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COVARIANCE FACTORISATION AND ABSTRACT REPRESENTATION OF GENERALISED RANDOM FIELDS

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This paper introduces a new concept of duality of generalised random fields using the geometric properties of Sobolev spaces of integer order. Under this duality condition, the covariance operators of a generalised random field and its dual can be factorised. The paper also defines a concept of generalised white noise relative to the geometries of the Sobolev spaces, and via the covariance factorisation, obtains a representation of the generalised random field as a stochastic equation driven by a generalised white noise. This representation is unique except for isometric isomorphisms on the parameter space.

1. INTRODUCTION

The problem of filtering and prediction plays a central role in the theory of stochastic processes and random fields. Its importance is due not only to the practical applications it has generated, but also because a large part of stochastic processes and random fields has been developed for its solution. The pioneering work of Kolmogorov [**6, 7, 8**] and Wiener [**14**] on prediction theory was concerned with a stationary stochastic process observed on a semi-infinite interval of time (discrete time in Kolmogorov's theory and continuous time in Wiener's theory). Wiener reduced the prediction problem to the solution of a Wiener-Hopf integral equation and gave the spectral factorisation method to solve this kind of integral equation. Kolmogorov reduced the problem to a Wold decomposition and gave a solution to the prediction problem directly. For stationary Gaussian processes, the prediction problem based on finite data was solved by Krein [**9, 10**]. In Krein's approach, the problem was reduced to that of finding the differential equation of oscillation of a nonhomogeneous string and its eigenfunctions from the given spectral density (that is, an inverse problem). An explicit prediction formula was derived for the case of rational spectral density using the theory of differential equations. A detailed account on Krein's theory was described in Dym and McKean [**4**]. A physical basis for Krein's prediction formula was given in Anderson [**1**].

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A direct extension of Wiener's theory was given in Zadeh and Ragazzini [**15**] for the prediction of stationary processes with rational spectral density observed on a finite interval. This extension was given a more rigorous treatment in Dolph and Woodbury [**3**], where the relationship between the covariance function of an autoregressive process and the Green function of a second-order self-adjoint linear differential equation was exploited. An extension of Dolph and Woodbury's theory to the nonstationarity case of time-dependent rational spectrum was obtained in Anh and Spencer [**2**]. Another line of significant development was detailed in Ramm [**11**] for random fields, where the random fields are characterised by the covariance kernels $R(x,y) = \int_{\Lambda} \left(P(\lambda)/Q(\lambda)\right)\Phi(x,y,\lambda)\,d\rho(\lambda)$, $P(\lambda), Q(\lambda)$ being positive polynomials, $\Lambda, d\rho, \Phi$ being respectively the spectrum, spectral measure and spectral kernel of an elliptic self-adjoint operator in $L^2(\mathbb{R}^n), n \geqslant 1$. Ramm's theory, which makes use of properties of Sobolev spaces of integer order, generalised the Wiener-Kolmogorov prediction theory to nonstationary random fields with rational spectra.

The above lines of developments represent a small subset of the works on filtering and prediction. A common feature of these approaches is the reduction of the problem to that of spectral decomposition (in the spectral domain) or equivalently covariance factorisation / Markovian representation / Wold decomposition (in the time / spatial domain). Solution to the latter problem leads, under certain conditions, to a sample path representation of the stochastic process or random field, commonly cast in the form of a stochastic differential equation (SDE) (for stochastic processes) or a stochastic partial differential equation (SPDE) (for random fields). This SDE / SPDE setting offers a convenient framework for filtering and prediction of stochastic processes and random fields.

A key question stands out in this field of research: What kind of conditions would allow a solution to the spectral decomposition / covariance factorisation problem? We have seen above that the rational form of the spectral density features prominently in the cited approaches. A more general condition is the Markovianness of stochastic processes and random fields. A general theory of generalised Markov random fields is detailed in Rozanov [**12**] and Rozanov [**13**]. In particular, the Markov property of generalised random fields arising as solutions of linear SPDEs has been extensively investigated. Here, the Markov property, based on the concept of splitting fields, is understood in the wide sense and shown to be equivalent to a duality condition (that is, the existence of a biorthogonal function in the sense of Rozanov [**13**]).

In this paper, we introduce a new concept of duality of generalised random fields (not necessarily Gaussian) in terms of the geometric properties of Sobolev spaces of integer order. This duality condition guarantees the factorisation of the covariance operator of the random field. We also introduce the concept of generalised white noise and reformulate the problem of representation by an abstract equation. Here, the representation will take the form of existence of an isomorphism L on the parameter Hilbert space W of the generalised random field X such that the geometry induced by the generalised random field XL coincides with the geometry of the parameter space W. We show that such an abstract representation exists and is unique under the duality condition defined in this paper. Another issue, which is intrinsic to the problem of filtering and prediction, is under what conditions L is a local operator, hence yielding an SPDE-type representation of the random field. This important issue will be taken up in a subsequent paper.

2. Generalised random fields

Let $T \subseteq \mathbb{R}^d$, $d \geqslant 1$, be a bounded domain. We denote by $\mathcal{D} = C_0^\infty(T)$ the space of all infinitely differentiable functions with compact supports in T, provided with the intersection topology induced by the sequence of norms associated with the Sobolev spaces of integer order, $H_2^n(T)$, $n \in \mathbb{N}$:

$$\mathcal{D} = \bigcap_{n=0}^{\infty} \mathcal{D}_n; \quad \mathcal{D}_n = \overline{C_0^\infty(T)}^{H_2^n(T)}, \quad n \in \mathbb{N}.$$

$\mathcal{D}'$denotes the space of distributions (continuous linear functionals) on $\mathcal{D}$:

$$\mathcal{D}' = \bigcup_{n=0}^{\infty} \mathcal{D}'_n,$$

with $\mathcal{D}'_n$ being the dual space of $\mathcal{D}_n$, for $n \in \mathbb{N}$. We denote by $\mathcal{D}_{-n}$ the space $\mathcal{D}'_n$, for $n \in \mathbb{N}$. We also denote by $\mathcal{D}_n^*$ the dual space of $\mathcal{D}_n$ provided with the inner product induced from the inner product in $\mathcal{D}_n$ by the Riesz Representation Theorem:

$$\langle \phi^*, \psi^* \rangle_{\mathcal{D}_n^*} = \langle \phi, \psi \rangle_{\mathcal{D}_n}, \quad \forall \phi, \psi \in \mathcal{D}_n,$$

with ϕ^* and ψ^* representing the respective dual elements of ϕ and ψ, and $\langle \cdot, \cdot \rangle_{\mathcal{D}_n}$ the restriction of the inner product $\langle \cdot, \cdot \rangle_{H_2^n(T)}$ to $\mathcal{D}_n$.

