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ON A CONNECTION BETWEEN THE GENERALIZED INCOMPLETE GAMMA FUNCTIONS AND THEIR EXTENSIONS

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Abstract

In this paper we have proved that the generalized incomplete gamma functions and their extensions are mutually related through integral and differential representations.

1. Introduction

Chaudhry and Zubair considered the generalized gamma functions [4]

$$\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1} e^{-t-b/t}\, dt, \tag{1}$$

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} e^{-t-b/t}\, dt, \tag{2}$$

found useful in a variety of transient heat conduction problems [4, 5, 13, 14].

The extensions

$$\gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_0^x t^{\alpha-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}(b/t) dt, \tag{3}$$

$$\Gamma_\nu(\alpha, x; b) = \left(\frac{2b}{\pi}\right)^{1/2} \int_x^\infty t^{\alpha-\frac{3}{2}} e^{-t} K_{\nu+\frac{1}{2}}(b/t) dt \quad (b>0, x>0, -\infty<\alpha<\infty) \tag{4}$$

of the generalized incomplete gamma functions (1) – (2) were introduced in connection with the generalization of the inverse Gaussian distribution [6]. It is to be noted that

$$\Gamma_0(\alpha, x; b) = \Gamma(\alpha, x; b), \qquad \text{and} \tag{5}$$

$$\gamma_0(\alpha, x; b) = \gamma(\alpha, x; b). \tag{6}$$

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Some applications of the functions (3) – (4) to the representation of Laplace and K–transforms were shown in [6]. Several properties of these functions including decomposition formulae, recurrence relations and special cases were also discussed. It was shown that when $\nu = n$ is an integer, the functions (3) – (4) can be simplified in terms of the generalized incomplete gamma functions (1) – (2). As a matter of fact, it was shown that

$$\Gamma_n(\alpha, x; b) = \sum_{m=0}^{n} \frac{(2b)^{-m}}{m!} \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \Gamma(\alpha+m, x; b). \tag{7}$$

For nonintegral values of ν we were not able to develop a relationship between the functions (1) – (2) and (3) – (4) and it was left as an open problem. The present paper is a continuation of our earlier work [4, 6].

In this paper we have found interesting relationships between the functions (1) – (2) and their extensions (3) – (4) for nonintegral values of ν. Following Erdélyi [8, 9], we shall define the Laplace, Hankel and K–transforms of a function $f(t)$ $(0 < t < \infty)$ respectively as

$$L\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt, \tag{8}$$

$$H_\nu\{f(t); y\} = \int_0^\infty f(t) J_\nu(yt)(yt)^{1/2} dt, \tag{9}$$

$$R_\nu\{f(t); y\} = \int_0^\infty f(t) K_\nu(yt)(yt)^{1/2} dt. \tag{10}$$

2. Some preliminaries

In this section we recall some results from [6].

THEOREM 2.1. *Let* $H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$ *be the Heaviside unit step function and*

$$f(t) = t^{-\alpha-1} e^{-b/t} H\left(t - \frac{1}{x}\right) \qquad b > 0,\ x > 0. \tag{11}$$

Then

$$R_\nu\{f(t); y\} = \left(\frac{\pi}{2}\right)^{1/2} b^{-\alpha} \gamma_{\nu-\frac{1}{2}}(\alpha, bx; by) \tag{12}$$

and

$$L\left\{t^{-\alpha-1} e^{-b/t} H\left(t - \frac{1}{x}\right); y\right\} = b^{-\alpha} \gamma(\alpha, bx; by) \quad (b>0, x>0). \tag{13}$$

THEOREM 2.2. *Let*

$$f(t) = t^{-\alpha-1}e^{-b/t}H\left(\frac{1}{x}-t\right)H(t) \qquad (b>0,\ x\geq 0,\ t>0). \tag{14}$$

Then

$$R_\nu\{f(t);y\} = \left(\frac{\pi}{2}\right)^{1/2} b^{-\alpha}\Gamma_{\nu-\frac{1}{2}}(\alpha,bx;by) \tag{15}$$

and

$$L\left\{t^{-\alpha-1}e^{-b/t}H\left(\frac{1}{x}-t\right)H(t);y\right\} = b^{-\alpha}\Gamma(\alpha,bx;by). \tag{16}$$

3. Integral representations

According to (7), the extension $\Gamma_\nu(\alpha,x;b)$ can be simplified in terms of the generalized gamma functions $\Gamma(\alpha,x;b)$ for integral values of ν. In this section we shall prove that these functions are related to each other through the integral representations for all $\nu>-1$. Some special cases of these results are found interesting.

