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TWISTS OF MATRIX ALGEBRAS AND SOME SUBGROUPS OF BRAUER GROUPS

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We consider twists of matrix algebras by some continuous characters. There are some subgroups of Brauer groups arising from these twists. Some results regarding these subgroups are proved.

1. INTRODUCTION

The idea of twists of vector spaces and algebras by 1-cocycles was considered by Weil in [**8**, pp.590–593]. In [**3**], we considered twists of central simple algebras by some special 1-cocycles which was motivated by the study of the endomorphism algebra of the abelian variety attached to a weight–2 newform on $\Gamma_0(N)$ (or on $\Gamma_1(N)$ with character ε) (see [**2, 3, 5, 6**]). There are some subgroups of Brauer groups arising from these twists. Our purpose in this article is to show some results on these subgroups of Brauer groups.

To begin with, let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of the field of rational numbers $\mathbb{Q}$, and let $G_{\mathbb{Q}} = \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ be the Galois group furnished with the Krull topology. For an arbitrary field F in $\overline{\mathbb{Q}}$, we take $\overline{\mathbb{Q}}$ as an algebraic closure of F and denote the Galois group of $\overline{\mathbb{Q}}$ over F by G_F. Throughout this article, E will be a finite Galois extension of F sitting inside $\overline{\mathbb{Q}}$ with Galois group $\mathrm{Gal}(E/F) = \Gamma$.

Let $\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$ be the group of all continuous characters from $G_{\mathbb{Q}}$ to E^*/F^*, where the multiplicative group E^*/F^* was given the discrete topology. Let $\mathrm{Br}(E/F)$ be the subgroup of the Brauer group $\mathrm{Br}(F)$ consisting of all classes of central simple F-algebras split by E. Then one has $\mathrm{Br}(E/F) \simeq H^2(\Gamma, E^*)$. The theory of twists gives a natural group homomorphism

$$\Phi\colon \mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*) \to \mathrm{Br}(E/F)$$

which can be defined by two ways (see Section 2). Let $H(E/F)$ be the image of Φ. Then $H(E/F)$ is a subgroup of $\mathrm{Br}(E/F)$. A natural question is to find out how large this group $H(E/F)$ is.

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Finally, I would like to express my hearty thanks to the referee for suggesting a neat proof for (ii).

In [**2**], we showed that if E is an abelian number field, then $H(E/F)$ is contained in $\mathrm{Br}(E/F) \cap S(F)$, where $S(F)$ is the Schur subgroup of the Brauer group $\mathrm{Br}(F)$ (see [**2**, Theorem 3.1]). In this paper, we obtain the following results:

(i) Let E be cyclic extension of F. For each $\alpha \in \mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$, there exists a positive integer N_α (depending on α) such that if the N_α-th root of unity is in F, then the class of $(\mathrm{End}_F E)(\alpha)$ is trivial. In particular, if F contains all the roots of unity in $\overline{\mathbb{Q}}$, then $H(E/F) = 0$.

(ii) If E is an arbitrary totally real number field, then $H(E/F)$ is contained in $\mathrm{Br}_2(F) \cap \mathrm{Br}(E/F)$, where $\mathrm{Br}_2(F)$ is the 2-torsion subgroup of $\mathrm{Br}(F)$.

(iii) If $F = \mathbb{Q}$ and $E = \mathbb{Q}\left(\sqrt{d}\right)$ is a real quadratic field whose fundamental unit has norm -1, then $H(E/F) = \mathrm{Br}(E/F)$.

2. Twists of matrix algebras

In this section, we shall recall some of the terminology and properties on the twists of matrix algebras. For a general reference, we refer to Sections 1, 2, and 4 in [**3**].

With the same notation as in Section 1, let $[E\colon F] = n$ and let $\mathrm{End}_F E$ be the matrix algebra of all F-linear operators of the F-vector space E. The multiplicative group E^*/F^* can be regarded as a subgroup of the automorphism group $\mathrm{Aut}_F(\mathrm{End}_F E)$ as follows.

For each e in E, e can be regarded as an element in $\mathrm{End}_F E$ via multiplication by e. In this way, E is a maximal subfield of $\mathrm{End}_F E$. For each e in E^*, one has the inner automorphism $a \longmapsto e\,a\,e^{-1}$ for all a in $\mathrm{End}_F E$. It is trivial if and only if e is in F^*.

