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TWO REMARKS ON PQ^{ϵ}-PROJECTIVITY OF RIEMANNIAN METRICS

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Abstract. We show that PQ^{ϵ}-projectivity of two Riemannian metrics introduced in [15] (P. J. Topalov, Geodesic compatibility and integrability of geodesic flows, *J. Math. Phys.* **44**(2) (2003), 913–929.) implies affine equivalence of the metrics unless $\epsilon \in \{0, -1, -3, -5, -7, ...\}$. Moreover, we show that for $\epsilon = 0$, PQ^{ϵ}-projectivity implies projective equivalence.

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1. Introduction.

1.1. PQ^{ϵ}-projectivity of Riemannian metrics. Let $g, \bar{g}$ be two Riemannian metrics on an m-dimensional manifold M. Consider $(1, 1)$-tensors P, Q that satisfy

$$\begin{aligned} g(P., .) = -g(., P.), \quad g(Q., .) = -g(., Q.) \\ \bar{g}(P., .) = -\bar{g}(., P.), \quad \bar{g}(Q., .) = -\bar{g}(., Q.) \\ PQ = \epsilon Id, \end{aligned} \tag{1}$$

where Id is the identity on TM and ϵ is a real number, $\epsilon \neq 1, m+1$. The following definition was introduced in [**15**].

DEFINITION 1. The metrics $g, \bar{g}$ are called PQ^{ϵ}-projective if for a certain 1-form Φ the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ satisfy

$$\bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \tag{2}$$

for all vector fields X, Y.

EXAMPLE 1. If the two metrics g and $\bar{g}$ are *affinely equivalent*, i.e. $\nabla = \bar{\nabla}$, then these are PQ^{ϵ}-projective with P, Q, ϵ arbitrary and $\Phi \equiv 0$.

EXAMPLE 2. Suppose that $\Phi(P.) = 0$ or $Q = 0$ and $\epsilon = 0$. It follows that equation (2) becomes

$$\bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X. \tag{3}$$

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By Levi-Civita [**4**], equation (3) is equivalent to the condition that g and $\bar{g}$ have the same geodesics considered as unparametrised curves, i.e. g and $\bar{g}$ are *projectively equivalent*. The theory of projectively equivalent metrics has a very long tradition in differential geometry, see for example [**5**, **6**, **8**, **10**, **13**] and the references therein.

EXAMPLE 3. Suppose that $P = Q = J$ and $\epsilon = -1$. It follows that J is an almost complex structure, i.e. $J^2 = -Id$, and by equation (1) the metrics g and $\bar{g}$ are required to be Hermitian with respect to J. Equation (2) now reads

$$\bar{\nabla}_X Y - \nabla_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(JX)JY - \Phi(JY)JX. \tag{4}$$

This equation defines the *h-projective equivalence* of the Hermitian metrics g and $\bar{g}$, and was introduced for the first time by Otsuki and Tashiro in [**12**, **14**] for the Kaehlerian metrics. The theory of h-projectively equivalent metrics was introduced as an analog of projective geometry in the Kèhlerian situation and has been studied actively over the years, see for example [**1–3**, **7**, **11**] and the references therein.

REMARK 1. PQ^ϵ-projectivity of the Riemannian metrics is a special case of the so-called F-planar mappings introduced and investigated in [**9**], whose defining equation, i.e. equation (1) in [**9**] clearly generalises equation (2) above.

1.2. Results. The aim of our paper is to give a proof of the following two theorems.

THEOREM 1. *Let Riemannian metrics g and $\bar{g}$ be PQ^ϵ-projective. If g and $\bar{g}$ are not affinely equivalent, the number ϵ is either zero or an odd negative integer, i.e.* $\epsilon \in \{0, -1, -3, -5, -7, ...\}$.

THEOREM 2. *Let Riemannian metrics g and $\bar{g}$ be PQ^ϵ-projective. If $\epsilon = 0$ then g and $\bar{g}$ are projectively equivalent.*

1.3. Motivation and open questions. As was shown in [**15**], PQ^ϵ-projectivity of the metrics $g, \bar{g}$ allows us to construct a family of commuting integrals for the geodesic flow of g (see Fact 2 and equation (9)). The existence of these integrals is an interesting phenomenon on its own. Besides, it appeared to be a powerful tool in the study of projectively equivalent and h-projectively equivalent metrics (Examples 2 and 3), see [**3**, **5–8**]. Moreover, it was shown in [**15**] that given one pair of PQ^ϵ-projective metrics, one can construct an infinite family of PQ^ϵ-projective metrics. Under some non-degeneracy condition, this gives rise to an infinite family of integrable flows.

