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ON THE ASPHERICITY OF KNOT COMPLEMENTS

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ABSTRACT Two examples of topological embeddings of S^2 in S^4 are constructed The first has the unusual property that the fundamental group of the complement is isomorphic to the integers while the second homotopy group of the complement is nontrivial The second example is a non-locally flat embedding whose complement exhibits this property locally

Two theorems are proved The first answers the question of just when good π_1 implies the vanishing of the higher homotopy groups for knot complements in S^4 The second theorem characterizes local flatness for 2-spheres in S^4 in terms of a local π_1 condition

0. **Introduction.** In this paper we construct two examples of topological embeddings of S^2 into S^4 whose complements have unusual homotopy properties. In order to explain the context in which we are working, we begin with a question whose answer is well known. The question is this: If Σ is a (topologically) locally flat $(n-2)$-sphere in S^n such that $\pi_1(S^n - \Sigma) \cong \mathbb{Z}$, then do the higher homotopy groups of $S^n - \Sigma$ necessarily vanish? (In case they do, $S^n - \Sigma$ is said to be *aspherical* and will have the homotopy type of S^1.) The answer to this question is "yes" in case $n \leq 4$ and "no" in case $n \geq 5$. For $n = 3$ this follows from the sphere theorem of Papakyriakopoulos [14]. In case $n = 4$, there are two quite different arguments that can be used. Both proofs begin by noting that, since Σ is locally flat, there is a strong deformation retraction of $S^4 - \Sigma$ to a compact 4-manifold with $S^2 \times S^1$ boundary and so $S^4 - \Sigma$ has the homotopy type of a finite complex ([2, Corollary 2] or [19, Corollary 5.3]). The proof that $S^4 - \Sigma$ is aspherical is then completed either by using Milnor Duality [13] to show that all the homology with $\mathbb{Z}[\pi_1]$-coefficients of $S^4 - \Sigma$ vanishes or, alternatively, by using an argument similar to a proof of Wall—see [8] or [11], for example. On the other hand, for $n \geq 5$ there exist smooth knots $\Sigma \subset S^n$ such that $\pi_1(S^n - \Sigma) \cong \mathbb{Z}$ but $\pi_2(S^n - \Sigma) \neq 0$. In fact, it is possible [16] to make the first nontrivial homotopy group appear anywhere in the range 2 through $[(n-1)/2]$.

We address two related questions. The first of these questions is: What happens if we drop the hypothesis that Σ be locally flat? It is well known that the answer remains unchanged in case $n = 3$ and in case $n \geq 5$ (in fact, the proofs referred to above still work in those cases), but we show by example that the answer does change in dimension 4.

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EXAMPLE 1.1. There exists a topological embedding $h_1: S^2 \to S^4$ such that $\pi_1\big(S^4 - h_1(S^2)\big) \cong \mathbb{Z}$ but $\pi_2\big(S^4 - h_1(S^2)\big) \neq 0$. Furthermore, h_1 is locally flat except at one point.

If the non-locally flat point is removed from Example 1.1, an interesting embedding of $\mathbb{R}^2$ into $\mathbb{R}^4$ results.

EXAMPLE 1.2. There exists a proper, locally flat, topological embedding $h_0: \mathbb{R}^2 \to \mathbb{R}^4$ such that $\pi_1\big(\mathbb{R}^4 - h_0(\mathbb{R}^2)\big) \cong \mathbb{Z}$ but $\pi_2\big(\mathbb{R}^4 - h_0(\mathbb{R}^2)\big) \neq 0$.

When we drop the local flatness hypothesis, there is also a local version of the question considered in this paper. Before we can state it precisely it is necessary to make some definitions.

DEFINITIONS. Suppose N is an $(n-2)$-manifold topologically embedded in the interior of the n-manifold M and $x \in N$. We say that N is *locally 1-alg at* x if for every neighborhood U of x in M there exists a neighborhood V of x in U such that any loop in $V - N$ which is null-homologous in $V - N$ is also null-homotopic in $U - N$. (The "alg" stands for "abelian local groups." A duality argument shows that the image of $\pi_1(V - N)$ in $\pi_1(U - N)$ is isomorphic to $\mathbb{Z}$. Other authors, *e.g.* [6], use the terminology "$M - N$ has good local π_1 at x" to mean the same thing.) We say that N is *locally homotopically unknotted at* x if N is locally 1-alg at x and if, in addition, for every neighborhood U of x in M there exists a neighborhood V of x in U such that the image of $\pi_i(V - N)$ in $\pi_i(U - N)$ is trivial for every $i \geq 2$. ∎

We can now state the second question considered in this paper: If N is an $(n-2)$-manifold topologically embedded in the interior of the n-manifold M and N is locally 1-alg at $x \in N$, then is N locally homotopically unknotted at x? Again the answer to this question is "yes" in case $n = 3$ [1] and "no" in high dimensions. In order to construct an example for $n \geq 6$ we proceed as follows. Let $\Sigma^{n-3} \subset S^{n-1}$ be a locally flat PL$(n-3)$-sphere such that $\pi_1(S^{n-1} - \Sigma^{n-3}) \cong \mathbb{Z}$ but $\pi_2(S^{n-1} - \Sigma^{n-3}) \neq 0$. Take Σ' to be the suspension of Σ in S^n. Then Σ' is a PL$(n-2)$-sphere in S^n which is locally 1-alg at every point but is not locally homotopically unknotted at either suspension point. In §2, below, we also construct an example of a 3-sphere in S^5 which is locally 1-alg but not locally homotopically unknotted; however the example is topological and, in fact, we prove that no such 5-dimensional example can be piecewise linear (as the high dimensional example above is). Our second result shows that the answer to the question is also "no" in dimension 4.

EXAMPLE 2.1. There exists a topological embedding $h_2: S^2 \to S^4$ and a point $x \in h_2(S^2)$ such that $h_2(S^2)$ is locally 1-alg at x but $h_2(S^2)$ is not locally homotopically unknotted at x.

This example is surprising and is the main result of this paper. It had generally been believed that the kind of argument used to show that a locally flat 2-sphere in S^4 which has π_1 of the complement equal to $\mathbb{Z}$ is aspherical could be localized to show that locally

1-alg implies locally homotopically unknotted in dimension 4. This is stated explicitly in Lemma 9.3B of [6], for example, and is also implicit in [5, Theorem 10]. Our construction in §1 actually contains a counterexample to the proof of [6, Lemma 9.3B]—see Remark 2, below.

The question of whether locally 1-alg implies locally homotopically unknotted is an important question because the proof in [6, §9.3] shows that a surface which is locally homotopically unknotted at every point is locally flat. Thus the real question is whether or not local flatness can be characterized in terms of the local π_1 condition or if it is necessary to assume, in addition, that the higher local homotopy groups vanish in order to conclude that the surface is locally flat.

In §3 of this paper we consider the problem of giving necessary and sufficient conditions for the complement of a topological 2-knot in S^4 to be aspherical. Our main result in that section (Theorem 3.1) gives such conditions; it shows that the only way in which a 2-knot complement in S^4 with fundamental group $\mathbb{Z}$ can fail to be aspherical is if π_2 of the complement is nontrivial and that this will only happen if the structure of π_2 as a module over $\mathbb{Z}[\pi_1]$ is quite complicated.