In particular, for each α in $\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$, α can be thought of as a continuous group homomorphism $\alpha\colon G_{\mathbb{Q}} \to \mathrm{Aut}_F(\mathrm{End}_F E)$. Let $\alpha_F = \alpha|_{G_F}$. Regarding α (respectively α_F) as 1-cocycle on $G_{\mathbb{Q}}$ (respectively G_F) with values in $\mathrm{Aut}_{\overline{F}}(\mathrm{End}_F E \otimes_F \overline{F})$, we have the twists $(\mathrm{End}_F E)(\alpha)$ and $(\mathrm{End}_F E)(\alpha_F)$, which are isomorphic central simple F-algebras (see [**3**, Propositions (1.2), (2.1)]). Let $M(n, \overline{F})$ be the algebra of $n \times n$ matrices over $\overline{F}$. Then, by the Skolem—Noether theorem, one can identify $\mathrm{Aut}_{\overline{F}}(M(n, \overline{F}))$ with the projective group $PGL(n, \overline{F})$. Let

$$\Delta_n\colon H^1(G_F, PGL(n, \overline{F})) \to H^2\left(G_F, \overline{F}^*\right) \simeq \mathrm{Br}(F)$$

be the coboundary operator defined by the following exact sequence:

$$1 \to \overline{F}^* \to GL(n, \overline{F}) \to PGL(n, \overline{F}) \to 1.$$

As is well known, $H^1(G_F, GL(n, \overline{F}))$ is trivial. So Δ_n is injective. Let $[\alpha]$ be the class determined by α in $H^1(G_F, PGL(n, \overline{F}))$. Then the cohomology class

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of $(\mathrm{End}_F E)(\alpha)$ in $H^1(G_F, PGL(n, \overline{F}))$ equals $[\alpha]$. Hence the cohomology class of $(\mathrm{End}_F E)(\alpha)$ in $H^2\left(G_F, \overline{F}^*\right)$ equals $\Delta_n([\alpha])$. It is easily seen that $(\mathrm{End}_F E)(\alpha)$ is split by E. So $\Delta_n([\alpha])$ is in $H^2(\Gamma, E^*) \simeq \mathrm{Br}(E/F)$. We have the following natural map:

$$\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*) \longmapsto H^1(G_F, PGL(n, \overline{F})) \longmapsto H^2(\Gamma, E^*) \overset{\sim}{\longmapsto} \mathrm{Br}(E/F)$$

by
$$\alpha \longmapsto [\alpha] \longmapsto \Delta_n([\alpha]) \longmapsto [(\mathrm{End}_F E)(\alpha)].$$

On the other hand, suppose we are given α in $\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$. For each γ in Γ, one defines $\chi_\gamma(g) = \gamma(\dot{\alpha}(g))/\dot{\alpha}(g)$ for all g in $G_{\mathbb{Q}}$, where $\dot{\alpha}(g)$ is a lift of $\alpha(g)$ to E^*. It is obvious that each χ_γ is a well-defined E^*-valued Dirichlet character on $G_{\mathbb{Q}}$. Let $X(E) = \mathrm{Hom}(G_{\mathbb{Q}}, E^*)$ be the Γ-module of all E^*-valued Dirichlet characters. For χ in $X(E)$ of conductor f, we define the Gauss sum $\tau(\chi)$ by

$$\tau(\chi) = \sum_{1 \leqslant a \leqslant f} \chi(a)\zeta_f^a,$$

where ζ_f is the primitive f-th root of unity $e^{2\pi i/f}$ in $\overline{\mathbb{Q}}^*$. For χ, χ' in $X(E)$, the Jacobi sum $j(\chi, \chi')$ is defined by $j(\chi, \chi') = \tau(\chi)\tau(\chi')/\tau(\chi\chi')$. It is easy to check that the Jacobi sum takes values in E^* (see [**3**, Section 4]).

The map $\theta_\alpha : \gamma \mapsto \chi_\gamma$ is easily seen to be a 1-cocycle on Γ with values in $X(E)$ (that is an element of $H^1(\Gamma, X(E))$). As is well known [**3**, Proposition (4.5)], the map $j : \Gamma \times \Gamma \to E^*$ defined by $j(\gamma, \delta) = j(\chi_\gamma^{-1}, \chi_\delta^{-\gamma})$ is a 2-cocycle in $H^2(\Gamma, E^*)$. Further, $(\mathrm{End}_F E)(\alpha)$ is isomorphic to the crossed product algebra $(E/F, j)$. We then have the following natural map

$$\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*) \longmapsto H^1(\Gamma, X(E)) \longmapsto H^2(\Gamma, E^*) \overset{\sim}{\longmapsto} \mathrm{Br}(E/F)$$

by
$$\alpha \longmapsto (\theta_\alpha : \gamma \longmapsto \chi_\gamma) \longmapsto j = j(\chi_\gamma^{-1}, \chi_\delta^{-\gamma}) \longmapsto [(E/F, j)].$$

