A NOTE ON PRE-INTERIOR OPERATORS

TENG-SUN LIU *

(Received 18 March 1965)

1

Let X be a non-empty set and denote by PX the set algebra consisting of all subsets of X. An operator $i : PX \to PX$ is said to be *pre-interior* if

(i) $iX = X$;

(ii) $i(A \cap B) = iA \cap iB$ for all A, B in PX.

In this note we first establish the following propositions.

PROPOSITION 1. *Let i be a pre-interior operator on X, then*

$T = \{A \in PX : A \subset iA\}$ *is a topology on* X;

$I = \{A \in PX : icA = X\}$ *is an ideal in* PX. [1]

PROPOSITION 2. *Let (X, T) be a topological space and let I be an ideal in PX, then the mapping $i : PX \to PX$ defined by*

$iA = \{x \in X$: *there exists* $V \in T_x$ [2] *such that* $V - A \in I\}$

is a pre-interior operator on X.

The operator i obtained in Proposition 2 induces, by Proposition 1, a topology M on X. Our discussion will be about this derived topology.

2

To prove the first assertion of Proposition 1, we observe first that (ii) implies that T is closed under finite intersections. It also follows from (ii) that the operator i is monotone and hence T is closed under arbitrary unions. By (i) and the definition of T it is clear that X and the empty set $\emptyset$ are in T. Thus T is a topology on X. For the second assertion we note that (i) implies $\emptyset \in I$ and (ii) implies that I is closed under finite unions and the formation of subsets. Thus I is an ideal in PX.

* This research was sponsored in part by the United States Army Research Office. The author wishes to express his thanks to Professor A. Ionescu Tulcea and Professor C. Ionescu Tulcea for many helpful discussions.

[1] cA denotes the complement of A.

[2] T_x denotes the set of all $V \in T$ such that $x \in V$.

The operator i defined in Proposition 2 obviously satisfies (i) since $\emptyset \in I$. Since i is clearly monotone, to show (ii) it suffices to prove $iA \cap iB \subset i(A \cap B)$. Suppose $x \in iA \cap iB$. Then there exist $U \in T_x$ and $V \in T_x$ such that $U - A \in I$ and $V - B \in I$. Let $W = U \cap V$, then $W \in T_x$ and $W - A \cap B = (W - A) \cup (W - B) \in I$. Hence $x \in i(A \cap B)$. Thus $iA \cap iB \subset i(A \cap B)$ and i is pre-interior.

3

From now on we suppose (X, T) is a topological space, I is an ideal in PX, i is the pre-interior operator on X induced by T and I and M is the topology on X induced by i. Let $d : PX \to PX$ be defined by

$$dA = \{x \in X : \text{for all } V \in T_x,\ V \cap A \notin I\}.$$

We see that $x \in ciA$ if and only if for all $V \in T_x$, $V - A \notin I$, hence if and only if $x \in dcA$. Thus $dA = cicA$. Hence by duality we know that the operator d has the following properties.

(i′) $d\emptyset = \emptyset$;

(ii′) $d(A \cup B) = dA \cup dB$ for all A, B in PX.

Proposition 3.

(1) *M is finer than T.*

(2) $\text{Int}_M A = A \cap iA$ *and hence* $\text{Cl}_M A = A \cup dA$.

(3) *d is the derived operator for M if and only if $\{x\} \in I$ for all $x \in X$.*

Proof. The assertion (1) is obvious. Since iA is in T, it is in M. Therefore $iA \subset iiA$. It follows that $A \cap iA \subset iA = iA \cap iiA = i(A \cap iA)$, hence $A \cap iA$ is M-open. On the other hand, if $x \in \text{Int}_M A$, then there exists $U \subset A$ such that $x \in U$ and $U \subset iU$. From $U \subset A$ we obtain $iU \subset iA$. Thus $x \in A \cap iA$. This proves (2).

Denote the derived operator for M by f, we have by (2) the following equivalences:

$$x \in fA \Leftrightarrow x \in \text{Cl}_M (A - \{x\}) \Leftrightarrow x \in d(A - \{x\}).$$

Therefore $f = d$ if and only if for all $x \in X$ and $A \in PX$, $x \in dA$ implies $x \in d(A - \{x\})$. But this is the case when and only when $\{x\} \in I$ for all $x \in X$. This proves (3).

Proposition 4. *The following statements are pairwise equivalent.*

(1) $i\emptyset = \emptyset$.

(2) *i is dominated by d.*

(3) $T \cap I = \{\emptyset\}$.

PROOF. Since $d\emptyset = \emptyset$, (2) implies (1). Since $V \in T \cap I$ implies $V \subset i\emptyset$, we see that (1) implies (3). Finally suppose $T \cap I = \{\emptyset\}$, then for any $A \subset X$, $x \in iA - dA$ would imply that there exists $U \in T_x$ and $V \in T_x$ such that $U - A \in I$ and $V \cap A \in I$. But then $U \cap V \in T_x$ and $U \cap V \subset (U - A) \cup \cup (V \cap A) \in I$. This is a contradiction. Hence $iA - dA = \emptyset$ and therefore i is dominated by d. Thus (3) implies (2) and the proof is completed.