For a given complete probability space $(\Omega, \mathcal{A}, P)$, $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ represents the Hilbert space of real-valued zero-mean random variables defined on $(\Omega, \mathcal{A}, P)$ with finite second-order moments, and with inner product defined by

$$\langle X, Y \rangle_{\mathcal{L}^2(\Omega)} = E[XY]; \quad X, Y \in \mathcal{L}^2(\Omega, \mathcal{A}, P).$$

Definition 2.1: A random function $X(\cdot)$ from $\mathcal{D}$ into $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ is said to be a *generalised random field* if it is linear and continuous in the mean-square sense with respect to the $\mathcal{D}-$topology.

For a generalised random field $X : \mathcal{D} \to \mathcal{L}^2(\Omega, \mathcal{A}, P)$, we denote by $H(X)$ the closed span of $\{X(\varphi) : \varphi \in \mathcal{D}\}$ in $\mathcal{L}^2(\Omega, \mathcal{A}, P)$. $H(X)$ is provided with the inner product $\langle \cdot, \cdot \rangle_{H(X)}$, which is the restriction of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}^2(\Omega)}$ to $H(X)$. From Definition 2.1, the generalised covariance function $B(\varphi, \xi) = E[X(\varphi) X(\xi)]$, with $\varphi, \xi \in \mathcal{D}$, is a bilinear continuous operator on $\mathcal{D} \times \mathcal{D}$.

DEFINITION 2.2: For $n \in \mathbb{Z}$, a random function $X(\cdot)$ from $\mathcal{D}_n$ into $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ is said to be an *n-generalised random field* if it is linear and continuous in the mean-square sense with respect to the $\mathcal{D}_n$-topology.

In this case, $H(X)$ represents the closed span of $\{X(\varphi) : \varphi \in \mathcal{D}_n\}$.

From Definition 2.2, it is clear that, as n increases, test functions in the space $\mathcal{D}_n$ are more regular, and a higher degree of singularity is allowed for the random field X. Definition 2.1 corresponds to the limiting case $n = \infty$. For $m \geqslant n$, the restriction of an n-generalised random field to $\mathcal{D}_m$ is an m-generalised random field.

The covariance functional $B(\cdot,\cdot)$ of a generalised random field X admits a representation as

$$B(\varphi, \xi) = \left\langle (R_n\varphi)^*, \xi \right\rangle_{\mathcal{D}_n}, \quad \varphi, \xi \in \mathcal{D},$$

with R_n being a symmetric positive continuous linear operator from some $\mathcal{D}_n$, $n \in \mathbb{Z}$, into $\mathcal{D}_{-n}$. For each $m \geqslant n$, a corresponding operator R_m from $\mathcal{D}_m$ into $\mathcal{D}_{-m}$ is then defined for X (see Gelfand and Vilenkin [5, p.74]). The random field X can then be extended by continuity on the space $\mathcal{D}_n$, for any $n \in \mathbb{Z}$ for which the operator R_n exists. This extension is an n-generalised random field.

In the following development, we assume X to be an n-generalised random field, and denote $W = \mathcal{D}_n$ and $F = \mathcal{D}_n^*$. The following concept of duality between two generalised random fields is fundamental in the construction of the results in this paper.

DEFINITION 2.3: An n-generalised random field $\widetilde{X} : \mathcal{D}_n^* \longrightarrow \mathcal{L}^2(\Omega, \mathcal{A}, P)$ is said to be the *dual relative to* $\mathcal{D}_n$ of the n-generalised random field $X : \mathcal{D}_n \longrightarrow \mathcal{L}^2(\Omega, \mathcal{A}, P)$ if it satisfies:

(i) $H(X) = H(\widetilde{X})$, and

(ii) $\langle X(\varphi), \widetilde{X}(f) \rangle_{H(X)} = \langle \phi, f^* \rangle_{\mathcal{D}_n}$, for $\phi \in \mathcal{D}_n$, and $f \in \mathcal{D}_n^*$, with f^* being the dual of f with respect to the $\mathcal{D}_n$-topology.

From this definition, and considering the isoporphism between a Hilbert space and its dual, it is clear that the dual (relative to $\mathcal{D}_n^*$) of $\widetilde{X}$ is equal to X.

REMARK 2.1. The existence of a 'biorthogonal function', in the sense of Rozanov [**13**, Chapter 3, Section 1], implies, by continuity and using the isomorphism between dual Hilbert spaces, the existence of the dual n-generalised random field $\widetilde{X}$, for an n-generalised random field X. Conversely, the restriction to $\mathcal{D}_m^*, m > n$, of the dual relative to $\mathcal{D}_n$ of an n-generalised random field is its dual relative to $\mathcal{D}_m$. Therefore, the existence of the dual relative to $\mathcal{D}_n$, for some $n \in \mathbb{Z}$, implies the existence of the 'biorthogonal function'. However, our concept of duality relative to $\mathcal{D}_n$ is fundamental for deriving the covariance factorisation using the theory of Hilbert spaces.

We denote by $\mathcal{H}(X)$ the reproducing kernel Hilbert space (RKHS) of X, defined as the closed span of $\{B(\phi, \cdot) : \phi \in W\}$ with respect to the norm of $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ (note that here $B(\phi,\cdot)$ is identified with $X(\phi)$, for each $\phi \in W$). Similarly, $\mathcal{H}(\widetilde{X})$ will represent the RKHS of $\widetilde{X}$, defined as the closed span of $\{\widetilde{B}(f,\cdot) : f \in F\}$ with respect to the norm of $\mathcal{L}^2(\Omega, \mathcal{A}, P)$, with $\widetilde{B}$ being the covariance functional of $\widetilde{X}$. In $\mathcal{H}(X)$ and $\mathcal{H}(\widetilde{X})$ we consider the topologies induced by the $\mathcal{L}^2(\Omega)$-norm.