PROPOSITION 2.1. *The following diagram defined as above commutes.*

$$\begin{tikzcd}[column sep=small] & H^1(G_F, PGL(n, \overline{F})) \arrow[r] & H^2(\Gamma, E^*) \arrow[dr] & \\ \mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*) \arrow[ur] \arrow[dr] & & & \mathrm{Br}(E/F) \\ & H^1(\Gamma, X(E)) \arrow[r] & H^2(\Gamma, E^*) \arrow[ur] & \end{tikzcd}$$

PROOF: This follows immediately from Theorem (4.8) of [**3**]. ∎

Let us denote the composite map of the above diagram by Φ.

PROPOSITION 2.2. *Φ is a group homomorphism.*

PROOF: (via upper arrows) It suffices to show that $\Delta_n([\alpha\beta]) = \Delta_n([\alpha])\Delta_n([\beta])$ in $H^2(\Gamma, E^*)$ for all α, β in $\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$. This follows directly from the definition of Δ_n.

(via lower arrows) It suffices to show that $\theta_{\alpha\beta} = \theta_\alpha \theta_\beta$ in $H^1(\Gamma, X(E))$ for all α, β in $\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$. This follows from

$$\gamma(\dot{\alpha}\beta(g))/\dot{\alpha}\beta(g) = (\gamma(\dot{\alpha}(g))/\dot{\alpha}(g))\left(\gamma\left(\dot{\beta}(g)\right)/\dot{\beta}(g)\right)$$

for all γ in Γ and g in $G_{\mathbb{Q}}$.

Let $H(E/F)$ be the image of Φ. Then $H(E/F)$ is a subgroup of $\mathrm{Br}(E/F)$. In general, it appears to be not obvious to see how large this group $H(E/F)$ is.

3. E is an arbitrary totally real number field

Theorem 3.1. *If E is a totally real number field, then the group $H(E/F)$ is contained in $\mathrm{Br}_2(F) \cap \mathrm{Br}(E/F)$.*

Proof: Let α be in $\mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$. It is clear that α is of finite order. For each $g \in G_{\mathbb{Q}}$, $(\dot{\alpha}(g))^k \in F$ for some integer $k \geq 1$.

Since E is totally real and E is a finite Galois extension over F, the torsion part of E^*/F^* has exponent 2. Hence one must have $(\dot{\alpha}(g))^2 \in F$ and α is of order 1 or 2. Consequently, $(\mathrm{End}_F\, E)(\alpha)$ is of index 1 or 2. ∎

Corollary 3.2. *If E is a totally real number field and $F = \mathbb{Q}$, then $H(E/F)$ is contained in $S(Q) \cap \mathrm{Br}(E/Q)$.*

Proof: This follows from Theorem 3.1 and a well known theorem of Benard-Fields [**9**, Theorem 7.2]. ∎

Remark. Corollary 3.2 covers the case where E is a totally real and abelian number field in Theorem 3.1 of [**2**].

4. E is a cyclic extension of F

In this section, we assume that E is a cyclic extension of F with $[E\colon F] = n$ and $\Gamma = \langle\sigma\rangle$.

Note that the 2-cocycle $j(\gamma, \wedge) = j\left(\chi_\gamma^{-1}, \chi_\wedge^{-\gamma}\right)$ in $H^2(\Gamma, E^*)$ defined by the Jacobi sum is a normalised factor set in the sense of

$$j(\mathrm{id.}, \gamma) = j(\gamma, \mathrm{id.}) = 1 \text{ for all } \gamma \text{ in } \Gamma.$$

Lemma 4.1. $(\mathrm{End}_F\, E)(\alpha)$ *is isomorphic to the cyclic algebra $(E/F, \sigma, a)$, where $a = \prod_{i=0}^{n-1} \tau\left(\chi_\sigma^{-\sigma^i}\right)$ is an algebraic integer in F.*