THEOREM 3.1. *Suppose Σ is a 2-sphere topologically embedded in S^4 and $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$. Then the following are equivalent:*

(1) $S^4 - \Sigma$ has the homotopy type of S^1,

(2) $\pi_2(S^4 - \Sigma) = 0$,

(3) $\pi_2(S^4 - \Sigma)$ is a submodule of a free $\mathbb{Z}[\pi_1(S^4 - \Sigma)]$-module, and

(4) $\pi_2(S^4 - \Sigma)$ is finitely generated as a module over $\mathbb{Z}[\pi_1(S^4 - \Sigma)]$.

COROLLARY 3.2. *If $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$ and $S^4 - \Sigma$ is finitely dominated, then $S^4 - \Sigma$ has the homotopy type of S^1.*

COROLLARY 3.3. *If Σ is a* PL *(not necessarily locally flat) 2-sphere in S^4 such that $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$, then $S^4 - \Sigma$ has the homotopy type of S^1.*

COROLLARY 3.4. *If $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$, and $\pi_1(\epsilon) \cong \mathbb{Z}$, where ϵ is the end of $S^4 - \Sigma$, then $S^4 - \Sigma$ has the homotopy type of S^1.*

DEFINITION. A compact set X in the interior of a manifold M is said to be *globally 1-alg* if for every neighborhood U of X in M there exists a neighborhood V of X in U such that each loop in $V - X$ which is null-homologous in $V - X$ is null-homotopic in $U - X$. ∎

COROLLARY 3.5. *Let Σ be a 2-sphere topologically embedded in S^4. If Σ is globally 1-*alg *and $\pi_1(S^4 - \Sigma) \cong \mathbb{Z}$, then $S^4 - \Sigma$ has the homotopy type of S^1.*

Corollaries 3.4 and 3.5 were first proved in [11]. It should be noted that $\pi_1(\epsilon) \cong \mathbb{Z}$ is not a necessary condition: Guilbault [7] has given an example of a 2-sphere $\Sigma \subset S^4$ such that $S^4 - \Sigma$ has the homotopy type of S^1 even though Σ is not globally 1-alg. Corollary 3.3 appears in [17].

Example 2.1 raises the question of whether or not a surface in a 4-manifold that is locally 1-alg at *every* point will be locally homotopically unknotted. The example does not answer that question because it contains an infinite sequence of points converging to x at which the embedding fails to be locally 1-alg. As was mentioned above, the question is important because a positive answer would have a taming theorem as a corollary. (The proofs in [6, §9.3] show that an embedding of a surface into a 4-manifold which is locally homotopically unknotted at every point will be locally flat.) We do not answer the question in general, but do give positive answers in several special cases, the most important being the case in which the submanifold is a 2-sphere.

THEOREM 4.1. *Suppose Σ is a 2-sphere topologically embedded in the interior of the 4-manifold M^4. If the self intersection number of the homology class represented by Σ is 0 and Σ is locally 1-*alg *at x for every $x \in \Sigma$, then Σ is locally homotopically unknotted.*

By [6, §9.3] we have the following corollary.

COROLLARY 4.2. *If $\Sigma \subset S^4$ is a 2-sphere which is locally 1-*alg *at x for every $x \in \Sigma$, then Σ is locally flat.*

The conclusion of Theorem 4.1 is local. Since we are often able to take an embedding of some surface other than S^2 and make it agree locally with an embedding of S^2, we can use Theorem 4.1 to obtain information about embeddings of more general surfaces. For example, we have the following two corollaries.

COROLLARY 4.3. *Suppose N is a surface topologically embedded in the interior of the 4-manifold M. If N is locally 1-*alg *at x for every $x \in N$ and N is locally flat except possibly at a closed 0-dimensional set, then N is locally flat at every point.*

COROLLARY 4.4. *Suppose Σ is a 2-sphere topologically embedded in the interior of the 4-manifold M. If Σ is locally 1-*alg *at every point and locally flat at at least one point, then Σ is locally flat at every point.*

1. **Construction of the main example.** In this section we construct Example 1.1. The construction proceeds indirectly: we begin by constructing an open subset W of S^4 having the correct homotopy properties to be the complement of the example we seek and then show how to move W a little so that $\Sigma = S^4 - W$ is a topological 2-sphere. We believe that the example is most easily understood if it is thought of as being analogous to the Whitehead continuum in S^3. In that case, too, it is easiest to begin by constructing the Whitehead manifold which is the complement in S^3 of the Whitehead continuum. Like the Whitehead manifold, our open manifold W is described as the union of a nested sequence of compact manifolds $\{W_n\}$. This similarity with the Whitehead continuum is not accidental; in fact, we will see that Σ can be obtained by starting with the standard 2-sphere S^2 in S^4 and attaching a copy of the Whitehead continuum to S^2. Specifically, we find a copy Wh of the Whitehead continuum which is cellular in S^4 and intersects S^2 in just one point but is tightly wound around S^2. Then we construct Σ as follows:

$$\Sigma = (S^2 \vee \mathrm{Wh}) / \mathrm{Wh} \subset S^4 / \mathrm{Wh} \cong S^4.$$

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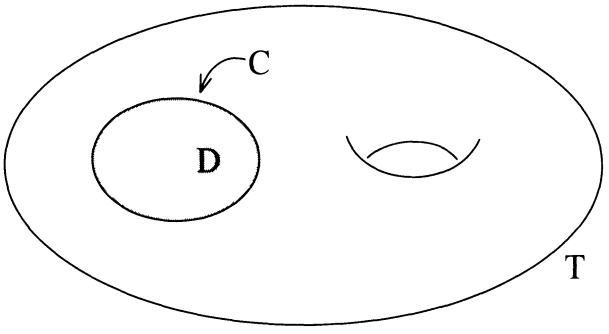

FIGURE 1

We begin by fixing some notation. Let T denote an unknotted solid torus in $\mathbb{R}^3$, let D be a flat disk in the interior of T and $C = \partial D$.

We think of S^4 as being $\mathbb{R}^3 \times (-\infty, \infty) \cup \{\infty\}$ and describe level pictures of the manifolds W_n. Define W_2 to be a regular neighborhood of $T \times [-1, 0] \cup C \times [0, 1] \cup D \times \{1\}$ in S^4. Notice that W_2 collapses to a copy of $S^1 \vee S^2$. We next define $W_1 \subset \operatorname{Int} W_2$ Let D_1 and D_2 be two close parallel copies of D in $\operatorname{Int} T$ and let B be a band which joins D_1 to D_2 as pictured in Figure 2.

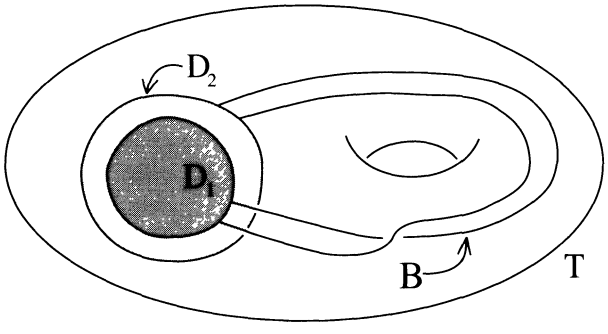

FIGURE 2

Define $C_1 = \partial D_1, C_2 = \partial D_2, D' = D_1 \cup B \cup D_2$ and $C' = \partial D'$. Choose $\epsilon > 0$ small enough so that $T \times [0, \epsilon]$ is contained in the interior of W_2. Then W_1 is defined to be a regular neighborhood of

$$T \times [-1, 0] \cup C' \times [0, \epsilon] \cup B \times \{\epsilon\} \cup (C_1 \cup C_2) \times [\epsilon, 1] \cup (D_1 \cup D_2) \times \{1\}.$$

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The regular neighborhood should be chosen so that $W_1 \subset \operatorname{Int} W_2$. Notice that D' is isotopic to D in T, so $W_1 \cong W_2$. In fact, W_1 is embedded nicely enough so that there is an isotopy $h_t\colon S^4 \to S^4, 0 \le t \le 1$, with $h_0 = id$, and $h_1(W_1) = W_2$. We recursively define $W_n, n \ge 3$, by $W_n = h_1(W_{n-1})$ and define $W = \cup_{n=1}^{\infty} W_n$.