From the other side, the theories of projectively equivalent and h-projectively equivalent metrics appeared to be very useful mathematical theories of deep interest.

The results in our paper suggest to look for other examples in the case when $\epsilon = -1, -3, -5, ...$ If $\epsilon = -1$ but $P^2 \neq -Id$, a lot of examples can be constructed using the 'hierarchy construction' from [**15**]. It is interesting to ask whether every pair of PQ^{-1}-projective metrics is in the hierarchy of some h-projectively equivalent metrics?

Another attractive problem is to find interesting examples for $\epsilon = -3, -5, ...$ Besides the relation to integrable systems provided by [**15**], one could find other branches of differential geometry of similar interest as projective or h-projective geometry.

1.4. PDE for PQ^ϵ-projectivity. Given a pair of Riemannian metrics $g, \bar{g}$ and tensors P, Q satisfying equation (1), we introduce the $(1,1)$-tensor $A = A(g, \bar{g})$ defined by

$$A = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{m+1-\epsilon}} \bar{g}^{-1} g. \tag{5}$$

Here we view the metrics as vector bundle isomorphisms $g : TM \to T^*M$ and $\bar{g}^{-1} : T^*M \to TM$. We see that A is non-degenerate and self-adjoint with respect to g and $\bar{g}$. Moreover, A commutes with P and Q.

FACT 1. (Lemma 2 in [**15**], see also Theorems 5 and 6 in [**9**]). *Two metrics g and $\bar{g}$ are PQ^ϵ-projective if for a certain vector field Λ, the $(1,1)$-tensor A defined in (5) is a solution of*

$$\begin{aligned}(\nabla_X A)Y = g(Y, X)\Lambda + g(Y, \Lambda)X + g(Y, QX)P\Lambda \\ + g(Y, P\Lambda)QX \text{ for all } X, Y \in TM.\end{aligned} \tag{6}$$

Conversely, if A is a g-self-adjoint positive solution of (6), which commutes with P and Q, the Riemannian metric

$$\bar{g} = (\det A)^{-\frac{1}{1-\epsilon}} g A^{-1}$$

is PQ^ϵ-projective to g.

REMARK 2. Taking the trace of the $(1,1)$-tensors in equation (6) acting on the vector field Y, we obtain

$$\Lambda = \frac{1}{2(1-\epsilon)} \operatorname{grad} \operatorname{trace} A. \tag{7}$$

Hence, (6) is a linear first-order PDE on the $(1,1)$-tensor A.

REMARK 3. From Fact 1 it follows that the metrics $g, \bar{g}$ are affinely equivalent if and only if $\Lambda \equiv 0$ on the whole M.

REMARK 4. Relation between the 1-form Φ in equation (2) and the vector field Λ in equation (6) is given by $\Lambda = -Ag^{-1}\Phi$ (again $g^{-1} : T^*M \to TM$ is considered as a bundle isomorphism), see [**15**]. Recall from Example 2 that projective equivalence is a special case of PQ^ϵ-projectivity with $\Phi(P.) = 0$ or $Q = 0$ and $\epsilon = 0$. In view of Fact 1, we now have that g and $\bar{g}$ are projectively equivalent if and only if $A = A(g, \bar{g})$ given by equation (5) (with $\epsilon = 0$), satisfies equation (6) with $P\Lambda = 0$ or $Q = 0$, i.e.

$$(\nabla_X A)Y = g(Y, X)\Lambda + g(Y, \Lambda)X \text{ for all } X, Y \in TM. \tag{8}$$

2. Proof of the results.

2.1. Topalov's integrals. We first recall the following.

FACT 2. (Proposition 3 in [**15**]). *Let g and $\bar{g}$ be PQ^ϵ-projective metrics and let A be defined by (5). We identify TM with T^*M by g, and consider the canonical symplectic*

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structure on $TM \cong T^*M$. *Then the functions* $F_t : TM \to \mathbb{R}$,

$$F_t(X) = |\det(A - tId)|^{\frac{1}{1-\epsilon}} g((A - tId)^{-1}X, X), \quad X \in TM \tag{9}$$

are commuting quadratic integrals for the geodesic flow of g.