PROPOSITION 5. *Suppose $i\emptyset = \emptyset$ and Y is a regular space. Then, if $g : X \to Y$ is M-continuous, g is T-continuous.* (cf. [5]).

PROOF. Take any x in X and let W be an open neighborhood of $g(x)$. Let W_1 be an open neighborhood of $g(x)$ such that $W_1 \subset \overline{W}_1 \subset W$. Consider $g^{-1}(Y - W_1)$, since it is M-closed, there exists $U \in T_x$ such that $U \cap g^{-1}(Y - W_1) \in I$. If $U \cap g^{-1}(Y - W) \neq \emptyset$, let x' be a point in it. Then since $g^{-1}(\overline{W}_1)$ is M-closed there exists $U' \in T_x$, such that $U' \cap g^{-1}(\overline{W}_1) \in I$. But then $U \cap U' \in I$ since $U \cap U' \subset (U \cap U' \cap g^{-1}(\overline{W}_1)) \cup (U \cap U' \cap g^{-1}(Y - W_1))$. Also $U \cap U' \neq \emptyset$. This contradicts the assumption $i\emptyset = \emptyset$. Therefore $U \cap g^{-1}(Y - W) = \emptyset$ and $U \subset g^{-1}(W)$. Thus g is T-continuous at x. Since x is arbitrarily taken, the proof is completed.

We have seen in Proposition 3 that M is always finer than T. Let M' be the topology on X generated by T and the complements of elements in I. Since for every $A \in I$, $dA \subset A$ and hence A is M-closed, we have proved the following proposition.

PROPOSITION 6. $T \subset M' \subset M$.

COROLLARY 1. *If $i\emptyset = \emptyset$, then the spaces (X, T), (X, M') and (X, M) have the same continuous functions.*

PROOF. Use Propositions 5 and 6.

PROPOSITION 7. *If $iA - A \in I$ for all $A \in M$ (hence if $iA - A \in I$ for all $A \in PX$), then $M = M'$.*

PROOF. $A \in M \Rightarrow A = iA \cap c(iA - A) \in M'$.

PROPOSITION 8. *$M = T$ if and only if every A in I is T-closed.*

PROOF. If $M = T$, then since every A in I is M-closed, it is T-closed. Conversely suppose every A in I is T-closed. If $B \in M$, then for any $x \in B$, x is in iB. Hence there exists $V \in T_x$ such that $V - B \in I$. Since $V - B$ is T-closed, $V \cap B \in T$. Thus $B \in T$. This proves $M = T$.

COROLLARY 2. *The topology induced by M and I is M itself.*

PROOF. Every $A \in I$ is M-closed.

4

We close this note with some examples.

Example 1. If $I = PX$, then M is the discrete topology. If $I \in \{\emptyset\}$, then $M = T$.

Example 2. Let (X, T) be a topological space. If I is the ideal of all nowhere dense subsets of X, then $i\emptyset = \emptyset$ and $iA = A \in I$ for all $A \subset X$. Thus $M' = M$ and T and M' have the same continuous functions. It can be shown that in (X, M') the ideal of nowhere dense sets is still I.

Example 3. Let (X, T) be second countable and let I be a σ-ideal in PX, then $iA - A \in I$ for all $A \subset X$ (cf. [5]).

Example 4. Let X be a locally compact group and let I be the ideal of all subsets of X with Haar measure 0. Then $i\emptyset = \emptyset$ and $iA - A \in I$ for all $A \subset X$. If X is σ-compact and separable, then $M = M'$ is of the first category (cf. [4]).

Example 5. Let $X = R \cup \{p\}$ where R is an infinite set and $p \notin R$. Let F be an ultrafilter on R containing all subsets of R whose complements are finite. We define a topology T on X by stipulating a set U to be in T if and only if $U \subset R$ or $p \in U$ and $U - \{p\} \in F$. Next we define an ideal I in PX by $A \in I$ if and only if $A - \{p\} \notin F$. Then the topology M induced by T and I is T itself. This can be deduced either from the fact that T is a maximal T_1-topology (see [3]) or from Proposition 8.

References

[1] P. R. Halmos, Measure Theory, Van Nostrand, New York, 1950.
[2] J. L. Kelley, General Topology, Van Nostrand, New York, 1955.
[3] Teng-sun Liu, A Note on Maximal T_1-topologies, Portugal Math. 18 (1959), 235–236.
[4] Teng-sun Liu, On Vanishing Algebras, Proc. Amer. Math. Soc. 14 (1963), 162–165.
[5] N. F. G. Martin, Generalized Condensation Points, Duke Math. J. 28 (1961), 507–514.

University of Pennsylvania and University of Massachusetts