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ON CLASSICAL KRULL DIMENSION OF GROUP-GRADED RINGS

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For any ring R graded by a finite group, we give a bound on the classical Krull dimension of R in terms of the dimension of the initial component R_e. It follows that if R_e has finite classical Krull dimension, then the same is true of the whole ring R, too.

Let G be a finite group with identity e. A ring R is said to be *G-graded* if $R = \bigoplus_{g \in G} R_g$ is a direct sum of additive subgroups R_g and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

There are many results relating properties of a group-graded ring $R = \bigoplus_{g \in G} R_g$ and its initial component R_e, where e is the identity of the group (see [**5**, **7**, **8**] and [**9**]). Ring-theoretic dimensions of group-graded rings have been considered by several authors (see, for example, Bell [**1**], Chin and Quinn [**2**], Cohen and Montgomery [**3**], Năstăsescu [**6**]).

Rings with Krull dimension form an important class and have many nice properties (see [**5**]). Suppose that the set $S = \operatorname{Spec}(R)$ of prime ideals of R satisfies a.c.c. Define the sets S_α inductively. Let S_0 be the set of all maximal elements in S; and for each ordinal α denote by S_α the set of all $s \in S$ such that $t \in S$, $t > s$ implies $t \in S_\beta$ for some $\beta < \alpha$. Then there exists the least ordinal α such that $S_\alpha = S$. This ordinal is called the *classical Krull dimension* of R. If it is finite, then it is also equal to the right Krull dimension of R defined on the lattice of right ideals of R (see [**5**, Chapter 6]).

Denote by $\text{cl-K-dim}(R)$ the classical Krull dimension of R. For any ordinal α and positive integer n, we introduce ordinals α_n, setting $\alpha_1 = \alpha + 1$, $\alpha_{n+1} = (\alpha+1)(\alpha_n + 1)$. We shall use the results on prime ideals due to Cohen and Montgomery [**3**] and prove the following theorem.

THEOREM 1. *Let G be a finite group with identity e and $|G| = n$, and let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring. If R_e has classical Krull dimension α, then R has classical Krull dimension, too, and* $\text{cl-K-dim}(R) < \alpha_n$.

This theorem is related to an open question [**4**, Problem 5].

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It is interesting to note that the analogous assertion is not valid for Krull dimension defined on the lattice of right ideals. Indeed, if we take any group G with identity e and an element $g \neq e$ in G, and take a ring R with zero multiplication which has no Krull dimension, then we can view R as a group-graded ring with $R_e = 0$, $R_g = R$, and $R_h = 0$ for all $h \in G \backslash \{e, g\}$. [**2**, Example 2.4] shows that our theorem does not transfer to rings graded by infinite groups, even in the case of the infinite cyclic group.

We need the following lemma (see [**3**, Theorems 7.1 and 7.3], or [**9**, Theorem 17.9]).

LEMMA 2. [**3**]. *Let G be a finite group with identity e, and let R be a G-graded ring.*

(i) *If P is a prime ideal of R, then there exist $n \leqslant |G|$ primes $Q_1, Q_2, \dots, Q_n$ of R_e minimal over $P \cap R_e$, and we have $P \cap R_e = Q_1 \cap Q_2 \cap \dots \cap Q_n$.*

(ii) *If $P \subseteq Q$ are prime ideals of R and $P \neq Q$, then $P \cap R_e \neq Q \cap R_e$.*

PROOF OF THEOREM 1: Suppose to the contrary that R contains a strictly increasing chain of prime ideals $P_1 \subset P_2 \subset \dots \subset P_{\alpha_n}$. Lemma 2(i) tells us that, for each $\gamma \leqslant \alpha_n$, there exists a finite set S_γ of prime ideals of R_e minimal over $R_e \cap P_\gamma$ and such that

$$\bigcap_{P \in S_\gamma} P = R_e \cap P_\gamma$$

and $|S_\gamma| \leqslant |G| = n$.

Put $S = \bigcup_{\gamma \leqslant \alpha_n} S_\gamma$. If a prime ideal contains an intersection of a finite number of ideals, then it contains at least one of them. Therefore, for any $\delta < \varepsilon \leqslant \alpha_n$ and $P \in S_\varepsilon$, there exists $Q \in S_\delta$, such that $Q \subseteq P$.

