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ON SEMICOMMUTING AUTOMORPHISMS OF RINGS

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ABSTRACT. Let R be a prime ring possessing a nontrivial automorphism T such that $x^Tx = \pm xx^T$. If R is not of characteristic 3 or R has nonzero center, then R is a commutative integral domain.

Let R be an associative ring. An automorphism T of R is called a commuting automorphism of R if $x^Tx = xx^T$ for each $x \in R$. In [1], Divinsky showed that a semisimple artinian ring must be commutative if it possesses a nontrivial commuting automorphism. Luh [2] extended this result to arbitrary prime rings. Recently, Mayne [3] generalized this result further by proving that a prime ring R possessing a nontrivial automorphism T such that $x^Tx - xx^T$ is in the center of R is necessarily commutative.

In this paper, we will call an automorphism T of R a semicommuting automorphism if $x^Tx = \pm xx^T$ for each $x \in R$. Clearly, commuting automorphisms are semicommuting automorphisms. We will show that a prime ring with characteristic $\neq 3$ which possesses a nontrivial semicommuting automorphism is necessarily commutative. It is still an open problem whether this result is true for rings with characteristic 3. However, we will show that the answer is affirmative if the ring R has a nonzero center (particularly if R has 1).

Throughout this paper, R denotes an associative ring, T denotes a nontrivial semicommuting automorphism of R, $\Omega_0 = \{x \in R \mid x^Tx = 0\}$, $\Omega_+ = \{x \in R \mid x^Tx = xx^T\}$, and $\Omega = \{x \in R \mid x^Tx = -xx^T\}$. It is clear that $\Omega_0 \subseteq \Omega_+ \cap \Omega_-$ and $\Omega_+ \cup \Omega_- = R$, and that $x \in \Omega_i$ implies $mx \in \Omega_i$ for any integer m and any $i \in \{0, +, -\}$.

For easy reference, we exhibit the proof of the following lemma given in [2].

LEMMA 1. *Let R be a prime ring possessing a nontrivial commuting automorphism T. Then R is a commutative integral domain.*

Proof. For $x, y \in R$, let $[x, y]$ denote $xy - yx$. Polarizing $[x^T, x] = 0$ gives $[y^T, x] = [y, x^T]$ and hence $[(xy)^T, x] = [xy, x^T]$. Since $[(xy)^T, x] = x^T[y^T, x]$ and $[xy, x^T] = x[y, x^T] = x[y^T, x]$, we obtain $(x - x^T)[y^T, x] = 0$. Since T is surjective $(x - x^T)[y, x] = 0$ for all $x, y \in R$. By noting that $z[y, x] = [zy, x] - [z, x]y$ for $z \in R$, we get $(x - x^T)z[y, x] = 0$ for all $z \in R$. This shows that if $z \notin Z$, the

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center of R, then $x=x^T$. Now let $x_0 \in R$ be such that $x_0 - x_0^T \neq 0$. Then by the primeness of R, $[y, x_0]=0$ for all $y \in R$ and, consequently $x_0 \in Z$.

Now suppose there exists $y \in R \backslash Z$. Then $x_0 + y \notin Z$. Thus, $y^T = y$ and $(x_0+y)^T = x_0 + y$ which imply that $x_0^T = x_0$, a contradiction. This completes the proof.

Assume R is a prime ring of characteristic 2. Then $\Omega_+ = \Omega_- = R$. Then, by Lemma 1, R is commutative. In view of this we will assume R is not of characteristic 2 throughout the balance of this paper.

Lemma 2. *Suppose R is a ring which possesses a nontrivial semicommuting automorphism T and $x, y \in \Omega_+$. Then $x+y \in \Omega_+$ if and only if $x-y \in \Omega_+$.*

Proof. Assume $x+y \in \Omega_+$. Then $[(x+y)^T, x+y]=0$ reduces to

$$y^T x + x^T y = y x^T + x y^T. \tag{1}$$

Suppose $x-y \notin \Omega_+$. Then $(x-y)^T(x-y) = -(x-y)(x-y)^T$. It follows that $2(x^Tx - y^Tx - x^Ty + y^Ty)=0$ by (1); so $x^Tx - y^Tx - x^Ty + y^Ty = 0$. Thus $(x-y)^T(x-y)=0$ and $x-y \in \Omega_0 \subseteq \Omega_+$, a contradiction. Similarly $x-y \in \Omega_+$ implies $x+y \in \Omega_+$.

