and let T be the putative free generating set for F_2 given in the statement of Lemma 2. An endomorphism θ of F_2 is now defined by prescribing its action on S. Let

$$x = [b_1, m_1 a_1^{\varepsilon_1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}] \tag{2}$$

be an arbitrary member of S; then set

$$\left.\begin{aligned} x\theta &= x && \text{if } \varepsilon_1 = +1 \\ &= [b_1, a_1, m_1 a_1^{-1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}] && \text{if } \varepsilon_1 = -1 \end{aligned}\right\}. \tag{3}$$

It is easy to see that θ maps S one-to-one onto T. It is now sufficient to show that that θ is an automorphism of F_2.

We now prove, by induction over m, that for any $m \geqq 1$, $[b, a, ma^{-1}]$ can be written as a product of elements of the forms $[b, a]$ and $[b, la^{-1}]$ $(1 \leqq l \leqq m)$ in which the factor $[b, ma^{-1}]$ appears exactly once and that with exponent ± 1. The initial case ($m = 1$) is verified by exhibiting the well known law

$$[b, a, a^{-1}] = [b, a]^{-1}[b, a^{-1}]^{-1}.$$

Assuming now that $[b, a, ma^{-1}]$ can be written in the required form, consider

$$\begin{aligned} [b, a, (m+1)a^{-1}] &= [b, a, ma^{-1}, a^{-1}] \\ &= [b, a, ma^{-1}]^{-1}[b, a, ma^{-1}]^{a^{-1}}. \end{aligned}$$

The first term here, by the inductive hypothesis, can be written as a product of elements of the forms $[b, a]$ and $[b, la^{-1}]$ $(1 \leqq l \leqq m)$, all of which are of the required form; note that $[b, (m+1)a^{-1}]$ does not appear as a factor in this product. The second term can be written as another such product conjugated by a^{-1} and thus as a product of elements of the forms $[b, a]^{a^{-1}}$ and $[b, la^{-1}]^{a^{-1}}$ $(1 \leqq l \leqq m)$ in which the factor $[b, ma^{-1}]^{a^{-1}}$ appears exactly once and that with exponent ± 1. Since

$$[b, a]^{a^{-1}} = [b, a^{-1}]^{-1}$$

and

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$$[b, la^{-1}]^{a^{-1}} = [b, la^{-1}][b, (l+1)a^{-1}]$$

the statement is proved.

A similar argument now shows, this time by induction over r, that for any $m \geqq 1$ and $r \geqq 0$,

$$[b, a, ma^{-1}, c_1, c_2, \cdots, c_r]$$

can be written as a product of elements of the forms

$$[b, a, c_1, c_2, \cdots, c_q] \qquad (0 \leqq q \leqq r)$$

and

$$[b, la^{-1}, c_1, c_2, \cdots, c_q] \qquad (0 \leqq q \leqq r,\ 1 \leqq l \leqq m)$$

in which the factor $[b, ma^{-1}, c_1, c_2, \cdots, c_r]$ appears exactly once and that with exponent ± 1.

Let us consider S to be well-ordered in such a way as to preserve weight and so that

$$[b, m_1 a_1^{+1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}] < [b, m_1 a_1^{-1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}].$$

Then S is a well-ordered free generating set for F_2 and θ is an endomorphism of F_2 with the property that, for each member x of S, $x\theta$ can be written as a product of members of S none of which exceed x itself and which contains x as a factor exactly once and that with exponent ± 1: if x is of the form (2) and $\varepsilon_1 = +1$, then $x\theta = x$ and this result is trivially true. If $\varepsilon_1 = -1$ then $x\theta$ is given by (3) and the result has been established by the second inductive proof above.

It follows that θ is an automorphism of F_2, as required.

LEMMA 3. *Let H be a subgroup of F with a well-ordered free generating set which is the disjoint union of two subsets X and Y such that every element of X precedes every element of Y. Let X' be the set of all elements of the standard form with $b \in X \cup Y$ and each $a_i \in X$ and let Y' be the set of all elements of the standard form with $b \in Y \cup X'$ and each $a_i \in Y$. Then the members of $X' \cup Y'$ are distinct as written and form a free generating set for the derived group of H.*

PROOF. Writing Z for the set of all elements of the standard form where b and each a_i belong to $X \cup Y$, it follows from Lemma 1 (i) that the elements of Z are distinct as written and freely generate the derived group of H; it remains to prove that $X' \cup Y' = Z$.