It is clear that $W_n \searrow S^1 \vee S^2$ and that the inclusion $W_n \subset W_{n+1}$ induces the identity map on the S^1 factor. It follows that $\pi_1(W) \cong \mathbb{Z}$. The band B has a half-twist in it (see Figure 2) and so the inclusion induced map $H_2(W_n) \to H_2(W_{n+1})$ is trivial. Thus $H_2(W) = 0$. However, we will see that the fact that the band goes around the S^1-factor of W_{n+1} makes $\pi_2(W)$ nontrivial.

In order to describe $\pi_2(W)$ more explicitly, we fix some notation. We use J to denote $\pi_1(W)$; so $J \cong \mathbb{Z}$ and we write a typical element of J as t^n, where t is a generator of J and $n \in \mathbb{Z}$. Let Λ denote the group ring $\mathbb{Z}[J]$. Then Λ consists of all Laurent polynomials in t with integer coefficients. Since $W_n \searrow S^1 \vee S^2$, $\pi_2(W_n)$ is naturally isomorphic to Λ. The important thing to notice is that it is possible to choose a Λ-generator for $\pi_2(W_n)$ in such a way that the inclusion map $W_n \hookrightarrow W_{n+1}$ induces multiplication by $t-1$; *i.e.*, the diagram

$$\begin{array}{ccc} \pi_2(W_n) & \longrightarrow & \pi_2(W_{n+1}) \\ \cong\downarrow & & \downarrow\cong \\ \Lambda & \xrightarrow{t-1} & \Lambda \end{array}$$

commutes. It is easiest to see this by looking at the universal covers $\widetilde{W}_n \subset \widetilde{W}_{n+1}$.

It is now clear that $\pi_2(W) \neq 0$. In particular, the generator of $\pi_2(W_1)$ represents $(t-1)^n$ in $\pi_2(W_{n+1})$. Thus the generator of $\pi_2(W_1)$ does not die in $\pi_2(W_n)$ for any n and therefore does not die in $\pi_2(W)$.

We now turn our attention to the complements. Define $Q_n = \overline{S^4 - W_n}$ and $X = \cap_{n=1}^{\infty} Q_n = S^4 - W$. The first thing to notice is that, since W_n is a regular neighborhood of $S^1 \vee S^2$ and both factors are unknotted in S^4, we have that $Q_n \searrow S^2 \vee S^1$; the S^2-factor of Q_n links the S^1-factor of W_n and the the S^2-factor of Q_n links the S^1-factor of W_n. The S^1-factor of Q_2 can be pictured as a loop A_2 in the level $\mathbb{R}^3 \times \{\frac{1}{2}\}$ such that A_2 links $C \times \{\frac{1}{2}\}$ and the S^2-factor can be pictured as the suspension in $\mathbb{R}^4$ of a circle B_2 in $\mathbb{R}^3 \times \{\frac{1}{2}\}$ such that B_2 links $T \times \{\frac{1}{2}\}$. This is indicated schematically in Figure 3.

In general, Q_n is a regular neighborhood of $A_n \cup \mathcal{S}(B_n)$, where $A_n \cup B_n$ is a figure-8 in a level, $\mathcal{S}$ denotes suspension, and all the B_n's are the same. We wish to understand the embedding of Q_{n+1} in Q_n. We concentrate on $Q_2 \subset Q_1$. In Figure 4 we show A_1, A_2 and $B_1 = B_2$.

Now A_1 is the spine of the complement of C'. Thus, to understand the embedding of Q_2 in Q_1, we must redraw the picture so that C' and A_1 are straightened out. This is shown in Figure 5.

We are now in a position to give a geometric description of $X = S^4 - W$. We see that $X = \cap_{n=1}^{\infty} Q_n$ where each Q_n a regular neighborhood of a copy of $S^1 \vee S^2$. Each $S^1 \vee S^2 = A_n \vee \mathcal{S}(B_n)$ where $A_n \cup B_n$ is a figure-8 in a hyperplane. Furthermore, $B_{n+1} = B_n$. The embedding of $A_{n+1} \cup B_{n+1}$ in a neighborhood of $A_n \cup B_n$ is indicated in Figure 6.

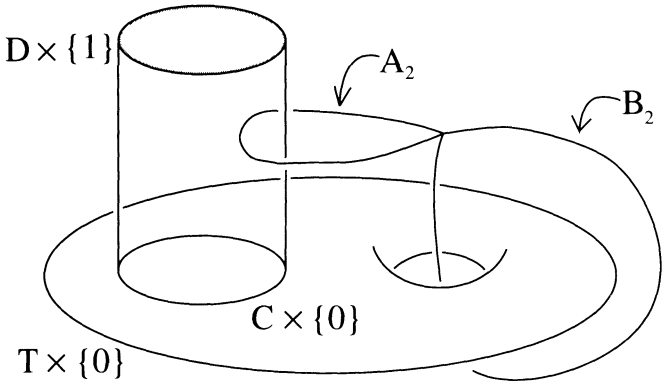

FIGURE 3

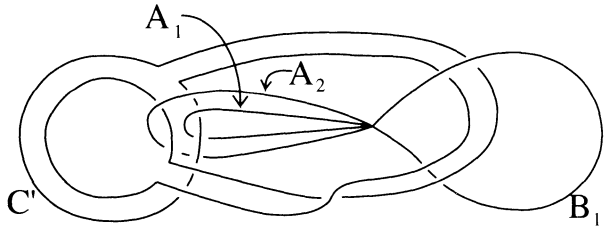

FIGURE 4

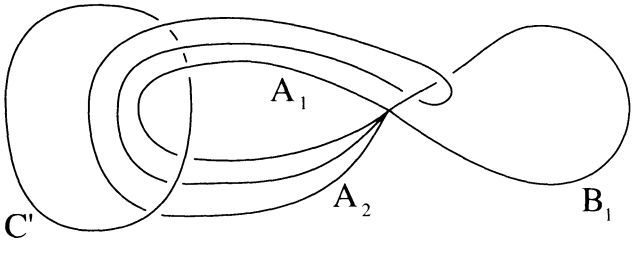

FIGURE 5

We now re-embed the Q_n's so that they are closer and closer regular neighborhoods of the spines $A_n \vee \mathcal{S}(B_n)$. After that adjustment, X consists of a locally flat 2-sphere, $\mathcal{S}(B_1)$, together with a 1-dimensional continuum which we call Wh; *i.e.*, $X = \mathcal{S}(B_1) \cup \mathrm{Wh}$ and $\mathcal{S}(B_1) \cap \mathrm{Wh}$ consists of a single point. Since A_{n+1} is null-homotopic in a regular neighborhood of A_n, Wh is cell-like. In fact, Wh is cellular in S^4 since it is the intersection