REMARK 5. Note that the function F_t in equation (9) is not defined in the points $x \in M$ such that $t \in \operatorname{spec} A_{|x}$. It will be clear from the proof of Theorem 1 that in the nontrivial case one can extend the functions F_t to these points as well.

2.2. Proof of Theorem 1. Suppose that g and $\bar{g}$ are PQ^ϵ-projective Riemannian metrics, and let $A = A(g, \bar{g})$ be the corresponding solution of equation (6) defined by equation (5). Since A is self-adjoint with respect to the positively definite metric g, the eigenvalues of A in every point $x \in M$ are real numbers. We denote these by $\mu_1(x) \leq \cdots \leq \mu_m(x)$; depending on the multiplicity, some of the eigenvalues might coincide. The functions μ_i are continuous on M. Denote by $M^0 \subseteq M$ the set of points where the number of different eigenvalues of A is maximal on M. Since the functions μ_i are continuous, M^0 is open in M. Moreover, it was shown in [**15**] that M^0 is dense in M as well. The implicit function theorem now implies that μ_i are differentiable functions on M^0.

From Remark 3 and equation (7) we immediately obtain that g and $\bar{g}$ are affinely equivalent if and only if all eigenvalues of A are constant. Suppose that g and $\bar{g}$ are not affinely equivalent, that is there is a non-constant eigenvalue ρ of A with multiplicity $k \geq 1$. Let us choose a point $x_0 \in M^0$ such that $d\rho_{|x_0} \neq 0$, define $c := \rho(x_0)$ and consider the hypersurface $H = \{x \in U : \rho(x) = c\}$, where $U \subseteq M^0$ is a geodesically convex neighbourhood of x_0. We think that U is sufficiently small such that $\mu(x) \neq c$ for all eigenvalues μ of A different from ρ and all $x \in U$.

LEMMA 1. *There is a smooth nowhere vanishing* $(0, 2)$-*tensor* T *on* U *such that on* $U \setminus H$, T *coincides with*

$$\operatorname{sgn}(\rho - c)|\det(A - cId)|^{\frac{1}{k}} g((A - cId)^{-1}., .). \tag{10}$$

Proof. Let us denote by $\rho = \rho_1, \rho_2, ..., \rho_r$ different eigenvalues of A on M^0 with multiplicities $k = k_1, k_2, ..., k_r$, respectively. Since the eigenspace distributions of A are differentiable on M^0, we can choose a local frame $\{U_1, ..., U_m\}$ on U such that g and A are given by matrices

$$g = \operatorname{diag}(1, ..., 1) \text{ and } A = \operatorname{diag}(\underbrace{\rho, ..., \rho}_{k \text{ times}}, ..., \underbrace{\rho_r, ..., \rho_r}_{k_r \text{ times}})$$

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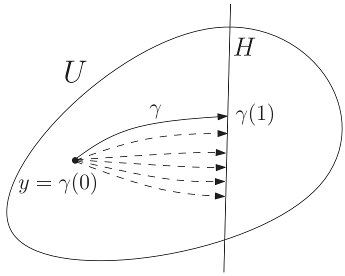

Figure 1. Case $\frac{1}{1-\epsilon} - \frac{1}{k} > 0$: We connect the point $y \in U \setminus H$ with the points in H by geodesics. The value of the integral F_c is zero on each of these geodesics.

with respect to this frame. The tensor (10) can now be written as

$$\begin{aligned}
&\operatorname{sgn}(\rho - c)|\det(A - cId)|^{\frac{1}{k}} g(A - cId)^{-1} = \\
&(\rho - c)\prod_{i=2}^{r} |\rho_i - c|^{\frac{k_i}{k}} \operatorname{diag}\Big(\underbrace{\frac{1}{\rho - c}, \dots, \frac{1}{\rho - c}}_{k \text{ times}}, \dots, \underbrace{\frac{1}{\rho_r - c}, \dots, \frac{1}{\rho_r - c}}_{k_r \text{ times}}\Big) \\
&= \prod_{i=2}^{r} |\rho_i - c|^{\frac{k_i}{k}} \operatorname{diag}\Big(\underbrace{1, \dots, 1}_{k \text{ times}}, \dots, \underbrace{\frac{\rho - c}{\rho_r - c}, \dots, \frac{\rho - c}{\rho_r - c}}_{k_r \text{ times}}\Big). \qquad (11)
\end{aligned}$$