For $\delta < \varepsilon \leqslant \alpha_n$, $Q \in S_\delta$, and $P \in S_\varepsilon$, we shall write

$$Q \ll P$$

if and only if, for all μ, $\delta < \mu < \varepsilon$, we can fix $I_\xi \in S_\mu$ so that $I_\mu \subseteq I_\nu$ whenever $\delta \leqslant \mu \leqslant \nu \leqslant \varepsilon$, where $I_\delta = Q$ and $I_\varepsilon = P$.

We shall show by induction on $\gamma \leqslant \alpha_n$ that, for each $P \in S_\gamma$, there exists $Q \in S_1$ such that $Q \ll P$. The case of $\gamma = 1$ is trivial. Suppose that this has been proved for all $\delta < \gamma$. Take any ideal $P \in S_\gamma$.

If γ is not a limit ordinal, then there exists $\gamma - 1$ and we can take $P' \in S_{\gamma-1}$ such that $P' \subseteq P$. By the induction assumption $Q \ll P'$ for some $Q \in S_1$. It follows that $Q \ll P$.

Consider the case where γ is a limit ordinal. Denote by L the set of all $Q \in \bigcup_{\delta < \gamma} S_\delta$ such that $Q \subseteq P$. By induction on δ we shall define ideals $Q_\delta \in S_\delta$, for all $\delta < \gamma$.

Given that S_1 is finite and every ideal Q, where $Q \in S_\nu \cap L \neq \emptyset$, $1 < \nu \leqslant \gamma$, contains at least one ideal of $S_1 \cap L$, it follows that there exists $Q_1 \in S_1 \cap L$ such that for any $\mu < \gamma$ we can find $\mu < \nu < \gamma$ and $Q \in S_\nu$ satisfying $Q_1 \ll Q$. By the definition of $\ll$, for any $\mu < \gamma$, we can find $Q \in S_\mu$ satisfying $Q_1 \ll Q$. Put

$$L_1 = L \cap \Big\{ Q \in \bigcup_{1<\nu<\gamma} S_\nu \mid Q_1 \ll Q \Big\}.$$

We have ensured that L_1 intersects all S_ν, for $1 < \nu < \gamma$.

Suppose that for some $\delta < \gamma$ ideals Q_ε have been defined for all $\varepsilon < \delta$, and suppose that these ideals form an ascending chain. In addition, assume that the sets

$$L_\varepsilon = L \cap \Big\{ Q \in \bigcup_{\varepsilon<\nu<\gamma} S_\nu \mid Q_\varepsilon \ll Q \Big\}$$

intersect all S_ν for $\varepsilon < \nu < \gamma$. Obviously, $M \subseteq L$ and $M = \bigcap_{\varepsilon<\delta} L_\varepsilon \cap S_\delta$ is not empty, because all sets $L_1 \supseteq L_2 \supseteq \dots \supseteq L_\delta \supseteq \dots$ are nonempty. As in the paragraph above, given that M is finite, there exists $Q_\delta \in M$ such that for any $\delta < \mu < \gamma$ we can find $\mu < \nu < \gamma$ and $Q \in S_\nu$ satisfying $Q_\delta \subseteq Q$. Thus the ascending chain of ideals Q_δ, $\delta < \gamma$, has been defined.

Since $Q_\delta \subseteq P$ for all $\delta < \gamma$, we see that $Q_1 \ll P$, as required.

Next, we are going to reduce the set S. Take any $P^{(1)} \in S_{\alpha_n}$ and fix a chain of ideals $P^{(1)}_\gamma \in S_\gamma$ such that $P^{(1)}_\mu \subseteq P^{(1)}_\nu$ for all $\mu \leqslant \nu \leqslant \alpha_n$. Given that cl-K-dim$(R_e) = \alpha$ and $\alpha_n = (\alpha+1)(\alpha_{n-1}+1)$, there exists $0 \leqslant \delta < \alpha_n$ such that

$$P^{(1)}_{\delta+1} = P^{(1)}_{\delta+2} = \dots = P^{(1)}_{\delta+\alpha_{n-1}+1} \subseteq R_e.$$