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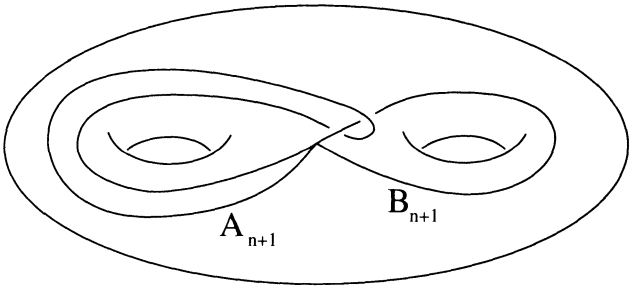

FIGURE 6

of a sequence of neighborhoods with 1-dimensional spines. We can therefore shrink Wh to a point and the resulting manifold is still homeomorphic to S^4. We define Σ_1 to be the image of X in the decomposition space $S^4/\text{ Wh}$. Since $S^4/\text{ Wh} - \Sigma_1 = S^4 - X = W$, it is clear that Σ_1 has the properties required of Example 1.1. ∎

REMARK 1. It should be noted that Σ_1 is locally flat except at the image of Wh; at that point Σ_1 cannot be locally flat. In fact, Σ_1 cannot even be locally 1-alg at the exceptional point. This is because Hollingsworth and Rushing [9] have shown that any 2-sphere which is locally 1-alg at every point is globally 1-alg. Thus Corollary 3.5 would be contradicted if Σ_1 were locally 1-alg at the exceptional point. ∎

REMARK 2. Our construction contains a counterexample to the proof of [6, Lemma 9.3B]. The manifolds $\{W_n\}$ above satisfy conditions (1)–(3) on p. 142 of [6] as well as (i)–(iv) on p. 143, but it is not the case that the inclusion induced map $\pi_2(W_n) \longrightarrow \pi_2(W_{n+k})$ is trivial for any $k \geq 1$. The error in [6] is the assertion (on p. 146) that "condition (iv) is equivalent to $\ldots I(\text{image}) = \text{image}$." In general it will only be the case that, in the terminology of [6], image $\subset I$(whole group). This is analyzed more carefully in the proof of Theorem 3.1, below. ∎

REMARK 3. There exists a high dimensional version of our example. Let $X = S^{n-3}(B_1) \cup \text{Wh} \subset S^n$, where B_1 and Wh are the subsets of S^3 described above and $S^{n-3}(B_1)$ denotes the $(n-3)$-fold suspension of B_1. Then

$$\Sigma^{n-2} = X/\text{ Wh} \subset S^n/\text{ Wh} \cong S^n$$

is a topologically embedded $(n-2)$-sphere in S^n, $n \geq 4$. The complement of Σ^{n-2} has the property that $\pi_1(S^n - \Sigma^{n-2}) \cong \mathbb{Z}$ and $\pi_i(S^n - \Sigma^{n-2}) = 0$ for $2 \leq i \leq n-3$, but $\pi_{n-2}(S^n - \Sigma^{n-2})$ is nontrivial. This contrasts sharply with Levine's Theorem [10] which asserts the following: If S^{n-2} is a smooth $(n-2)$-sphere in S^n such that $\pi_1(S^n - S^{n-2}) \cong \mathbb{Z}$ and $\pi_i(S^n - S^{n-2}) = 0$ for $2 \leq i < n/2$, then S^{n-2} is unknotted, $n \geq 6$. ∎

REMARK 4. The embedding $h_0\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ of Example 1.2 is constructed by removing the point Wh from (S^4, Σ_1). More generally, we can remove the point Wh from $\Sigma^{n-2} \subset S^n$

(described in Remark 3, above), to construct an interesting example of an embedding of $\mathbb{R}^{n-2}$ in $\mathbb{R}^n$. Again, this example contrasts sharply with Levine's Theorem. ∎

EXAMPLE 1.2′. *There exists a proper, smooth embedding* $h: \mathbb{R}^{n-2} \to \mathbb{R}^n$, $n \geq 4$, *such that* $\pi_1(\mathbb{R}^n - h(\mathbb{R}^{n-2})) \cong \mathbb{Z}$ *and* $\pi_i(\mathbb{R}^n - h(\mathbb{R}^{n-2})) = 0$ *for* $2 \leq i \leq n-3$ *but* $\pi_{n-2}(\mathbb{R}^n - h(\mathbb{R}^{n-2})) \neq 0$.

2. **The second example.** In this section we construct Example 2.1. The method of constructing the local example from the global one is the same as that described by Daverman on p. 372 of [3].

Let D' be a 4-cell in S^4 such that D' intersects Σ_1 in a locally flat disk missing the exceptional point of Σ_1 and let $D = \overline{S^4 - D'}$. Then $(D, D \cap \Sigma_1)$ is a knotted topological (4,2)-ball pair. Remove from S^4 a sequence $D_1, D_2, D_3, \ldots$ of pairwise disjoint 4-cells such that, for each n, $(D_n, D_n \cap S^2)$ is an unknotted (4,2)-ball pair and $\{D_n\}_{n=1}^{\infty}$ converges to a point $x \in S^2$. Replace each $(D_n, D_n \cap S^2)$ with a (4,2)-ball pair (D_n, C_n) such that $\partial C_n = \partial(D_n \cap S^2)$ and $(D_n, C_n) \cong (D, D \cap \Sigma_1)$. Then let

$$\Sigma_2 = \left[S^2 - \left(\bigcup_{n=1}^{\infty}(D_n \cap S^2)\right)\right] \cup \left(\bigcup_{n=1}^{\infty} C_n\right)$$

Since $\pi_1(D - \Sigma_1) \cong \mathbb{Z}$, Van Kampen's theorem implies that Σ_2 is locally 1-alg at x. A Mayer-Vietoris argument in the universal cover of $S^4 - \Sigma_2$ shows that the local π_2 of $S^4 - \Sigma_2$ is not trivial at x. Thus Σ_2 has the properties required of Example 2.1. ∎

In the Introduction we described an example of a PL embedding of S^{n-2} into S^n, $n \geq 6$, which shows that locally 1-alg does not imply locally homotopically unknotted. We have just constructed an example of such an embedding for $n = 4$ as well. For the sake of completeness, we now construct an example for $n = 5$. Let $\Sigma^3 \subset S^5$ be a locally flat PL 3-sphere such that $\pi_1(S^5 - \Sigma^3) \cong \mathbb{Z}$ and $\pi_2(S^5 - \Sigma^3) \neq 0$. By removing an unknotted ball pair from (S^5, Σ^3), we obtain a knotted ball pair $(D^5, D^5 \cap \Sigma^3)$. Proceeding as in the construction of Example 2.1, we start with the unknotted sphere pair (S^5, S^3), remove a null-sequence of balls from it, and replace each with a copy of $(D^5, D^5 \cap \Sigma^3)$ to produce a topological embedding of S^3 into S^5 having the following properties.

EXAMPLE 2.2. There exists a topological embedding $h_3: S^3 \to S^5$ and a point $x_3 \in h_3(S^3)$ such that $h_3(S^3)$ is locally 1-alg at x_3 but $h_3(S^3)$ is not locally homotopically unknotted at x_3.

Notice that $h_3(S^3)$ is locally flat and PL at every point other than x_3 and that it is the local π_2 that is bad at x_3. We could construct an example with bad local π_3 by applying the infinite construction to the example mentioned in Remark 3 at the end of §1. The example constructed in that way would have an infinite sequence of non-locally flat points. Obviously our 5-dimensional examples are not PL as the high dimensional examples were. Our next theorem shows that there can be no PL embedding of S^3 in S^5 having the properties mentioned in Example 2.2, but that PL embeddings of S^3 into S^5 follow the low dimensional pattern.