3. Covariance factorisation

Under the existence of the dual $\widetilde{X}$ (relative to $\mathcal{D}_n$) of an n-generalised random field X, we prove that the covariance operator R_n (respectively, $\widetilde{R}_n$ for $\widetilde{X}$) can be factorised. This factorisation leads to an abstract representation of the n-generalised random field X in terms of a generalised white noise (see Section 4).

We first introduce some operators. Assume X and $\widetilde{X}$ are dual (relative to $\mathcal{D}_n$) n-generalised random fields. We define the following two isometric isomorphisms, $J : H(X) \longrightarrow \mathcal{H}(X)$ and $\widetilde{J} : H(\widetilde{X}) \longrightarrow \mathcal{H}(\widetilde{X})$ by

$$Y \longrightarrow JY, \text{ with } (JY)(\phi) = EYX(\phi), \quad \forall \phi \in W$$

and

$$Z \longrightarrow \widetilde{J}Z, \text{ with } (\widetilde{J}Z)(f) = EZ\widetilde{X}(f), \quad \forall f \in F,$$

respectively. Also, we define the following two operators:

$$K : \mathcal{H}(X) \longrightarrow F, \text{ with } f \longrightarrow Kf = f$$

and

$$\widetilde{K} : \mathcal{H}(\widetilde{X}) \longrightarrow W, \text{ with } \phi \longrightarrow \widetilde{K}\phi = \phi.$$

Note that, by definition, $\mathcal{H}(X) \subseteq F$ and $\mathcal{H}(\widetilde{X}) \subseteq W$, as sets of functionals.

We denote

$$U := KJ : H(X) \longrightarrow F,$$

and

$$\widetilde{U} := \widetilde{K}\widetilde{J} : \widetilde{H}(\widetilde{X}) \longrightarrow W.$$

The following lemma gives the relationship between an n-generalised random field X and its dual $\widetilde{X}$ relative to $\mathcal{D}_n$ (if it exists), and between their respective covariance operators, R_n and $\widetilde{R}_n$.

Lemma 3.1. *Assume that X and $\widetilde{X}$ are dual (relative to $\mathcal{D}_n$) n-generalised random fields. Then*

(i) *$\mathcal{H}(X) = F$ and $\mathcal{H}(\widetilde{X}) = W$, as sets of functions;*

(ii) *$\widetilde{U}X$ is the identity operator on the space W, and, reciprocally, $X\widetilde{U}$ is the identity operator on the space $H(X)$; similarly, $U\widetilde{X}$ is the identity operator on the space F, and, reciprocally, $\widetilde{X}U$ is the identity operator on the space $H(X)$;*

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(iii) *with operator A defined as*

$$A := \widetilde{U}U^{-1} : F \longrightarrow W$$

the following relationships hold:

(a) $X(\phi) = \widetilde{X}(A^{-1}\phi)$, $\forall \phi \in W$, *and* $\widetilde{X}(f) = X(Af)$, $\forall f \in F$;

(b) $R_n\phi = A^{-1}\phi$, $\forall \phi \in W$, *and* $\widetilde{R}_n f = Af$, $\forall f \in F$; *then,* $\widetilde{R}_n R_n$ *is the identity operator on the space* W, *and* $R_n\widetilde{R}_n$ *is the identity operator on the space* F.

PROOF: (i) By definition, $\mathcal{H}(X) \subseteq F$. Now, let $f \in F$. Then, by the duality of X and $\widetilde{X}$ relative to $\mathcal{D}_n$,

$$f(\phi) = \langle f^*, \phi\rangle_{\mathcal{D}_n} = \left\langle \widetilde{X}(f), X(\phi)\right\rangle_{H(X)}, \quad \forall \phi \in W.$$

Thus, there exists $Z = \widetilde{X}(f) \in H(X)$ such that

$$f(\phi) = EZX(\phi), \quad \forall \phi \in W.$$

That is, $f \in \mathcal{H}(X)$.

Similarly, it can be proved that $\mathcal{H}(\widetilde{X}) = W$.

(ii) Let $\phi \in W$. From the duality condition,

$$\left\langle \widetilde{X}(f), X(\phi)\right\rangle_{H(X)} = \langle f^*, \phi\rangle_{\mathcal{D}_n} = \langle f, \phi^*\rangle_{\mathcal{D}_n^*}, \quad \forall f \in F.$$

That is,

$$E\widetilde{X}(f)X(\phi) = \left[\widetilde{J}(X(\phi))\right](f) = \left[\widetilde{U}(X(\phi))\right](f) = \phi(f), \quad \forall f \in F.$$

Hence, $\widetilde{U}(X(\phi)) = \phi$, for each $\phi \in W$. Then, $\widetilde{U}X$ is the identity operator on W. Reciprocally, let $Z \in H(\widetilde{X}) = H(X)$. Then, $\widetilde{U}Z = \widetilde{J}Z = \phi_Z \in \mathcal{H}(\widetilde{X}) = W$ is defined by

$$(\widetilde{U}Z)(f) = EZ\widetilde{X}(f), \quad \forall f \in F.$$

Again, from the duality condition,

$$\begin{aligned}\left\langle \widetilde{X}(f), X(\widetilde{U}Z)\right\rangle_{H(X)} &= \left\langle \widetilde{X}(f), X(\phi_Z)\right\rangle_{H(X)} = \langle f, \phi_Z^*\rangle_{\mathcal{D}_n^*} = EZ\widetilde{X}(f) \\ &= \left\langle Z, \widetilde{X}(f)\right\rangle_{H(X)}.\end{aligned}$$

Then

$$\left(X(\widetilde{U}Z) - Z\right) \underset{H(X)}{\perp} \widetilde{X}(f), \quad \forall f \in F.$$

As $\widetilde{X}(F)$ is dense in $H(\widetilde{X}) = H(X)$, $X(\widetilde{U}Z) = Z$. Hence, $X\widetilde{U}$ is the identity operator on $H(X)$.

Similarly, it can be proved that $U\widetilde{X}$ an $\widetilde{X}U$ are identity operators on F and $H(X)$, respectively.

(iii) (a) For each $f \in F$,

$$X(Af) = X\left[\left(\widetilde{U}U^{-1}\right)(f)\right] = \left(X\widetilde{U}\right)\left[U^{-1}(f)\right] = \widetilde{X}(f).$$

The other assertion is similarly obtained.