Using a similar proof, we obtain

Lemma 2′. *Suppose R is a ring which possesses a nontrivial semicommuting automorphism T and $x, y \in \Omega_-$. Then $x+y \in \Omega_-$ if and only if $x-y \in \Omega_-$.*

Lemma 3. *Let R be a ring which possesses a nontrivial semicommuting automorphism T. If $x \in \Omega_- \backslash \Omega_+$ then $(x^Tx)^2 = 0$.*

Proof. Clearly, $x^2 \in \Omega_+$. Now consider $x + x^2$ and $x - x^2$.

Suppose $x+x^2 \notin \Omega_-$. Since $x^2x^T = x^Tx^2$ and $x(x^2)^T = (x^2)^Tx$, linearizing $[(x+x^2)^T, x+x^2]=0$ gives $[x^T, x]=0$. So $x \in \Omega_+$, a contradiction. Hence $x+x^2 \in \Omega_-$. Similarly $x - x^2 \in \Omega_-$. By Lemma 2′, since $(x+x^2)+(x-x^2) = 2x \in \Omega_-$ $2x^2 = (x+x^2)-(x-x^2) \in \Omega_-$. So $x^2 \in \Omega_-$ and $(x^2)^Tx^2 = -x^2(x^2)^T$. But $(x^2)^Tx^2 = x^2(x^2)^T$ since $x^2 \in \Omega_+$. We have $x^2(x^2)^T = -x^2(x^2)^T$ and hence $(x^Tx)^2 = 0$ as we desired.

Lemma 4. *Let R be a prime ring of characteristic $\neq 3$. If R possesses a nontrivial semicommuting automorphism T and if $\Omega_- = R$, then R is commutative.*

Proof. Let $x \in R$. If $x \in \Omega_+$ then $x^Tx = xx^T = -xx^T$ which implies that $xx^T = 0$ and hence $(x^Tx)^2 = 0$. If $x \notin \Omega_+$ then by Lemma 3, $(x^Tx)^2 = 0$. Hence $(x^Tx)^2 = 0$ for all $x \in R$.

Now, let $x, y \in R$. From $((x+y)^T(x+y))^2=0$, we get

$$\begin{aligned} &y^Txx^Tx+x^Tyx^Tx+y^Tyx^Tx+x^Txy^Tx+y^Txy^Tx \\ &\quad +x^Tyy^Tx+y^Tyy^Tx+x^Txx^Ty+y^Txx^Ty+x^Tyx^Ty \\ &\quad +y^Tyx^Ty+x^Txy^Ty+y^Txy^Ty+x^Tyy^Ty=0. \end{aligned} \tag{2}$$

In (2), by replacing y by $2y$, we get

$$\begin{aligned} &2y^Txx^Tx+2x^Tyx^Tx+4y^Tyx^Tx+2x^Txy^Tx+4y^Txy^Tx \\ &\quad +4x^Tyy^Tx+8y^Tyy^Tx+2x^Txx^Ty+4y^Txx^Ty+4x^Tyx^Ty \\ &\quad +8y^Tyx^Ty+4x^Txy^Ty+8y^Txy^Ty+8x^Tyy^Ty=0. \end{aligned} \tag{3}$$

Subtracting four times equation (2) from (3), we have

$$\begin{aligned} &-2y^Txx^Tx-2x^Tyx^Tx-2x^Txy^Tx+4y^Tyy^Tx \\ &-2x^Txx^Ty+4y^Tyx^Ty+4y^Txy^Ty+4x^Tyy^Ty=0 \end{aligned} \tag{4}$$

By exchanging x and y in (4), we get

$$\begin{aligned} &-2x^Tyy^Ty-2y^Txy^Ty-2y^Tyx^Ty+4x^Txx^Ty \\ &-2y^Tyy^Tx+4x^Txy^Tx+4x^Tyx^Ty+4y^Txx^Tx=0. \end{aligned} \tag{5}$$

Now add two times equation (5) to (4).
It follows that

$$x^Txx^Ty+x^Txy^Tx+x^Tyx^Tx+y^Txx^Tx=0. \tag{6}$$

Premultiply by x^Tx and postmultiply by x^T on both sides of (6).
We obtain

$$x^Txx^Tyx^Txx^T=0 \tag{7}$$

Since (7) holds for all $x, y \in R$ and R is prime, $x^Txx^T=0$ for all $x \in R$. Likewise $xx^Tx=0$ for all $x \in R$.