THEOREM 3.1.

$$\Gamma_\nu(\alpha,x;y) = \frac{2^{-\nu}y^{-\nu}}{\Gamma(\nu+1)}\int_y^\infty(\xi^2-y^2)^\nu\Gamma(\alpha-\nu-1,x;\xi)d\xi, \quad (y\geq 0,\ \nu>-1). \tag{17}$$

PROOF. Let

$$f(t) = t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(t). \tag{18}$$

Then, according to (15),

$$R_\nu\{f(t);y\} = \left(\frac{\pi}{2}\right)^{1/2}\Gamma_{\nu-\frac{1}{2}}(\alpha,x;y). \tag{19}$$

Moreover, according to (16) we have

$$L\left\{t^{\frac{1}{2}+\nu}f(t);\xi\right\} = \Gamma(\alpha-\nu-1/2,x;\xi). \tag{20}$$

However, according to [9, p. 122]

$$R_\nu\{f(t);y\} = \frac{\pi^{1/2}2^{-\nu}y^{\frac{1}{2}-\nu}}{\Gamma\left(\frac{1}{2}+\nu\right)}\int_y^\infty(\xi^2-y^2)^{\nu-\frac{1}{2}}L\left\{t^{\frac{1}{2}+\nu}f(t);\xi\right\}d\xi \quad \operatorname{Re}\nu>-\frac{1}{2} \tag{21}$$

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And from (19) – (21)

$$\left(\frac{\pi}{2}\right)^{1/2}\Gamma_{\nu-\frac{1}{2}}(\alpha, x; y)=\frac{\pi^{1/2}2^{-\nu}y^{\frac{1}{2}-\nu}}{\Gamma\left(\frac{1}{2}+\nu\right)}\int_y^\infty(\xi^2-y^2)^{\nu-\frac{1}{2}}\Gamma(\alpha-\nu-1/2, x; \xi)d\xi. \quad (22)$$

Multiplying both sides in (22) by $\left(\frac{2}{\pi}\right)^{1/2}$ and replacing ν.by $\nu+\frac{1}{2}$ completes the proof.

COROLLARY 3.1.

$$\Gamma(\alpha, x; y) = \int_y^\infty \Gamma(\alpha-1, x; \xi)d\xi \qquad y \geq 0. \quad (23)$$

PROOF. This follows from (17) when $\nu = 0$. It should be noted that (23) can be proved directly from the definition (2). In particular, when $y = 0$ in (23) an interesting relation

$$\Gamma(\alpha, x) = \int_0^\infty \Gamma(\alpha-1, x; \xi)d\xi \quad (24)$$

between the classical incomplete gamma function $\Gamma(\alpha, x)$ and the generalized gamma function $\Gamma(\alpha - 1, x; \xi)$ is found. Several special cases of (24) can be listed. For example, the substitution $\alpha = 0$ leads to

$$-\mathrm{Ei}(-x) = \int_0^\infty \Gamma(-1, x; \xi)d\xi \quad (25)$$

while the substitution $\alpha = 1/2$ leads to (*cf.* [4])

$$\int_0^\infty \left[e^{-2\sqrt{\xi}}\mathrm{Erfc}\left\{\sqrt{x}-\sqrt{\xi/x}\right\}-e^{2\sqrt{\xi}}\mathrm{Erfc}\left\{\sqrt{x}+\sqrt{\xi/x}\right\}\right]\frac{d\xi}{\sqrt{\xi}} = 2\,\mathrm{Erfc}\left(\sqrt{x}\right). \quad (26)$$

THEOREM 3.2.

$$\gamma_\nu(\alpha, x; y) = \frac{2^{-\nu}y^{-\nu}}{\Gamma(\nu+1)}\int_y^\infty(\xi^2-y^2)^\nu\gamma(\alpha-\nu-1, x; \xi)d\xi \quad (y \geq 0, \nu > -1). \quad (27)$$

PROOF. This is similar to the proof of (17). In particular, substituting $\nu = 0$ in (27), we get

$$\gamma(\alpha, x; y) = \int_y^\infty \gamma(\alpha-1, x; \xi)d\xi, \quad (28)$$

which can be verified directly from (1).