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PROOF: By $(\mathrm{End}_F E)(\alpha) \simeq (E/F, j)$, we may write

$$(\mathrm{End}_F E)(\alpha) \simeq \bigoplus_{i=0}^{n-1} E v_{\sigma^i},$$

where $$v_{\sigma^i} \cdot x = \sigma^i(x) \cdot v_{\sigma^i}, \qquad x \in E,$$

and $$v_{\sigma^i} \cdot v_{\sigma^j} = j(\sigma^i, \sigma^j) v_{\sigma^{i+j}}, \qquad \text{for } 0 \le i,\ j \le n-1.$$

Then we have

$$\begin{aligned} v_\sigma^2 &= v_\sigma \cdot v_\sigma = j(\sigma, \sigma) \cdot v_{\sigma^2}, \\ v_\sigma^3 &= (j(\sigma, \sigma) v_{\sigma^2}) \cdot v_\sigma = j(\sigma, \sigma) j(\sigma^2, \sigma) \cdot v_{\sigma^3}, \\ &\vdots \\ v_\sigma^n &= \prod_{i=0}^{n-1} j(\sigma^i, \sigma) \cdot v_{\sigma^n} = \prod_{i=0}^{n-1} j(\sigma^i, \sigma). \end{aligned}$$

Hence

$$(\mathrm{End}_F E)(\alpha) \simeq \bigoplus_{i=0}^{n-1} E \cdot v_\sigma^i,$$

where $$v_\sigma \cdot x = \sigma(x) \cdot v_\sigma, \ x \in E,$$

and $$v_\sigma^n = \prod_{i=0}^{n-1} j(\sigma^i, \sigma).$$

Since v_σ^n lies in the centre of $\bigoplus_{i=0}^{n-1} E \cdot v_\sigma^i$, so $\prod_{i=0}^{n-1} j(\sigma^i, \sigma) \in F^*$.

On the other hand, by definition and the 1-cocycle property of θ_α, we have

$$\begin{aligned} \prod_{i=0}^{n-1} j(\sigma^i, \sigma) &= \prod_{i=0}^{n-1} \frac{\tau(\chi_{\sigma^i}^{-1}) \tau\left(\chi_\sigma^{-\sigma^i}\right)}{\tau\left(\chi_{\sigma^i}^{-1} \chi_\sigma^{-\sigma^i}\right)} \\ &= \prod_{i=0}^{n-1} \frac{\tau(\chi_{\sigma^i}^{-1}) \tau\left(\chi_\sigma^{-\sigma^i}\right)}{\tau(\chi_{\sigma^{i+1}}^{-1})} \\ &= \prod_{i=0}^{n-1} \tau\left(\chi_\sigma^{-\sigma^i}\right) = a, \end{aligned}$$ which *a priori* is an algebraic

integer in $\overline{\mathbb{Q}}^*$. Since $\prod_{i=0}^{n-1} j(\sigma^i, \sigma) \in F^*$, so a is an algebraic integer in F. ∎

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Lemma 4.2. *The algebra* $(\mathrm{End}_F E)(\alpha)$ *is of index* 1 *if and only if* $a \in N_{E/F}(E^*)$.

Proof: This follows from a well known fact (see [**4**, Theorem 30.4]) on cyclic algebras. ∎

Theorem 4.3. *For each* $\alpha \in \mathrm{Hom}_{\mathrm{con.}}(G_{\mathbb{Q}}, E^*/F^*)$, *there exists a positive integer* N_α *(depending on* α*) such that if* $\zeta_{N_\alpha} \in F$, *then* $(\mathrm{End}_F E)(\alpha)$ *is a matrix algebra over* F.

Proof: Let N_α be the product of all the conductors of the finite set of Dirichlet characters $\{\chi_\gamma\}_{\gamma \in \Gamma}$ defined by α. Under the hypothesis, $a = \prod_{i=0}^{n-1} \tau\left(\chi_\sigma^{-\sigma^i}\right) = N_{E/F}\left(\tau\left(\chi_\sigma^{-1}\right)\right)$. Our assertion follows from the preceding lemmas. ∎

Corollary 4.4. *If* F *contains all the roots of unity in* $\overline{\mathbb{Q}}$, *then* $H(E/F) = 0$.

Proof: This follows immediately from Theorem 4.3. ∎

Remark. If F contains all the roots of unity in $\overline{\mathbb{Q}}$, it is well known that $\mathrm{Br}(F) = 0$ (see [**7**, p.162]). From this, one certainly has $H(E/F) = 0$. However, our proof here is elementary.