Since $\rho_i \neq c$ on $U \subseteq M^0$ for $i = 2, \dots, r$, we see that (11) is a smooth nowhere vanishing $(0, 2)$-tensor on U. □

LEMMA 2. *The multiplicity of the non-constant eigenvalues of A is equal to $1 - \epsilon$.*

Proof. Let us consider the integral $F_c : TM \to \mathbb{R}$ defined in equation (9). Using the tensor T from Lemma 1, we can write F_c as

$$F_c(X) = \underbrace{\operatorname{sgn}(\rho - c)|\det(A - cId)|^{\frac{1}{1-\epsilon} - \frac{1}{k}}}_{=: f_c} T(X, X), \quad X \in TM. \qquad (12)$$

Our goal is to show that $\frac{1}{1-\epsilon} - \frac{1}{k} = 0$.

First suppose that $\frac{1}{1-\epsilon} - \frac{1}{k} > 0$ and let $y \in U \setminus H$. We choose a geodesic $\gamma : [0, 1] \to U$ such that $y = \gamma(0)$ and $\gamma(1) \in H$, see Figure 1. Since $\rho(\gamma(t)) \xrightarrow{t \to 1} c$, we see from equation (12) that $f_c(\gamma(t)) \xrightarrow{t \to 1} 0$. It follows that $F_c(\dot\gamma(t)) \xrightarrow{t \to 1} 0$. On the other hand, since F_c is an integral for the geodesic flow of g (see Fact 2), the value $F_c(\dot\gamma(t))$ is independent of t, and hence $F_c(\dot\gamma(0)) = 0$. We have shown that $F_c(\dot\gamma(0)) = 0$ for all initial velocities $\dot\gamma(0) \in T_yM$ of geodesics connecting y with points of H. Since H is a hypersurface, it follows that the quadric $\{X \in T_yM : F_c(X) = 0\}$ contains an open subset that implies $F_c \equiv 0$ on T_yM. This is a contradiction to Lemma 1, since T is non-vanishing in y. We obtain that $\frac{1}{1-\epsilon} - \frac{1}{k} \leq 0$.

Let us now treat the case when $\frac{1}{1-\epsilon} - \frac{1}{k} < 0$. We choose a vector $X \in T_{x_0}M$ which is not tangent to H and satisfies $T(X, X) \neq 0$. Such a vector exists, since $T_{x_0}M \setminus T_{x_0}H$ is open in $T_{x_0}M$ and T is not identically zero on $T_{x_0}M$ by Lemma 1. Let us consider the geodesic γ with $\gamma(0) = x_0$ and $\dot\gamma(0) = X$, see Figure 2. Since $X \notin T_{x_0}H$, the geodesic

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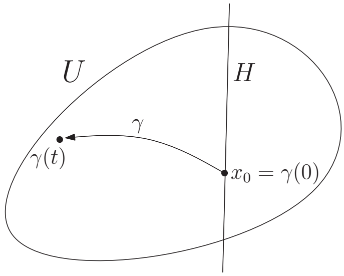

Figure 2. Case $\frac{1}{1-\epsilon} - \frac{1}{k} < 0$: For any geodesic γ starting in $x_0 \in H$ and leaving H, the value of integral F_c along this geodesic is infinite.

γ has to leave H for $t > 0$. In a point $\gamma(t) \in U \setminus H$ the value $F_c(\dot\gamma(t))$ will be finite. On the other hand, since $f_c(\gamma(t)) \overset{t\to 0}{\longrightarrow} \infty$ and $T(\dot\gamma(0), \dot\gamma(0)) \neq 0$, we have $F_c(\dot\gamma(t)) \overset{t\to 0}{\longrightarrow} \infty$. Again this contradicts the fact that the value of F_c must remain constant along $\dot\gamma$ by Fact 2. We have shown that $\frac{1}{1-\epsilon} - \frac{1}{k} = 0$, and finally Lemma 2 is proven. □

As a consequence of Lemma 2, if the metrics $g, \bar g$ are not affinely equivalent (i.e. at least one eigenvalue of A is non-constant), ϵ is an integer less or equal to zero. If $\epsilon \neq 0$, the condition $PQ = \epsilon Id$ in equation (1) implies that P is non-degenerate and by the first condition in equation (1), $g(P., .)$ is a non-degenerate 2-form on each eigenspace of A (note that A and P commute). This implies that for $\epsilon \neq 0$ the eigenspaces of A have even dimension, in particular, $1 - \epsilon \in \{2, 4, 6, 8, ...\}$. Theorem 1 is proven.