THEOREM 2.3. *If M^3 is a 3-dimensional* PL *submanifold of the* PL *5-manifold W^5 and M^3 is locally 1-*alg *at x for every $x \in M^3$, then M^3 is (topologically) locally flat.*

PROOF. Let $x \in M^3$ and let σ be the simplex of M^3 such that $x \in \operatorname{Int}\sigma$. If $\operatorname{lk}\sigma$ denotes the link of σ in W^5, then $(\operatorname{lk}\sigma, \operatorname{lk}\sigma \cap M^3)$ is a PL sphere pair of dimension $(4-\dim\sigma, 2-\dim\sigma)$. The fact that M^3 is locally 1-alg at x implies that $\pi_1(\operatorname{lk}\sigma - M^3) \cong \mathbb{Z}$. As long as $\dim\sigma \geq 2$, it is obvious that $(\operatorname{lk}\sigma, \operatorname{lk}\sigma \cap M^3)$ is an unknotted sphere pair. In case $\dim\sigma = 1$, this follows from the Loop Theorem [14] and in case $\dim\sigma = 0$, $(\operatorname{lk}\sigma, \operatorname{lk}\sigma \cap M^3)$ is a topologically unknotted (4,2)-sphere pair by [5, Theorem 6]. ∎

3. **Aspherical 2-sphere complements.** We next turn our attention to the problem of proving a positive theorem regarding aspherical 2-spheres in S^4. For the remainder of this section, we let Σ denote a 2-sphere topologically embedded in S^4 and $W = S^4 - \Sigma$. We assume that Σ is embedded in such a way that $\pi_1(W) \cong \mathbb{Z}$. In §1 we showed that $\pi_2(W)$ may be nontrivial and thus W is not necessarily aspherical. In this section we give conditions which imply that W is aspherical. It should be noted that, by the Whitehead theorem, W has the homotopy type of S^1 if and only if $\pi_i(W) = 0$ for $i \geq 2$. But the Hurewicz theorem implies that $\pi_i(W) = 0$ for $i \geq 2$ if $H_i(\widetilde{W}) = 0$ for $i \geq 2$, where $\widetilde{W}$ is the universal cover of W. Thus we will concentrate in this section on the problem of computing the homology of $\widetilde{W}$. The main result of this section (Theorem 3.1) shows that any nontrivial higher homology groups must appear at the H_2 level and that this will only happen if H_2 is quite complicated. Our other results in this section show that most of the algebraic properties of Example 1.1 are common to all examples.

We begin the proof of Theorem 3.1 by constructing two sequences of submanifolds of S^4 which will be useful in the proofs below. Let $U_1 \supset U_2 \supset U_3 \supset \cdots$ be a sequence of compact connected PL manifold neighborhoods of Σ such that $U_{n+1} \subset \operatorname{Int} U_n$ for each n and $\cap_{n=1}^{\infty} U_n = \Sigma$. Since Σ is an ANR, there is a retraction $r_n: U_n \to \Sigma$ for large n. We may assume that U_n is chosen in such a way that r_n is homotopic to the inclusion in U_{n-1}. Since Σ is 2-dimensional, Σ does not separate any neighborhood and so we may also assume that ∂U_n is connected for each n. (If not, connect up the boundary components with arcs lying in $U_n - \Sigma$ and remove regular neighborhoods of the arcs.) Define $W_n = \overline{S^4 - U_n}$. It follows from Alexander duality that $H_2(W) \cong H^1(\Sigma) = 0$. Each $H_2(W_n)$ is finitely generated (since W_n is a compact PL manifold) so we may also assume that the inclusion induced map of $H_2(W_n)$ into $H_2(W_{n+1})$ is trivial. In addition, we use the fact that $\pi_1(W) \cong \mathbb{Z}$ to arrange that any loop in W_n which is null-homotopic in W is null-homotopic in W_{n+1} and that $\pi_1(W_n) \to \pi_1(W)$ is onto. Let $p: \widetilde{W} \to W$ denote the covering projection.

LEMMA 3.6. *For each n, the image of $H_1(U_{n+1} - \Sigma)$ in $H_1(U_n - \Sigma)$ is canonically isomorphic to $H_1(W) \cong \mathbb{Z}$. It follows that both $p^{-1}(U_n - \Sigma)$ and $p^{-1}(\partial U_n)$ are connected and that $\widetilde{W}$ has one end.*

PROOF. Fix n. Let $\gamma: H_1(U_{n+1} - \Sigma) \to H_1(U_n - \Sigma)$ and $\eta: H_1(U_n - \Sigma) \to H_1(W)$ denote the inclusion induced homomorphisms. We will show that $\eta|\operatorname{im}\gamma: \operatorname{im}\gamma \to H_1(W) \cong \mathbb{Z}$

is an isomorphism. We begin by showing that $\operatorname{im}\gamma$ is cyclic. Consider the commutative diagram

$$\begin{array}{ccccc} & & H_1(U_{n+1}-\Sigma) & \longrightarrow & H_1(U_{n+1}) \\ & & \gamma\downarrow & & \downarrow\delta \\ H_2(U_n, U_n-\Sigma) & \xrightarrow{\alpha} & H_1(U_n-\Sigma) & \xrightarrow{\beta} & H_1(U_n). \end{array}$$

Since the inclusion $U_{n+1} \hookrightarrow U_n$ factors, up to homotopy, through a retraction onto Σ, we have that $\delta = 0$. Therefore $\beta\circ\gamma = 0$ and so $\operatorname{im}\gamma \subset \ker\beta = \operatorname{im}\alpha$. But $H_2(U_n, U_n-\Sigma) \cong \mathbb{Z}$ by Alexander duality, so $\operatorname{im}\gamma$ is cyclic.

Now consider the commutative diagram

$$\begin{array}{ccccc} H_1(U_{n+1}-\Sigma) & \xrightarrow{\zeta} & H_1(S^4-\Sigma) & \longrightarrow & H_1(S^4-\Sigma, U_{n+1}-\Sigma) \\ \gamma\downarrow & & \| & & \\ H_1(U_n-\Sigma) & \xrightarrow{\eta} & H_1(S^4-\Sigma). & & \end{array}$$

By excision, $H_1(S^4-\Sigma, U_{n+1}-\Sigma) \cong H_1(S^4, U_{n+1})$. Since U_{n+1} is connected, it follows that $H_1(S^4, U_{n+1}) = 0$. Therefore ζ is onto and hence $\eta\circ\gamma$ is onto. This implies that $\eta|\operatorname{im}\gamma\colon \operatorname{im}\gamma \to H_1(S^4-\Sigma) \cong \mathbb{Z}$ is an isomorphism since the only onto homomorphisms from a cyclic group to $\mathbb{Z}$ are isomorphisms.

In order to see that each $p^{-1}(U_n-\Sigma)$ is connected, it is enough to note that the composition $\pi_1(U_n-\Sigma) \to H_1(U_n-\Sigma) \to H_1(W) \cong \pi_1(W)$ is onto. A similar proof shows that $p^{-1}(\partial U_n)$ is connected. Since ∂U_n is connected, the only other ingredient we need is a representative of the generator t of $H_1(W)$ in ∂U_n. We have already seen that there is a representative of t in $U_n-\Sigma$ and the choice of W_n shows that there is a representative of t in W_n. Hence, a Mayer-Vietoris sequence argument shows that there is a class in $H_1(\partial U_n)$ which is homologous in W to t.