(b) Let $\xi \in W$. From (iii)(a) and the duality of X and $\widetilde{X}$ relative to $\mathcal{D}_n$,

$$\begin{aligned} 0 &= \left\langle X(\xi) - \widetilde{X}\left(A^{-1}\xi\right), X(\phi)\right\rangle_{H(X)} \\ &= \left\langle X(\xi), X(\phi)\right\rangle_{H(X)} - \left\langle \widetilde{X}\left(A^{-1}\xi\right), X(\phi)\right\rangle_{H(X)} \\ &= \left\langle (R_n\xi)^*, \phi\right\rangle_{\mathcal{D}_n} - \left\langle \left(A^{-1}\xi\right)^*, \phi\right\rangle_{\mathcal{D}_n} = \left\langle \left(R_n\xi - A^{-1}\xi\right)^*, \phi\right\rangle_{\mathcal{D}_n}, \quad \forall\phi \in W. \end{aligned}$$

Then, $R_n\xi = A^{-1}\xi$, for each $\xi \in W$. Analogously, $\widetilde{R}_n f = Af$, for each $f \in F$. The remaining assertions are similarly deduced. ∎

The next corollary gives additional information about the relationships between the spaces $\mathcal{H}(X)$ and F, and $\mathcal{H}(\widetilde{X})$ and W, respectively.

Corollary 3.2. *Under the condition of Lemma 3.1, the spaces $\left(W, \langle\cdot,\cdot\rangle_{\mathcal{D}_n}\right)$ (respectively, $\left(F, \langle\cdot,\cdot\rangle_{\mathcal{D}_n^*}\right)$) and $\left(\mathcal{H}(\widetilde{X}), \langle\cdot,\cdot\rangle_{\mathcal{H}(\widetilde{X})}\right)$ (respectively, $\left(\mathcal{H}(X), \langle\cdot,\cdot\rangle_{\mathcal{H}(X)}\right)$) are topologically equivalent.*

Proof: Let $\{\phi_n\}_{n\in\mathbf{N}}$ be a sequence in W convergent to $\phi \in W$ in the $\mathcal{D}_n$-norm. From the continuity of X and Lemma 3.1, the following implications hold:

$$\begin{aligned} \phi_n \xrightarrow{\mathcal{D}_n} \phi &\Longrightarrow X(\phi_n) \xrightarrow{H(X)} X(\phi) \Longleftrightarrow \widetilde{J}X(\phi_n) \xrightarrow{\mathcal{H}(\widetilde{X})} \widetilde{J}X(\phi) \\ &\Longleftrightarrow \widetilde{K}^{-1}(\phi_n) \xrightarrow{\mathcal{H}(\widetilde{X})} \widetilde{K}^{-1}(\phi). \end{aligned}$$

Conversely, suppose that $\{\phi_n\}_{n\in\mathbf{N}} \in \mathcal{H}(\widetilde{X})$ converges to $\phi \in \mathcal{H}(\widetilde{X})$ in the $\mathcal{H}(\widetilde{X})$-norm. Then, from the above equivalences,

$$X\left(\widetilde{K}\phi_n\right) \xrightarrow{H(X)} X\left(\widetilde{K}\phi\right),$$

that is,

$$X\left(\widetilde{K}\phi_n - K\phi\right) \xrightarrow{H(X)} 0.$$

By continuity of the inner product $\langle\cdot,\cdot\rangle_{H(X)}$, and the duality condition, we have, for each $f \in F$,

$$\begin{aligned} 0 &= \left\langle \lim_{n\to\infty} X\left(\widetilde{K}\phi_n - \widetilde{K}\phi\right), \widetilde{X}(f)\right\rangle_{H(X)} = \lim_{n\to\infty}\left\langle X\left(\widetilde{K}\phi_n - \widetilde{K}\phi\right), \widetilde{X}(f)\right\rangle_{H(X)} \\ &= \lim_{n\to\infty}\left\langle \widetilde{K}\phi_n - \widetilde{K}\phi, f^*\right\rangle_{\mathcal{D}_n} = \left\langle \lim_{n\to\infty}\left(\widetilde{K}\phi_n - \widetilde{K}\phi\right), f^*\right\rangle_{\mathcal{D}_n}. \end{aligned}$$

Then,

$$\left(\lim \widetilde{K}\phi_n - \widetilde{K}\phi\right) \underset{\mathcal{D}_n}{\perp} f^*,$$

for all $f \in F$. Hence,

$$\widetilde{K}\phi_n \xrightarrow{\mathcal{D}_n} \widetilde{K}\phi.$$

The proof for F and $\mathcal{H}(X)$ is similar. ∎

From Corollary 3.2, the operators K and $\widetilde{K}$ are isomorphisms between $\mathcal{H}(X)$ and F, and $\mathcal{H}(\widetilde{X})$ and W, respectively. Also the operator A is an isomorphism (non isometric) between F and W.

Since, from Lemma 3.1, the operators R_n and $\widetilde{R}_n$ are injective, the binary operators

$$\langle \phi, \psi \rangle_B = B(\phi, \psi), \quad \forall \phi, \psi \in W,$$
$$\langle f, g \rangle_{\widetilde{B}} = \widetilde{B}(f, g), \quad \forall f, g \in F,$$

define, under the duality condition, inner products on the spaces W and F, respectively. In the following corollary, it is proved that the topologies defined by these inner products are dual, and hence the operators R_n and $\widetilde{R}_n$ are adjoint to each other with respect to these inner products; that is,

$$\langle R_n\phi, \psi_B^* \rangle_{\widetilde{B}} = \left\langle \phi, \widetilde{R}_n \psi_B^* \right\rangle_B, \quad \forall \phi, \psi \in W,$$

with ψ_B^* being the dual element of $\psi \in W$ with respect to the topology induced by $\langle \cdot, \cdot \rangle_B$.