2.3. Proof of Theorem 2. Let $g, \bar g$ be two PQ^ϵ-projective metrics and let A be the corresponding solution of equation (6) defined by equation (5). As it was already stated in the proof of Theorem 1, the eigenspace distributions of A are differentiable in a neighbourhood of almost every point of M. First let us prove the following.

LEMMA 3. *Let X be an eigenvector of A corresponding to the eigenvalue ρ. If μ is another eigenvalue of A and $\rho \neq \mu$, then $X(\mu) = 0$. In particular,* grad μ *is an eigenvector of A corresponding to the eigenvalue μ.*

REMARK 6. Lemma 3 is known for projectively equivalent (Example 2) and h-projectively equivalent (Example 3) metrics. For projectively equivalent metrics, it is a classical result that was already known to Levi-Civita [**4**]. For h-projectively equivalent metrics, it follows from [**1**, **7**].

Proof. Let Y be an eigenvector field of A corresponding to the eigenvalue μ. For arbitrary $X \in TM$, we obtain $\nabla_X(AY) = \nabla_X(\mu Y) = X(\mu)Y + \mu\nabla_X Y$ and $\nabla_X(AY) = (\nabla_X A)Y + A\nabla_X Y$. Combining these equations and replacing the expression $(\nabla_X A)Y$ by equation (6) we obtain

$$(A - \mu Id)\nabla_X Y = X(\mu)Y - g(Y, X)\Lambda - g(Y, \Lambda)X - g(Y, QX)P\Lambda - g(Y, P\Lambda)QX. \tag{13}$$

Now let X be an eigenvector of A corresponding to the eigenvalue ρ and suppose that $\rho \neq \mu$. Since A is g-self-adjoint, the eigenspaces of A corresponding to different eigenvalues are orthogonal to each other. Moreover, since A and Q commute, Q leaves

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the eigenspaces of A invariant. Using equation (13) we obtain

$$(A - \mu Id)\nabla_X Y + g(Y, \Lambda)X + g(Y, P\Lambda)QX = X(\mu)Y.$$

Since the left-hand side is orthogonal to the μ-eigenspace of A, we necessarily have $X(\mu) = 0$. We have shown that $g(\operatorname{grad}\mu, X) = X(\mu) = 0$ for any eigenvalue μ and any eigenvector field X corresponding to an eigenvalue which is different from μ. This forces $\operatorname{grad}\mu$ to be contained in the eigenspace of A corresponding to μ. □

Now suppose $\epsilon = 0$. Let us denote the non-constant eigenvalues of A by $\rho_1, \ldots, \rho_l$. Using Lemma 2, the corresponding eigenspaces are 1-dimensional and Lemma 3 implies that these are spanned by the gradients $\operatorname{grad}\rho_1, \ldots, \operatorname{grad}\rho_l$ respectively. Since P and A commute, P leaves the eigenspaces of A invariant, hence $P\operatorname{grad}\rho_i = p_i\operatorname{grad}\rho_i$ for some real number p_i. Now P is skew with respect to g and we obtain $0 = g(\operatorname{grad}\rho_i, P\operatorname{grad}\rho_i) = p_i g(\operatorname{grad}\rho_i, \operatorname{grad}\rho_i)$, which implies that

$$P\operatorname{grad}\rho_i = 0.$$

On the other hand, by equation (7)

$$\Lambda = \frac{1}{2}\operatorname{grad}\operatorname{trace}A = \frac{1}{2}(\operatorname{grad}\rho_1 + \ldots + \operatorname{grad}\rho_l).$$

Combining the last two equations, we obtain $P\Lambda = 0$. It follows from Remark 4 that g and $\bar{g}$ are projectively equivalent and hence Theorem 2 is proved.

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