For each n, we can find a compact connected polyhedron $K_n \subset \widetilde{W}$ such that $W_n \subset p(K_n)$. A typical neighborhood of ∞ in $\widetilde{W}$ has the form

$$V_{n,k} = p^{-1}(U_n-\Sigma) \cup \Big(\bigcup_{|j|\geq k} t^j K_n\Big)$$

Since $V_{n,k}$ is connected for every n and k, $\widetilde{W}$ has one end. ∎

As in §1, we use J to denote $\pi_1(W) \cong \mathbb{Z}$, Λ to denote the integral group ring $\mathbb{Z}[J]$, and t to denote a generator of J. Let $p\colon \widetilde{W} \to W$ be the universal cover. If A is a subset of W such that $\pi_1(A) \to \pi_1(W) \cong \mathbb{Z}$ is onto, we write $H_k(A;\Lambda)$ for $H_k\big(p^{-1}(A);\mathbb{Z}\big)$. Here we are thinking of Λ-coefficients as local coefficients in the sense of Steenrod [15]. In case A is a compact n-dimensional submanifold of W, there is a Poincaré duality theorem for $H_*(A;\Lambda)$. It asserts that $H_k(A;\Lambda) \cong H^{n-k}(A, \partial A;\Lambda)$. In order to avoid confusion it should be remembered that $H^i(A;\Lambda)$ is the same as $H^i_c\big(p^{-1}(A);\mathbb{Z}\big)$.

LEMMA 3.7. *For each n,* $H_3(W_n;\Lambda) = 0$, $H^3(W_n;\Lambda) = 0$, *and* $H_2(W_n;\Lambda)$ *is a submodule of a finitely generated free* Λ*-module.*

REMARK. Since $H_*(\widetilde{W})$ is the direct limit of the images of the groups $H_*(W_n;\Lambda)$, we can conclude from the Lemma that every complement W of the sort studied in this section will possess many of the same algebraic properties that the complement of Example 1.1 had. In particular, $H_i(\widetilde{W}) = 0$ for $i \geq 3$ and $H_2(\widetilde{W})$ is the direct limit of a sequence of modules, each of which is a submodule of a finitely generated free module. In Theorem 3.1 we show that the only way in which $H_2(\widetilde{W})$ itself can be a submodule of a free module is if it is trivial. ∎

PROOF OF LEMMA 3.7. First observe that the pair $(W, U_n - \Sigma)$ is 1-connected. By Poincaré duality and excision we have $H^3(W_n;\Lambda) \cong H_1(W_n, \partial W_n;\Lambda) \cong H_1(W, U_n - \Sigma;\Lambda) = 0$. Similarly, $H_3(W_n;\Lambda) \cong H^1(W_n, \partial W_n;\Lambda) \cong H^1(W, U_n - \Sigma;\Lambda) = 0$.

Since W_n collapses to a compact 3-dimensional polyhedron, the chain complex for $H_*(W_n;\Lambda)$ can have the form

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Because $H_3(W_n;\Lambda) = 0$, ∂_3 is injective and $\operatorname{im}\partial_3$ is a finitely generated free Λ-module. We can think of $[\partial_3]$ as an element of $H^3(W_n;\operatorname{im}\partial_3)$. Moreover $H^3(W_n;\operatorname{im}\partial_3) = 0$ since $\operatorname{im}\partial_3$ is a finitely generated free Λ-module and $H^3(W_n;\Lambda) = 0$. So there exists $\theta \in \operatorname{Hom}_\Lambda(C_2, \operatorname{im}\partial_3)$ such that $\partial_3 = \delta_3(\theta) = \theta \circ \partial_3$. Let $Z_2 = \ker\partial_2$. Then $\theta|Z_2$ splits the sequence

$$0 \longrightarrow \operatorname{im}\partial_3 \hookrightarrow Z_2 \longrightarrow H_2(W_n;\Lambda) \longrightarrow 0.$$

Thus $H_2(W_n;\Lambda)$ is isomorphic to a submodule of $Z_2 \subset C_2$. ∎

PROOF OF THEOREM 3.1. It is obvious that condition (1) implies each of the others and that condition (2) implies each of (3) and (4). It follows from Lemma 3.7 along with the Whitehead and Hurewicz theorems that (2) implies (1). We will show that (3) implies (2) and that (4) implies (2).

Suppose, first, that $H_2(\widetilde{W})$ is a submodule of a free Λ-module. The short exact sequence

$$0 \longrightarrow C_*(\widetilde{W}) \xrightarrow{t-1} C_*(\widetilde{W}) \xrightarrow{p_*} C_*(W) \longrightarrow 0$$

of chain complexes gives rise to the exact sequence

$$0 = H_3(W) \longrightarrow H_2(\widetilde{W}) \xrightarrow{t-1} H_2(\widetilde{W}) \xrightarrow{p_*} H_2(W) = 0$$

of homology groups. Thus $(t-1)\colon H_2(\widetilde{W}) \longrightarrow H_2(\widetilde{W})$ is onto. This means that each element of $H_2(\widetilde{W})$ is infinitely divisible by $(t-1)$, but the only element of a free module with that property is 0. Thus $H_2(\widetilde{W})$ cannot be a submodule of a free module unless it is trivial.

Second, suppose that $H_2(\widetilde{W})$ is finitely generated over Λ. We fix a finite set of generators and choose W_1 in such a way that $p^{-1}(W_1)$ contains representatives of this set of

generators for $H_2(W, \Lambda)$ Each $H_2(W_n, \Lambda)$ is finitely generated, so we can choose W_{n+1} large enough that each element of $H_2(W_n, \Lambda)$ is homologous in W_{n+1} to some combination of the fixed set of generators This will mean that the image of $H_2(W_n, \Lambda)$ in $H_2(W_{n+1}, \Lambda)$ will be equal to the image of $H_2(W_1, \Lambda)$ in $H_2(W_{n+1}, \Lambda)$ In this way we arrange that, for each n, the image of $H_2\big(p^{-1}(W_n)\big)$ in $H_2\big(p^{-1}(W_{n+2})\big)$ is the same as that of $H_2\big(p^{-1}(W_{n+1})\big)$ in $H_2\big(p^{-1}(W_{n+2})\big)$ As in the previous paragraph, the rows in the following diagram are exact

$$\begin{array}{ccccc} H_2\big(p^{-1}(W_n)\big) & \xrightarrow{t\ 1} & H_2\big(p^{-1}(W_n)\big) & \xrightarrow{p_*} & H_2(W_n) \\ \downarrow & & \downarrow{\scriptstyle \alpha_n} & & \downarrow 0 \\ H_2\big(p^{-1}(W_{n+1})\big) & \xrightarrow{t-1} & H_2\big(p^{-1}(W_{n+1})\big) & \xrightarrow{p_*} & H_2(W_{n+1}) \\ \downarrow & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow 0 \\ H_2\big(p^{-1}(W_{n+2})\big) & \xrightarrow{t\ 1} & H_2\big(p^{-1}(W_{n+2})\big) & \xrightarrow{p_*} & H_2(W_{n+2}) \end{array}$$

This means that $\operatorname{im}\alpha_n \subset (t-1)\big(H_2\big(p^{-1}(W_{n+1})\big)\big)$ and so

$$\operatorname{im}(\alpha_{n+1}\circ\alpha_n) \subset (t-1)\operatorname{im}\alpha_{n+1} = (t-1)\operatorname{im}(\alpha_{n+1}\circ\alpha_n)$$