Corollary 3.3. *Under the conditions of Lemma 3.1,* $(W, \langle \cdot, \cdot \rangle_B) = \left(\mathcal{H}(\widetilde{X}), \langle \cdot, \cdot \rangle_{\mathcal{H}(\widetilde{X})}\right)$ *and* $(F, \langle \cdot, \cdot \rangle_{\widetilde{B}}) = \left(\mathcal{H}(X), \langle \cdot, \cdot \rangle_{\mathcal{H}(X)}\right)$ *are dual Hilbert spaces.*

Proof: We first show the identity between the spaces $(W, \langle \cdot, \cdot \rangle_B) = \left(\mathcal{H}(\widetilde{X}), \langle \cdot, \cdot \rangle_{\mathcal{H}(\widetilde{X})}\right)$, and $(F, \langle \cdot, \cdot \rangle_{\widetilde{B}}) = \left(\mathcal{H}(X), \langle \cdot, \cdot \rangle_{\mathcal{H}(X)}\right)$. Let $\phi, \xi \in W$. From the proof of Lemma 3.1(i),

$$\langle \phi, \xi \rangle_B = B(\phi, \xi) = \left\langle X(\phi), X(\xi) \right\rangle_{H(X)} = \langle \phi, \xi \rangle_{\mathcal{H}(\widetilde{X})}.$$

The second identity is similarly proved. Now, denote by ϕ_B^* the dual element of $\phi \in W$ with respect to the topology induced by $\langle \cdot, \cdot \rangle_B$. For each $\psi \in W$,

$$\phi_B^*(\psi) = \langle \phi, \psi \rangle_B = B(\phi, \psi) = \left\langle (R_n\phi)^*, \psi \right\rangle_{\mathcal{D}_n} = (R_n\phi)(\psi).$$

That is, $\phi_B^* = R_n\phi$ in F, for each $\phi \in W$. Similarly, $f_{\widetilde{B}}^* = \widetilde{R}_n f$ in W, for each $f \in F$. According to the Riesz Representation Theorem, to prove the assertion in the corollary we need to show that

$$\langle \phi, \psi \rangle_B = \langle \phi_B^*, \psi_B^* \rangle_{\widetilde{B}}, \quad \forall \phi, \psi \in W.$$

In effect,

$$\langle \phi_B^*, \psi_B^* \rangle_{\widetilde{B}} = \langle R_n\phi, R_n\psi \rangle_{\widetilde{B}} = \widetilde{B}\,(R_n\phi, R_n\psi) = \left\langle \left(\widetilde{R}_n R_n \phi\right)^*, R_n\psi \right\rangle_{\mathcal{D}_n^*}$$
$$= \langle \phi^*, R_n\psi \rangle_{\mathcal{D}_n^*} = \left\langle \phi, (R_n\psi)^* \right\rangle_{\mathcal{D}_n} = B\,(\phi, \psi) = \langle \phi, \psi \rangle_B.$$

From this identity it is also clear that R_n and $\widetilde{R}_n$ are adjoint operators with respect to the inner products $\langle \cdot, \cdot \rangle_B$ and $\langle \cdot, \cdot \rangle_{\widetilde{B}}$. ∎

The geometries of the spaces $(W, \langle \cdot, \cdot \rangle_{\mathcal{D}_n})$, $(W, \langle \cdot, \cdot \rangle_B)$, $(F, \langle \cdot, \cdot \rangle_{\mathcal{D}_n^*})$ and $(F, \langle \cdot, \cdot \rangle_{\widetilde{B}})$ are then related by the following identities:

$$\langle \phi, f^* \rangle_{\mathcal{D}_n} = \langle \phi^*, f \rangle_{\mathcal{D}_n^*} = \langle R_n\phi, f \rangle_{\widetilde{B}} = \langle \phi, \widetilde{R}_n f \rangle_B, \quad \forall \phi \in W,\ \forall f \in F.$$

In the next corollary we show that the operators R_n and $\widetilde{R}_n$ are essentially self-adjoint. Let I_W and I_F denote the following isomorphisms:

$$I_W : W \longrightarrow W^* = F,\ I_W(\phi) = \phi^*,\ \forall \phi \in W,\ \text{and}$$
$$I_F : F \longrightarrow F^* = W,\ I_F(f) = f^*,\ \forall f \in F.$$

COROLLARY 3.4. *Under the conditions of Lemma 3.1, $I_F R_n$ and $\widetilde{R}_n I_W$ are self-adjoint operators on W, and $R_n I_F$ and $I_W \widetilde{R}_n$ are self-adjoint operators on F.*

PROOF: Let $\phi, \psi \in W$. Then,

$$\left\langle (I_F R_n)(\phi), \psi \right\rangle_{\mathcal{D}_n} = \left\langle (R_n\phi)^*, \psi \right\rangle_{\mathcal{D}_n} = B\,(\phi, \psi) = \left\langle \phi, (R_n\psi)^* \right\rangle_{\mathcal{D}_n}$$
$$= \left\langle \phi, (I_F R_n)(\psi) \right\rangle_{\mathcal{D}_n}.$$

Now, let $f, g \in F$. Then,

$$\left\langle (R_n I_F)(f), g \right\rangle_{\mathcal{D}_n^*} = \left\langle (R_n f^*), g \right\rangle_{\mathcal{D}_n^*} = \left\langle (R_n f^*)^*, g^* \right\rangle_{\mathcal{D}_n} = B\,(f^*, g^*)$$
$$= \left\langle f^*, (R_n g^*)^* \right\rangle_{\mathcal{D}_n} = \langle f, R_n g^* \rangle_{\mathcal{D}_n^*} = \left\langle f, (R_n I_F)\, g \right\rangle_{\mathcal{D}_n^*}.$$

The other cases are similarly proved in terms of $\widetilde{B}$ instead of B. ∎

The following theorem gives a factorisation of each of the operators R_n and $\widetilde{R}_n$, and shows that dual random fields X and $\widetilde{X}$ are esentially inverse adjoint to each other.