5. $F = \mathbb{Q}$ and $E = \mathbb{Q}\left(\sqrt{d}\right)$ is a real quadratic field

In this section, we assume that $F = \mathbb{Q}$ and $E = \mathbb{Q}\left(\sqrt{d}\right)$ is a real quadratic field whose fundamental unit has norm -1. It is well known that $S(\mathbb{Q}) = \mathrm{Br}_2(\mathbb{Q})$ [**9**, Theorem 7.2]. So, it is clear that $H\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right)$ is a subgroup of $S(\mathbb{Q})$. (In [**2**], this was proved for arbitrary abelian extensions of $\mathbb{Q}$.)

Theorem 5.1. $H\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right) = \mathrm{Br}\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right)$.

Proof: Let $\mathrm{Gal}\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right) = \{\sigma, \mathrm{id.}\}$. Under our assumption, the $\mathbb{Q}\left(\sqrt{d}\right)^*$-valued character χ_σ is a quadratic character. By Lemma (4.1), each class in $H\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right)$ is represented by a cyclic algebra of the form $\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}, \sigma, \tau\left(\chi_\sigma^{-1}\right)^2\right)$. The value $\tau\left(\chi_\sigma^{-1}\right)^2$ is a well known integer for all quadratic characters (see [**1**, pp.349, Theorem 7]).

On the other hand, by $\mathrm{Br}\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right) \simeq \mathbb{Q}^*/N\left(\mathbb{Q}\left(\sqrt{d}\right)^*\right)$, where N is the norm of $\mathbb{Q}\left(\sqrt{d}\right)$ to $\mathbb{Q}$, each class in $\mathrm{Br}\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right)$ is represented by a cyclic algebra of the form $\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}, \sigma, b/a\right)$, where a, b are relatively prime integers.

We may write b/a as the following three possible forms:

(i) $b/a = \pm(d/c)^2$, where $c, d \in \mathbb{Z}$. In this case, since $b/a \in N\left(\mathbb{Q}\left(\sqrt{d}\right)^*\right)$,

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so the class of $\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q},\, \sigma,\, b/a\right)$ in $\mathrm{Br}\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right)$ is trivial.

(ii) $b/a = \pm(d/c)^2 \cdot 1/(p_1 \cdots p_t)$, where $c, d \in \mathbb{Z}$ and $p_1, \cdots, p_t$ are distinct prime numbers. In this case, one can find an $\alpha \in \mathrm{Hom}_{\mathrm{con.}}\left(G_{\mathbb{Q}},\, \mathbb{Q}\left(\sqrt{d}\right)^*/\mathbb{Q}^*\right)$ such that χ_σ is a primitive quadratic character of conductor $p_1 \cdots p_t$ with $\tau\left(\chi_\sigma^{-1}\right)^2 = \pm[(c/d)^2 \cdot (p_1 \cdots p_t)^2] \cdot b/a$. Since $\pm[(c/d)^2 \cdot (p_1 \cdots p_t)^2] \in N\left(\mathbb{Q}\left(\sqrt{d}\right)^*\right)$, the cyclic algebra $\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q},\, \sigma,\, \tau\left(\chi_\sigma^{-1}\right)^2\right)$ is isomorphic to $\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q},\, \sigma,\, b/a\right)$.

(iii) $b/a = \pm(d/c)^2 \cdot (q_1 \cdots q_s)/(p_1 \cdots p_t)$, where $c, d \in \mathbb{Z}$ and $q_1, \cdots, q_s$, $p_1, \cdots, p_t$ are distinct prime numbers. In this case, one can find an $\alpha \in \mathrm{Hom}_{\mathrm{con.}}\left(G_{\mathbb{Q}},\, \mathbb{Q}\left(\sqrt{d}\right)^*/\mathbb{Q}^*\right)$ such that χ_σ is a primitive quadratic character of conductor $p_1 \cdots p_t \cdot q_1 \cdots q_s$ with $\tau\left(\chi_\sigma^{-1}\right)^2 = \pm[(c/d)^2 \cdot (p_1 \cdots p_t)^2] \cdot b/a$. Again, the cyclic algebra $\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q},\, \sigma,\, \tau\left(\chi_\sigma^{-1}\right)^2\right)$ is isomorphic to $\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q},\, \sigma,\, b/a\right)$.

From (i), (ii), (iii), one concludes that $H\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right) = \mathrm{Br}\left(\mathbb{Q}\left(\sqrt{d}\right)/\mathbb{Q}\right)$. ∎

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