Therefore every element of $\operatorname{im}(\alpha_{n+1}\circ\alpha_n)$ is infinitely divisible by $(t-1)$ By Lemma 3 7, $H_2\big(p^{-1}(W_{n+2})\big)$ is a submodule of a free Λ-module and thus $\operatorname{im}(\alpha_{n+1}\circ\alpha_n)$ is a submodule of a free Λ-module This means that $\operatorname{im}(\alpha_{n+1}\circ\alpha_n) = 0$ and hence $H_2(\widetilde{W}) = 0$ The proof of the theorem is complete ∎

PROOF OF COROLLARY 3 4 Since $\pi_1(\epsilon) = \mathbb{Z}$ and the natural map $\pi_1(\epsilon) \longrightarrow \pi_1(W)$ is an isomorphism (Lemma 3 6), $\widetilde{W}$ has one simply connected end This means that $H^3(\widetilde{W}) \cong H_1^\infty(\widetilde{W}) = 0$ ($H_*^\infty(\widetilde{W})$ denotes the homotopy based on infinite chains) It follows that $H_2(\widetilde{W})$ is a projective module over Λ (by the argument of [11, Lemma 2 4] or [18, Lemma 2 1]) and so the Theorem applies ∎

4 **Surfaces which are locally 1-alg at every point.** In this final section of the paper we consider the question of whether or not a topologically embedded surface in a 4-manifold which is locally 1-alg at every one of its points is locally homotopically unknotted (and therefore locally flat) We are not able to answer that question in general, but do prove a positive result in the important special case in which the surface is a 2-sphere In this section we will make use of the following technical lemma from homological algebra

LEMMA 4 5 *Suppose W is a* PL *manifold with $\pi_1(W) \cong \mathbb{Z}$ and $A \subset B$ are two polyhedral subsets of W such that the pairs $(W, W-A)$ and $(W, W-B)$ are $(q-1)$-connected for some q If the inclusion induced homomorphism $H_q(W, W-B, \Lambda) \longrightarrow H_q(W, W-A, \Lambda)$ is the zero homomorphism, then the restriction homomorphism $H^q(W, W-A, \Lambda) \longrightarrow H^q(W, W-B, \Lambda)$ is also zero*

SKETCH OF PROOF First notice that $(W, W-A)$ and $(W, W-B)$ are simple homotopy equivalent to pairs with no relative n-cells, $n < q$ (*cf* [18] Lemma 1 1) Thus the cellular

chain complexes have the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{q+1}(W, W-B;\Lambda) & \xrightarrow{\partial} & C_q(W, W-B;\Lambda) & \longrightarrow 0 \\ & & \downarrow & & \downarrow & \\ \cdots & \longrightarrow & C_{q+1}(W, W-A;\Lambda) & \xrightarrow{\partial} & C_q(W, W-A;\Lambda) & \longrightarrow 0 \end{array}$$

Take $\mathrm{Hom}_\Lambda(\cdot, \Lambda)$ of the diagram above. Then the proof is completed by a straightforward argument using the definitions of homology and cohomology. ∎

To begin the proof of Theorem 4.1, we assume that Σ is a 2-sphere topologically embedded in the interior of the 4-manifold M^4 in such a way that the self intersection number is 0. Our next two lemmas do not require the hypothesis that Σ be locally 1-alg at every point, but only the weaker hypothesis that Σ is globally 1-alg. Thus that is all we will assume for the moment. (It is shown in [9] that if Σ is locally 1-alg at every point, then Σ is globally 1-alg. Hence the global hypothesis is strictly weaker than the local one.)

Since Σ is globally 1-alg and the self intersection number is 0, the main theorem of [12] implies that there is a collar C of the end of $M - \Sigma$. Specifically, there exists a compact neighborhood N of Σ and a homeomorphism $\phi\colon S^1 \times S^2 \times [0,1) \longrightarrow C = N - \Sigma$. Let $p\colon \tilde{C} \longrightarrow C$ be the universal cover of C. As before, we use J to denote $\pi_1(C)$ and $\Lambda = \mathbb{Z}[J]$. For $0 < a < 1$ we use C_a to denote the subcollar $C_a = \phi\big(S^1 \times S^2 \times [a,1)\big)$.

LEMMA 4.6. *Suppose B is a* PL *4-ball in the interior of $C \cup \Sigma$, $0 < a < 1$, and $B - (\mathrm{Int}\, C_a \cup \Sigma)$ is* PL *with respect to the* PL *structure on C induced by ϕ. If $P = \overline{B - (C_a \cup \Sigma)}$, then $H_3(P;\Lambda) = 0$, $H^3(P;\Lambda) = 0$, and $H_2(P;\Lambda)$ is a submodule of a finitely generated free Λ-module.*

PROOF. We first prove that $C - P$ is connected. Notice that $C - P = C_a \cup (C - B)$ and that it is obvious that C_a is connected. Given two points y and z in $C - B$, there exists an arc $A \subset C$ joining them. Put A in general position with respect to ∂B. Then $A \cap \partial B$ will consist of a finite number of points which divide A into a finite number of subarcs, each of which is entirely inside B or entirely outside B. We replace each subarc of A which lies inside B with an arc lying on ∂B. The new arc from y to z can be pushed into the complement of B. Since Σ is 2-dimensional, Σ does not separate any open set so we can adjust the arc slightly to miss Σ as well. Thus $C - B$ is connected and so is $C - P$.

Since $\pi_1(C-P) \longrightarrow \pi_1(C)$ is onto, we conclude that the pair $(C, C-P)$ is 1-connected. The Lemma can now be proved by the same argument as was used to prove Lemma 3.7. ∎

LEMMA 4.7. *Fix $x \in \Sigma$. Suppose B_1 is a 4-ball in $C \cup \Sigma$ such that $x \in \mathrm{Int}\, B_1$. Let D be a disk on Σ such that $x \in \mathrm{Int}\, D$ and $D \subset \mathrm{Int}\, B_1$ and let B_2 be a 4-ball such that $x \in \mathrm{Int}\, B_2$, $B_2 \cap \Sigma \subset \mathrm{Int}\, D$, and $B_2 \subset \mathrm{Int}\, B_1$. For every a, $0 < a < 1$, there exists a number b, $a < b < 1$, such that if $P_1 = \overline{B_1 - (C_b \cup \Sigma)}$ and $P_2 = \overline{B_2 - (C_a \cup \Sigma)}$, then the inclusion induced homomorphism $H_1(C - \mathrm{Int}\, P_1;\Lambda) \longrightarrow H_1(C - \mathrm{Int}\, P_2;\Lambda)$ is trivial.*

PROOF. Let x, B_1, D, B_2, and a be given. Choose V to be a connected PL manifold neighborhood of ∂D in $\mathrm{Int}\big((B_1 - B_2) \cap C_a\big)$. Then choose $b > a$ so that V separates

$C_b \cap B_2$ from $C_b \cap (C - B_1)$ in C_b. Define $P_1 = \overline{B_1 - (C_b \cup \Sigma)}$ and $P_2 = \overline{B_2 - (C_a \cup \Sigma)}$. We must show that each loop in $p^{-1}(C - \operatorname{Int} P_1)$ is null-homologous in $p^{-1}(C - \operatorname{Int} P_2)$.