Let us denote by $\mathcal{J}$ the isometric isomorphism

$$\mathcal{J} : H(X) \longrightarrow [H(X)]^*$$

given by the Riesz Representation Theorem; that is, for each $Y \in H(X)$,

$$\mathcal{J}(Y) = Y^*,\ \text{with}\ Y^*(Z) = \langle Y, Z \rangle_{H(X)},\ \text{for all}\ Z \in H(X).$$

THEOREM 3.5. *Let X and $\widetilde{X}$ be dual (relative to $\mathcal{D}_n$) n-generalised random fields. Then, the covariance operator R_n of X can be factorised as*

$$R_n = U\widetilde{U}^{-1}.$$

Correspondingly, the covariance operator $\widetilde{R}_n$ of $\widetilde{X}$ can be factorised as

$$\widetilde{R}_n = \widetilde{U}U^{-1}.$$

The operators

$$U^* = \mathcal{J}\widetilde{U}^{-1} = \left(\widetilde{U}\mathcal{J}^{-1}\right)^{-1},$$
$$\widetilde{U}^* = \mathcal{J}U^{-1} = \left(U\mathcal{J}^{-1}\right)^{-1}$$

are, respectively, the adjoint operators of U and $\widetilde{U}$. Also, the operators

$$J^* = \mathcal{J}\widetilde{J}^{-1} = \left(\widetilde{J}\mathcal{J}^{-1}\right)^{-1},$$
$$\widetilde{J}^* = \mathcal{J}J^{-1} = \left(J\mathcal{J}^{-1}\right)^{-1}$$

are, respectively, the adjoint operators of J and $\widetilde{J}$.

REMARK 3.1. According to the theorem and from Lemma 3.1(i), we can say that $R_n = J\widetilde{J}^{-1}$ and $\widetilde{R}_n = \widetilde{J}J^{-1}$ as mappings, and that J and $\widetilde{J}$ are essentially inverse adjoint operators, considering in the respective image spaces either the inner products $\langle\cdot,\cdot\rangle_B$ and $\langle\cdot,\cdot\rangle_{\widetilde{B}}$, or the inner products $\langle\cdot,\cdot\rangle_{\mathcal{D}_n}$ and $\langle\cdot,\cdot\rangle_{\mathcal{D}_n^*}$.

PROOF: The first part of the theorem is straighforward from Lemma 3.1(iii):

$$R_n\phi = A^{-1}\phi = \left(\widetilde{U}U^{-1}\right)^{-1}\phi, \quad \forall\phi \in W;$$
$$\widetilde{R}_n f = Af = \left(\widetilde{U}U^{-1}\right)f, \quad \forall f \in F.$$

Next, we show that U and $\mathcal{J}\widetilde{U}^{-1}$ are adjoint operators. For $Y \in H(X)$ and $\phi \in W$, we have

$$\left\langle (UY)^*, \phi\right\rangle_{\mathcal{D}_n} = \left\langle \widetilde{X}(UY), X(\phi)\right\rangle_{H(X)} = \left\langle Y, \widetilde{U}^{-1}(\phi)\right\rangle_{H(X)} = \left\langle Y, (U^*\phi)^*\right\rangle_{H(X)}.$$

The proof for $\widetilde{U}$ and $\mathcal{J}U^{-1}$ is similar. Finally, we prove that J and $\mathcal{J}\widetilde{J}^{-1}$ are also adjoint operators. For $Y \in H(X)$ and $\phi \in \mathcal{H}(\widetilde{X})$, we have

$$\begin{aligned}\left\langle (JY)^*_{\mathcal{H}(X)}, \phi\right\rangle_{\mathcal{H}(\widetilde{X})} &= \left\langle (UY)^*_{\widetilde{B}}, \widetilde{K}\phi\right\rangle_B = \left\langle \widetilde{R_n}(UY), \widetilde{K}\phi\right\rangle_B = \left\langle \widetilde{U}Y, \widetilde{K}\phi\right\rangle_B \\ &= \left\langle X(\widetilde{U}Y), X(\widetilde{K}\phi)\right\rangle_{H(X)} = \left\langle Y, \widetilde{U}^{-1}(\widetilde{K}\phi)\right\rangle_{H(X)} \\ &= \left\langle Y, \widetilde{J}^{-1}\phi\right\rangle_{H(X)} = \left\langle Y, (\mathcal{J}\widetilde{J}^{-1}\phi)^*\right\rangle_{H(X)} \\ &= \left\langle Y, (J^*\phi)^*\right\rangle_{H(X)}.\end{aligned}$$

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Here, $f^*_{\mathcal{H}(X)}$, for $f \in \mathcal{H}(X)$, denotes the dual element of f with respect to the topology induced by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}(X)}$.

The proof for $\widetilde{J}$ and $\mathcal{J}J^{-1}$ is similarly derived. □

The following well-known adjointness relationships are straightforward after Theorem 3.5:

$$\begin{aligned}
\left(U^{-1}\right)^* &= \widetilde{U}\mathcal{J}^{-1} = \left(\mathcal{J}\widetilde{U}^{-1}\right)^{-1} = \left(U^*\right)^{-1};\\
\left(\widetilde{U}^{-1}\right)^* &= U\mathcal{J}^{-1} = \left(\mathcal{J}U^{-1}\right)^{-1} = \left(\widetilde{U}^*\right)^{-1};\\
\left(J^{-1}\right)^* &= \widetilde{J}\mathcal{J}^{-1} = \left(\mathcal{J}\widetilde{J}^{-1}\right)^{-1} = \left(J^*\right)^{-1};\\
\left(\widetilde{J}^{-1}\right)^* &= J\mathcal{J}^{-1} = \left(\mathcal{J}J^{-1}\right)^{-1} = \left(\widetilde{J}^*\right)^{-1}.
\end{aligned}$$

4. Generalised white noise and abstract representation

In this section we study, for an n-generalised random field X, the problem of existence of an isomorphism L on $\left(W, \langle \cdot, \cdot \rangle_{\mathcal{D}_n}\right)$ such that the geometry induced by the n-generalised random field XL coincides with the geometry of the parameter space $\left(W, \langle \cdot, \cdot \rangle_{\mathcal{D}_n}\right)$:

$$\left\langle XL(\phi), XL(\psi)\right\rangle_{H(X)} = \langle \phi, \psi \rangle_{\mathcal{D}_n}, \quad \forall \phi, \psi \in \mathcal{D}_n.$$

The meaning of the above problem can be interpreted through the following definitions of generalised white noise relative to a Hilbert space structure and abstract equation. (Here, the concepts of generalised random field and duality introduced in Section 2, Definitions 2.2 and 2.3, are implicitly extended for a general Hilbert space H as parameter space.)