Now, we postmultiply by x^T on both sides of (6). We get

$$x^Txy^Txx^T=0.$$

Since y is arbitrary, T is surjective, and R is prime, it follows that $x^Tx=0$ and hence $x \in \Omega_0 \subseteq \Omega_+$ for all $x \in R$. R is therefore commutative by Lemma 1.

LEMMA 5. *Suppose R is a prime ring of characteristic $\neq 3$. If R possesses a nontrivial semicommuting automorphism T and if $x, y \in \Omega_+$, then either $x+y \in \Omega_+$ or $x, y, x+y, x-y \in \Omega_-$.*

Proof. Suppose $x+y \notin \Omega_+$. Then by Lemma 2, $x-y \notin \Omega_+$. We consider $2x+y$. If $2x+y \in \Omega_+$ then, by Lemma 2, $2x-y \in \Omega_+$. Since $(2x-y)+y=2x \in \Omega_+$, $2(x-y)=(2x-y)-y \in \Omega_+$. So $x-y \in \Omega_+$, a contradiction. Hence $2x+y \in \Omega_-$ and, consequently, $2x-y \in \Omega_-$. Since $3(x+y)+(x-y)=2(2x+y) \in \Omega_-$ and $3(x+y)$, $x-y \in \Omega_-$, $3(x+y)-(x-y) \in \Omega_-$; so $2x+4y \in \Omega_-$. Since $(2x+4y)+(x-y)=3(x+y) \in \Omega_-$, $x+5y=(2x+4y)-(x-y) \in \Omega_-$. Since

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$(x+5y)+(x-y)=2(x-2y)\in\Omega_-$, $6y=(x+5y)-(x-y)\in\Omega_-$ and hence $y\in\Omega_-$. Since $(x-y)-(x+y)=-2y\in\Omega$, $2x=(x-y)+(x+y)\in\Omega_-$, and $x\in\Omega_-$. This completes the proof.

As an immediate consequence of Lemma 5 we have

COROLLARY. *Let R be a prime ring of characteristic $\neq 3$. If R possesses a nontrivial semicommuting automorphism T and if $x\in\Omega_+\backslash\Omega_-$ and $y\in\Omega_+$ then $x+y\in\Omega_+$.*

THEOREM 1. *Let R be a prime ring of characteristic $\neq 3$. If R possesses a nontrivial semicommuting automorphism T, then R is a commutative integral domain.*

Proof. Suppose contrarily that R is not commutative. Then $R\neq\Omega_+$ and $R\neq\Omega_-$ by Lemma 1 and Lemma 4. There exist $x\in\Omega_+\backslash\Omega_-$ and $y\in\Omega_-\backslash\Omega_+$.

Suppose $x+y\in\Omega_-$. Then since $(x+y)-y=x\notin\Omega_-$, $(x+y)+y\notin\Omega_-$ by Lemma 2′, i.e. $x+2y\notin\Omega_-$. By the previous corollary, $x-(x+2y)\in\Omega_+$. So $y\in\Omega_+$, a contradiction.

Suppose $x+y\notin\Omega_-$. Then by the previous corollary, $(x+y)-x\in\Omega_+$, i.e. $y\in\Omega_+$. It is again a contradiction. This completes the proof.

We noted in passing that it is still unknown whether Theorem 1 remains true if the characteristic of R is 3. However, we have the following

THEOREM 2. *Let R be a prime of any characteristic. Suppose R possesses a nontrivial semicommuting automorphism T and R has non-zero center. Then R is a commutative integral domain.*

Proof. Let $0\neq z\in Z$, the center of R. Suppose contrarily that R is not commutative. Then there exists $x\in\Omega_-\backslash\Omega_+$. We consider $z+x$. If $z+x\in\Omega_+$, then, since $z^T\in Z$, linearizing $[(z+x)^T, z+x]=0$ gives $[x^T, x]=0$ which is a contradiction. Thus, $z+x\in\Omega_-$ and $(z+x)^T(z+x)=-(z+x)(z+x)^T$. By expansion, it reduces to $2(z^Tz+x^Tz+z^Tx)=0$. So $z^Tz+x^Tz+z^Tx=0$. Thus, $z^Txx^T=-(z^Tz+x^Tz)x^T=-x^T(z^Tz+x^Tz)=x^T(z^Tx)=z^Tx^Tx$. That is, $z^T(xx^T-x^Tx)=0$. By primeness of R, $x^Tx=xx^T$ since $0\neq z^T\in Z$.

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