Put $S^{(1)}_\mu = S_\mu \backslash \{P^{(1)}_\mu\}$ for $\delta \leqslant \mu \leqslant \delta + \alpha_{n-1} + 1$, and

$$S^{(1)} = \bigcup_{\delta<\mu\leqslant\delta+\alpha_{n-1}} S^{(1)}_\mu.$$

For any $\delta < \mu < \nu \leqslant \delta + \alpha_{n-1} + 1$, and any ideal $I \in S^{(1)}_\nu$ there exists $Q \in S_\mu$ such that $Q \subseteq I$. If $Q \notin S^{(1)}_\mu$, then $Q = P^{(1)}_\mu = P^{(1)}_\nu$; whence $I \supseteq P^{(1)}_\nu$, a contradiction. Therefore $Q \in S^{(1)}_\mu$.

Thus $S^{(1)}$ satisfies the same property we used for S, but now $\left|S^{(1)}_\gamma\right| \leqslant n-1$ for all γ.

Suppose that for some γ such that $\delta < \gamma \leqslant \delta + \alpha_{n-1}$ the set $S^{(1)}_\gamma$ is empty. Then, for any $\gamma < \mu \leqslant \delta + \alpha_{n-1} + 1$ and $Q \in S_\mu$, we have $P^{(1)}_\mu = P^{(1)}_\gamma = P_\gamma \subseteq Q$. Hence

$Q = P_\mu^{(1)}$ and so $S_\mu^{(1)} = \emptyset$. Therefore $P_\gamma = P_{\gamma+1}$ by Lemma 2(ii). This contradiction shows that all sets $S_\gamma^{(1)}$ are nonempty for $\delta < \gamma \leqslant \delta + \alpha_{n-1}$.

Let us apply the same argument as above to $S^{(1)}$. Take an ideal $P^{(2)}$ in $S^{(1)}_{\delta+\alpha_{n-1}}$. Find $P^{(2)}_{\delta+1} \in S_{\delta+1}$ with $P^{(2)}_{\delta+1} \ll P^{(2)}$. Take a chain

$$P^{(2)}_{\delta+1} \subseteq P^{(2)}_{\delta+2} \subseteq \ldots \subseteq P^{(2)}_{\delta+\alpha_{n-1}} = P^{(2)} \subseteq R_e,$$

where $P_\gamma^{(2)} \in S_\gamma^{(1)}$ for all $\delta < \gamma \leqslant \delta + \alpha_{n-1}$. Find a new ordinal δ_2 such that $P^{(2)}_{\delta_2+1} = P^{(2)}_{\delta_2+2} = \cdots = P^{(2)}_{\delta_2+\alpha_{n-2}+1}$. Put $S_\gamma^{(2)} = S_\gamma^{(1)} \backslash \{P_\gamma^{(2)}\}$,

$$S^{(2)} = \bigcup_{\delta_2 < \gamma \leqslant \delta_2 + \alpha_{n-2}} S_\gamma^{(2)}.$$

Then the set $S^{(2)}$ satisfies the same property we used for S, but now $\left|S_\gamma^{(2)}\right| \leqslant n-2$ for all γ. As above, all sets $S_\gamma^{(2)}$ will be nonempty for $\delta_2 < \gamma \leqslant \delta_2 + \alpha_{n-2}$.

If we repeat this reduction $n-1$ times, we get a set

$$S^{(n-1)} = \bigcup_{\delta_{n-1} < \gamma \leqslant \delta_{n-1} + \alpha_1} S_\gamma^{(n-1)}$$

satisfying the same conditions and such that $\left|S_\gamma^{(n-1)}\right| \leqslant 1$ for all $\delta_{n-1} < \gamma \leqslant \delta_{n-1} + \alpha_1$. As earlier we can show that all sets $S_\gamma^{(n-1)}$ are nonempty for $\delta_{n-1} < \gamma \leqslant \delta_{n-1} + \alpha_1$. Thus $\left|S_\gamma^{(n-1)}\right| = 1$ for all γ.

Given that $\alpha_1 = \alpha + 1$, we get $S_\gamma^{(n-1)} = S_{\gamma+1}^{n-1}$ for some $\delta_2 \leqslant \gamma < \delta_2 + \alpha_1$. It follows from Lemma 2(ii) that $P_\gamma = P_{\gamma+1}$. This contradiction completes the proof. $\square$

REMARK. For a finite cl-K-dim(R_e) our proof simplifies since several steps become redundant.

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