Definition 4.1: A generalised random field $\varepsilon(\cdot)$ with parameter space a Hilbert space $\left(H, \langle \cdot, \cdot \rangle_H\right)$ is called a *generalised white noise relative to H* if

$$\left\langle \varepsilon(u), \varepsilon(v)\right\rangle_{H(X)} = \langle u, v \rangle_H, \quad \forall u, v \in H.$$

Given a generalised white noise ε relative to a Hilbert space H, it is immediate that the generalised random field $\widetilde{\varepsilon} \equiv \varepsilon I_{H^*}$, with $I_{H^*} : H^* \to H$ being the isometric isomorphism between the dual space H^* of H and H established by the Riesz Representation Theorem, is a generalised white noise relative to H^*. Then, the proof of the following statement is straighforward.

Proposition 4.1. *For the generalised white noise ε relative to H, the generalised white noise $\widetilde{\varepsilon} \equiv \varepsilon I_{H^*}$ relative to H^* is its dual generalised random field (relative to H).*

Now, let X be a generalised random field on H. We say that X satisfies an abstract equation if it can be represented as

$$X(Lu) = \varepsilon(u), \quad \forall u \in H$$

with L being an isomorphism on $(H, \langle\cdot,\cdot\rangle_H)$, and $\varepsilon(\cdot)$ being a generalised white noise relative to H. (The above expression is interpreted as an identity in the space $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ for each $u \in H$.)

Depending on whether the generalised white noise ε in the above abstract equation is predetermined or not, we distinguish between weak-sense and strong-sense abstract representations.

Definition 4.2: We say that a generalised random field X defined on a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ has a *(weak-sense) abstract representation*

$$X(Lu) = \varepsilon_L(u), \quad \forall u \in H,$$

if there exists an isomorphism $L : H \longrightarrow H$ such that

$$\langle XL(u), XL(v)\rangle_{H(X)} = \langle u, v\rangle_H, \quad \forall u, v \in H;$$

that is, if $\varepsilon_L \equiv XL$ is a generalised white noise relative to H. We write $X(L, \varepsilon_L)$ to denote this abstract representation.

Definition 4.3: We say that a generalised random field X defined on a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ has a *(strong-sense) abstract representation* in terms of a given generalised white noise ε relative to H,

$$X(L_\varepsilon u) = \varepsilon(u), \quad \forall u \in H,$$

if there exists an isomorphism $L_\varepsilon : H \longrightarrow H$ such that

$$X(L_\varepsilon u) = \varepsilon(u)$$

in $\mathcal{L}^2(\Omega, \mathcal{A}, P)$ for each $u \in H$. We write $X(L_\varepsilon, \varepsilon)$ to denote this abstract representation.

In the following theorem we show that under the existence of its dual (relative to $\mathcal{D}_n$), an n-generalised random field has a (weak-sense) abstract representation, which is unique except for isometric isomorphisms on W.

Theorem 4.2. *Let X be an n-generalised random field such that its dual relative to $\mathcal{D}_n, \widetilde{X}$, exists. Then,*

(i) *X has a (weak-sense) abstract representation $X(L, \varepsilon_L)$;*

(ii) *if $X(L_1, \varepsilon_{L_1})$ and $X(L_2, \varepsilon_{L_2})$ are two (weak-sense) abstract representations for X, then the operator $V = L_1^{-1}L_2$ is an isometric isomorphism on W.*

Proof: (i) Let $\{X_n\}_{n\in\mathbf{N}}$ and $\{\gamma_n\}$ be two orthogonal bases of $H(X) = H(\widetilde{X})$ and W, respectively (note that $H(X)$ and W are separable Hilbert spaces). The linear operator $Q : (W, \langle\cdot,\cdot\rangle_{\mathcal{D}_n}) \longrightarrow (H(X), \langle\cdot,\cdot\rangle_{H(X)})$ defined by

$$\gamma_m \longrightarrow Q\gamma_m = X_m$$

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is an isometric isomorphism. Now, we define $L := \widetilde{U}Q$, with $\widetilde{U}$ being as defined in Section 3. Then, from Lemma 3.1,

$$\begin{aligned}\left\langle XL(\phi), XL(\psi)\right\rangle_{H(X)} &= \left\langle X\widetilde{U}Q(\phi), X\widetilde{U}Q(\psi)\right\rangle_{H(X)} = \left\langle Q(\phi), Q(\psi)\right\rangle_{H(X)} \\ &= \langle \phi, \psi\rangle_{\mathcal{D}_n}, \quad \forall \phi, \psi \in W.\end{aligned}$$

(ii) Assume

$$\begin{aligned} X(L_1\phi) &= \varepsilon_{L_1}(\phi), \quad \forall \phi \in W, \\ X(L_2\psi) &= \varepsilon_{L_2}(\psi), \quad \forall \psi \in W, \end{aligned}$$

with L_1 and L_2 being isomorphisms on W, and with ε_{L_1} and ε_{L_2} being generalised white noises relative to W. Then, with $V = L_1^{-1}L_2$,

$$\begin{aligned}\langle V\phi, V\psi\rangle_{\mathcal{D}_n} &= \langle L_1^{-1}L_2\phi, L_1^{-1}L_2\psi\rangle_{\mathcal{D}_n} = \left\langle X\left(L_1L_1^{-1}L_2\phi\right), X\left(L_1L_1^{-1}L_2\psi\right)\right\rangle_{\mathcal{L}^2(\Omega)} \\ &= \left\langle X(L_2\phi), X(L_2\psi)\right\rangle_{\mathcal{L}^2(\Omega)} = \langle \phi, \psi\rangle_{\mathcal{D}_n},\end{aligned}$$

for each $\phi, \psi \in W$. ∎

Corollary 4.3. *If $X(L_1, \varepsilon_{L_1})$ and $X(L_2, \varepsilon_{L_2})$ are two (weak-sense) abstract representations for X, then the cross-covariance functional $B_{\varepsilon_{L_1},\varepsilon_{L_2}}(\cdot,\cdot)$ between ε_{L_1} and ε_{L_2} is given by*

$$B_{\varepsilon_{L_1},\varepsilon_{L_2}}(\phi,\psi) = \left\langle \varepsilon_{L_1}(\phi), \varepsilon_{L_2}(\psi)\right\rangle_{\mathcal{L}^2(\Omega)} = \langle \phi, V\psi\rangle_{\mathcal{D}_n} = \langle V^{-1}\phi, \psi\rangle_{\mathcal{D}_n}, \ \forall \phi, \psi \in \mathcal{D}_n,$$

with $V = L_1^{-1}L_2$.