An element x of $X' \cup Y'$ is of the standard form

$$x = [b, m_1 a_1^{\varepsilon_1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}]$$

where either $b \in X \cup Y$ and each $a_i \in X$ or $b \in Y \cup X'$ and each $a_i \in Y$. Then x is obviously a member of Z unless $b \in X'$ and each $a_i \in Y$; but then b is itself of standard form,

$$b = [d, n_1 c_1^{\delta_1}, n_2 c_2^{\delta_2}, \cdots, n_s c_s^{\delta_s}]$$

say, where $d \in X \cup Y$ and each $c_i \in X$. Thus

$$x = [d, n_1 c_1^{\delta_1}, n_2 c_2^{\delta_2}, \cdots, n_s c_s^{\delta_s}, m_1 a_1^{\varepsilon_1}, m_2 a_2^{\varepsilon_2}, \cdots, m_r a_r^{\varepsilon_r}]$$

and, since $c_s \in X$ and $a_1 \in Y$, $c_s < a_1$ so this is also standard form and again $x \in Z$. This proves that $X' \cup Y' \subseteq Z$. Conversely, any element x of z is of the standard form (4) where b and each b_i belong to $X \cup Y$. If all the a_i belong to X then $x \in X'$ and if all the b_i belong to Y then so does b, since $b > a_1$, and thus $x \in Y'$. It remains to consider the case where some of the a_i belong to X and some belong to Y. Since members of X precede those of Y, there is an integer k $(1 \leqq k < r)$ such that $a_1, a_2, \cdots, a_k \in X$ and $a_{k+1}, a_{k+2}, \cdots, a_r \in Y$. Then writing

$$c = [b, m_1 a_1^{\varepsilon_1}, m_2 a_2^{\varepsilon_2}, \cdots, m_k a_k^{\varepsilon_k}],$$

$c \in X'$ and

$$x = [c, m_{k+1} a_{k+1}^{\varepsilon_{k+1}}, \ m_{k+2} a_{k+2}^{\varepsilon_{k+2}}, \cdots, m_r a_r^{\varepsilon_r}]$$

is thus a member of Y'. Hence $Z \subseteq X' \cup Y'$ and the lemma is proved.

3. Proof of the theorem

Let X_0 be the subset of F consisting of all basic commutators of weight two, that is, all elements of the form (1) with $r = 0$; let Y_0 be the subset of F consisting of all elements of the form (1) with $r \geqq 1$. By Lemma 2, the elements of $X_0 \cup Y_0$ are distinct as written and freely generate F_2; we will assume them to be well-ordered in any way which preserves weight. Under this order, elements of X_0 precede those of Y_0.

Subsets X_n and Y_n of F are now defined inductively for all positive integers n: assuming that the subsets X_{n-1} and Y_{n-1} have already been defined and well-ordered in such a way that element of X_{n-1} precede those of Y_{n-1} and that the members of $X_{n-1} \cup Y_{n-1}$ are distinct as written and freely generate $\delta^{n-1}F_2$, let X_n be the set of all elements of the standard form with $b \in X_{n-1} \cup Y_{n-1}$ and each $a_i \in X_{n-1}$ and let Y_n be the set of all elements of the standard form with $b \in Y_{n-1} \cup X_n$ and each $a_i \in Y_{n-1}$. It follows from Lemma 3 that the elements of $X_n \cup Y_n$ are distinct as written and freely generate $\delta^n F_2$. Well-order $X_n \cup Y_n$ in any way such that elements of X_n precede those of Y_n.

Defining, for each non-negative integer n, A_n and B_n to be the subgroups of F generated by X_n and Y_n respectively, it follows that $\delta^n F_2$ is the free product of A_n and B_n.

It is now clear that, in the union of all the sets $X_n \cup Y_n$, elements are distinct as written. The orders defined, for each individual n, on the sets $X_n \cup Y_n$ may now be extended to their union by specifying that whenever $m < n$, elements of $X_m \cup Y_m$

precede those of $X_n \cup Y_n$. In particular, elements of B_n precede those of A_{n+1}.

An argument similar to the proof of Lemma 3 yields the fact that the free generating set for F_3 given in Lemma 1 (i) is just $Y_0 \cup X_1$. Lemma 3 then shows by induction that, for any non-negative integer n, $\delta^n F_3$ is freely generated by $Y_n \cup X_{n+1}$. It follows that $\delta^n F_3$ is the free product of B_n and A_{n+1}.

References

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