Let γ be a loop in $p^{-1}(C - \operatorname{Int} P_1)$ and let $\gamma' = p(\gamma)$. Put γ' in general position with respect to ∂B_1. Then $\gamma' \cap \partial B_1$ will consist of a finite number of points. These points can be paired off so that each pair either bounds a subarc of γ' which lies outside B_1 or a subarc which lies in C_b. Let y_1, y_2 be a pair which bounds a subarc in C_b. We claim that there is an arc from y_1 to y_2 which lies in $C_a - B_2$. If the subarc of γ' which joins y_1 to y_2 lies outside B_2, we can use it. Otherwise the subarc must pass through V and so we can replace the portion from the first point of intersection with V to the last point of intersection with V with an arc in $V - \Sigma$ to obtain the arc we need to prove the claim. Use these arcs to write $\gamma' = \gamma_1 + \gamma_2$, where $\gamma_1 \subset C - B_2$ and $\gamma_2 \subset C_a$. By adding a multiple of the generator of $\pi_1(C_a)$ to γ_1 and its inverse to γ_2, we can arrange that both γ_1 and γ_2 are null-homotopic in C. This makes γ_2 null-homotopic in C_a, so we have γ' homotopic to γ_1 in $C - \operatorname{Int} P_2$.

Now γ_1 is null-homotopic in C. Since ∂B_2 is simply connected, this null-homotopy can be cut off on ∂B_2. In other words, there exists a continuous function $F\colon \Delta^2 \longrightarrow (C - \operatorname{Int} B_2) \cup \partial B_2$ such that $F(\partial \Delta^2) = \gamma_1$. There exist a finite number of pairwise disjoint disks $\Delta_1, \ldots, \Delta_n$ in Δ^2 such that

$$F^{-1}(\Sigma) \subset \bigcup_{i=1}^{n} \operatorname{Int} \Delta_i$$

and $F(\partial \Delta_i) \subset \operatorname{Int} C_b$ for each i. Choose a basepoint $z_0 \in \partial \Delta^2$ such that $F(z_0) \in C_b$ and let $A_1, \ldots, A_n$ be a collection of oriented arcs in Δ^2 such that A_i is an arc from z_0 to $\partial \Delta_i$ and $A_i \cap A_j = z_0$ for $i \neq j$. Let δ_i denote the loop $F(\partial \Delta_i)$ and let $\alpha_i = F(A_i)$. Then γ_1 is homotopic in $C - \operatorname{Int} P_2$ to the loop $\alpha_1 \delta_1 \alpha_1^{-1} \cdots \alpha_n \delta_n \alpha_n^{-1}$. Let β_i be an arc in C_b which joins the ends of α_i. By adding a multiple of the generator of $\pi_1(C_b)$ to β_i, we can arrange that the loop $\alpha_i \beta_i^{-1}$ is null-homotopic in C. Now γ_1 is homologous in $C - \operatorname{Int} P_2$ to the loop $\gamma_3 = \beta_1 \delta_1 \beta_1^{-1} \cdots \beta_n \delta_n \beta_n^{-1}$. Furthermore, the fact that $\alpha_i \beta_i^{-1}$ is null-homotopic in C means that this homology can be lifted to $\tilde{C}$. So γ is homologous in $p^{-1}(C - \operatorname{Int} P_2)$ to a lift $\tilde{\gamma}_3$ of γ_3. This completes the proof because $\tilde{\gamma}_3$ is contained in the simply connected space $p^{-1}(C_b) \subset p^{-1}(C - P_2)$. ∎

PROOF OF THEOREM 4.1. Since Σ is locally 1-alg at each of its points, it is globally 1-alg [9]. Thus we may apply the two lemmas above. Fix $x \in \Sigma$. We wish to show that C is locally i-connected at x for every $i \geq 2$.

Let P_1 and P_2 be as in Lemma 4.7. Consider the diagram

$$\begin{array}{ccccc} & & H_2(C, C - \operatorname{Int} P_1; \Lambda) & \longrightarrow & H_1(C - \operatorname{Int} P_1; \Lambda) \\ & & \downarrow & & \downarrow \beta \\ H_2(C; \Lambda) & \xrightarrow{\alpha} & H_2(C, C - \operatorname{Int} P_2; \Lambda) & \longrightarrow & H_1(C - \operatorname{Int} P_2; \Lambda) \end{array}$$

Since α is trivial and β is trivial (by Lemma 4.7), we have that the inclusion induced homomorphism $H_2(C, C - \operatorname{Int} P_1; \Lambda) \longrightarrow H_2(C, C - \operatorname{Int} P_2; \Lambda)$ is trivial. Moreover, since

the pairs $(C, C - \operatorname{Int} P_1)$ and $(C, C - \operatorname{Int} P_2)$ are 1-connected, it follows from Lemma 4.5 that the restriction homomorphism $H^2(C, C - \operatorname{Int} P_2; \Lambda) \to H^2(C, C - \operatorname{Int} P_1; \Lambda)$ is trivial. Therefore its dual, the inclusion induced homomorphism $H_2(P_2; \Lambda) \to H_2(P_1; \Lambda)$, is trivial as well.

Now let $P_1 \supset P_2 \supset P_3 \supset \cdots$ be a sequence of submanifolds of C, constructed as in Lemma 4.7. Then Lemmas 4.6 and 4.7 show that the inclusion induced homomorphisms $H_i(P_k; \Lambda) \to H_i(P_{k-1}; \Lambda)$ are trivial for $i \geq 2$. It follows from the Eventual Hurewicz Theorem [4, Lemma 3.1] that there is an n such that $\pi_i(P_k) \to \pi_i(P_{k-n})$ is trivial for $i \geq 2$.

Given a neighborhood U of x, we can find a 4-ball B_0 such that $x \in \operatorname{Int} B_0$ and $B_0 \subset U$. Let $B_0 \supset B_1 \supset \cdots \supset B_n$ be 4-balls as in Lemma 4.7. Consider a map $f: S^i \to B_n - \Sigma$, $i \geq 2$. Choose a so that $f(S^i) \subset B_n - C_a$ and construct $P_0, \ldots, P_n$ from $B_0, \ldots, B_n$ as in Lemma 4.7 with all the subcollars involved being subcollars of C_a. Then f is null-homotopic in P_0 by the previous paragraph. Hence f is null-homotopic in $U - \Sigma$ and the proof is complete. ∎

PROOF OF COROLLARY 4.3. Pick a point x in N. There exists a disk D such that $x \in \operatorname{Int} D \subset D \subset N$ and N is locally flat in a neighborhood of ∂D. Let A be an annulus such that $\partial D \subset A \subset D$ and N is locally flat at every point of A. Then there exists an embedding $H: A \times B^2 \to M$ such that $H(x, 0) = x$ for every $x \in A$. Choose a compact neighborhood W of D such that $H(A \times B^2) \subset W, H(\partial D \times B^2) \subset \partial W$ and $W \cap N = D$. Form M' by attaching a 2-handle to W along $\partial D \times B^2$ and let $\Sigma \subset M'$ consist of D together with the core of the 2-handle. By changing the framing of the 2-handle, if necessary, we can arrange that the self-intersection number of Σ is 0. By Theorem 4.1, Σ is locally flat and thus N is locally flat at each point of $\operatorname{Int} D$. ∎

PROOF OF COROLLARY 4.4. Remove a flat ball pair from a neighborhood of the locally flat point and then attach a 2-handle with appropriate framing so that the new 2-sphere has self-interesection number 0. Apply Theorem 4.1 to the new 2-sphere. ∎

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