Proof: For each $\phi, \psi \in W$,

$$\begin{aligned}\left\langle \varepsilon_{L_1}(\phi), \varepsilon_{L_2}(\psi)\right\rangle_{H(X)} &= \left\langle XL_1(\phi), XL_2(\psi)\right\rangle_{H(X)} = \left\langle XL_1(\phi), XL_1(V\psi)\right\rangle_{H(X)} \\ &= \langle \phi, V\psi\rangle_{\mathcal{D}_n}.\end{aligned}$$

Also, since from Theorem 4.2(ii) V is an isometric isomorphism on W,

$$\langle \phi, V\psi\rangle_{\mathcal{D}_n} = \langle V^{-1}\phi, V^{-1}V\psi\rangle_{\mathcal{D}_n} = \langle V^{-1}\phi, \psi\rangle_{\mathcal{D}_n}, \text{ for } \phi, \psi \in W.$$ ∎

Corollary 4.4. *Under the conditions of Theorem 4.2, if $X(L, \varepsilon_L)$ is a (weak-sense) abstract representation for X, then $\widetilde{X}\left(\widetilde{L}, \widetilde{\varepsilon}_{\widetilde{L}}\right)$, with*

$$\widetilde{L} := R_n L I_F = UQI_F = U\widetilde{Q}$$

is a (weak-sense) abstract representation for $\widetilde{X}$; in this case, $\widetilde{\varepsilon}_{\widetilde{L}} = \varepsilon_L I_F$. (Here, we denote $\widetilde{Q} = QI_F : F \longrightarrow H(\widetilde{X}) = H(X)$ and I_F denotes the isomorphism $I_F : F \to W$ defined in Section 3.)

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PROOF: For $f, g \in F$,

$$\langle \widetilde{X}\widetilde{L}(f), \widetilde{X}\widetilde{L}(g) \rangle_{H(X)} = \langle \widetilde{X}U\widetilde{Q}(f), \widetilde{X}U\widetilde{Q}(g) \rangle_{H(X)} = \langle Q(f^*), Q(g^*) \rangle_{H(X)}$$
$$= \langle f^*, g^* \rangle_{\mathcal{D}_n} = \langle f, g \rangle_{\mathcal{D}_n^*}.$$

Also,

$$\widetilde{\varepsilon}_{\widetilde{L}} = \widetilde{X}\widetilde{L} = \widetilde{X}R_n LI_F = XLI_F = \varepsilon_L I_F.$$ ∎

We now prove that a necessary and sufficient condition for an n-generalised random field X, with the existence of its dual, to have a (strong-sense) abstract representation in terms of a given generalised white noise ε on W is that X and ε generate the same subspaces in $\mathcal{L}^2(\Omega, \mathcal{A}, P)$. This abstract representation, relative to ε , is unique.

THEOREM 4.5. *Let X be an n-generalised random field such that its dual relative to $\mathcal{D}_n$ exists. Let ε be a generalised white noise relative to $\mathcal{D}_n$. Then,*

(i) *X has a (strong-sense) abstract representation in terms of $\varepsilon, X(L_\varepsilon, \varepsilon)$, if and only if $H(X) = H(\varepsilon)$;*

(ii) *if such a representation exists, it is unique; that is, L_ε is unique.*

PROOF: (i) First, we assume $H(X) = H(\varepsilon)$. We then define

$$X_m = \varepsilon(\gamma_m), \quad \forall m \in \mathbb{N},$$

for a given orthonormal basis $\{\gamma_m\}_{m\in\mathbf{N}}$ of $\left(W, \langle\cdot,\cdot\rangle_{\mathcal{D}_n}\right)$. As $\{X_m\}_{m\in\mathbf{N}}$ generates $H(\varepsilon) = H(X)$, and since, for $m, p \in \mathbb{N}$,

$$\langle X_m, X_p \rangle_{H(X)} = \left\langle \varepsilon(\gamma_n), \varepsilon(\gamma_p) \right\rangle_{H(X)} = \langle \gamma_n, \gamma_p \rangle_{\mathcal{D}_n} = \delta_{n,p},$$

we have that $\{X_m\}_{m\in\mathbf{N}}$ is an orthonormal basis of $H(X)$. Now, for $\phi = \sum\limits_{m=1}^{\infty} \phi_m \gamma_m \in W$, and with L defined as in the proof of Theorem 4.2(i),

$$X(L\phi) = X\left(\sum_{m=1}^{\infty} \phi_m L\gamma_m\right) = X\left(\sum_{m=1}^{\infty} \phi_m \widetilde{U} X_m\right) = \sum_{m=1}^{\infty} \phi_m X\widetilde{U}X_m = \sum_{m=1}^{\infty} \phi_m X_m$$
$$= \sum_{m=1}^{\infty} \phi_m \varepsilon(\gamma_m) = \varepsilon\left(\sum_{m=1}^{\infty} \phi_m \gamma_m\right) = \varepsilon(\phi).$$

Conversely, assume that $X(L\phi) = \varepsilon(\phi)$, $\forall \phi \in W$. As L is an isomorphism on W,

$$H(\varepsilon) = \overline{\mathrm{sp}}^{\mathcal{L}^2(\Omega)}\left\{\varepsilon(\phi); \phi \in W\right\} = \overline{\mathrm{sp}}^{\mathcal{L}^2(\Omega)}\left\{X(L\phi); \phi \in W\right\}$$
$$= \overline{\mathrm{sp}}^{\mathcal{L}^2(\Omega)}\left\{X(\psi); \psi \in W\right\} = H(X).$$

(ii) Assume that L_ε and L'_ε are two isomorphisms on W such that

$$X(L_\varepsilon \phi) = \varepsilon(\phi), \quad \forall \phi \in W, \text{ and}$$
$$X(L'_\varepsilon \phi) = \varepsilon(\phi), \quad \forall \phi \in W.$$

Then, from the linearity of X,

$$X\left(L_\varepsilon\phi - L'_\varepsilon\phi\right) = 0, \quad \forall\phi \in W.$$

As the random field X is injective,

$$L_\varepsilon\phi = L'_\varepsilon\phi, \quad \forall\phi \in W;$$

that is,

$$L_\varepsilon = L'_\varepsilon$$

□

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