The substitution $y = 0$ and $\alpha = 1/2$ in (28) leads to

$$\int_0^\infty \gamma(-1/2, x; \xi)d\xi = \sqrt{\pi}\,\mathrm{Erf}\left[\sqrt{x}\right]. \quad (29)$$

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THEOREM 3.3.

$$\Gamma_\nu(\alpha+\mu, x; b) = 2^{1-\mu}\,[\Gamma(\mu)]^{-1}\, b^{\nu+1}\int_b^\infty \xi^{-\mu-\nu}(\xi^2-b^2)^{\mu-1}\Gamma_{\nu+\mu}(\alpha, x; \xi)d\xi$$
$$(\nu\geq -1,\ \mu>0,\ b>0). \tag{30}$$

PROOF. Let $f(t) = t^{-\alpha-1}e^{-1/t}H(1/x-t)H(t)$. Then, according to (15),

$$g(y;\nu) = R_\nu\{f(t); y\} = \left(\frac{\pi}{2}\right)^{1/2}\Gamma_{\nu-\frac{1}{2}}(\alpha, x; y), \tag{31}$$

$$R_\nu\left\{t^{-\mu}f(t); b\right\} = \left(\frac{\pi}{2}\right)^{1/2}\Gamma_{\nu-\frac{1}{2}}(\alpha+\mu, x; b). \tag{32}$$

However (see [9, p. 126(7)]),

$$R_\nu\left\{t^{-\mu}f(t); b\right\} = 2^{1-\mu}\,[\Gamma(\mu)]^{-1}\, b^{\nu+\frac{1}{2}}\int_b^\infty \xi^{\frac{1}{2}-\mu-\nu}(\xi^2-b^2)^{\mu-1}g(\xi;\nu+\mu)d\xi$$
$$\left(b>0,\ \mu>0,\ \nu>-\frac{1}{2}\right). \tag{33}$$

From (31) – (33), we get

$$\Gamma_{\nu-\frac{1}{2}}(\alpha+\mu, x; b) = \frac{2^{1-\mu}b^{\nu+\frac{1}{2}}}{\Gamma(\mu)}\int_b^\infty \xi^{\frac{1}{2}-\mu-\nu}(\xi^2-b^2)^{\mu-1}\Gamma_{\nu+\mu-\frac{1}{2}}(\alpha, x; \xi)d\xi. \tag{34}$$

Replacing ν by $\nu+\frac{1}{2}$ in (34) completes the proof.

COROLLARY 3.2.

$$\Gamma(\alpha+\mu, x; b) = \frac{2^{1-\mu}}{\Gamma(\mu)}\int_b^\infty \xi^{1-\mu}(\xi^2-b^2)^{\mu-1}\Gamma_{\mu-1}(\alpha, x; \xi)d\xi \quad (\mu>0, b\geq 0). \tag{35}$$

PROOF. This follows from (30) when $\nu = -1$ and the fact that

$$\Gamma(\alpha, x; b) = \Gamma_{-1}(\alpha, x; b) = \Gamma_0(\alpha, x; b). \tag{36}$$

In particular, substituting $\mu = 1$ in (35), we get

$$\Gamma(\alpha+1, x; b) = \int_b^\infty \Gamma(\alpha, x; \xi)d\xi, \tag{37}$$

which can be verified directly from (2).

The substitution $b = 0$ in (35) leads to

$$\Gamma(\alpha+\mu, x) = \frac{2^{1-\mu}}{\Gamma(\mu)}\int_0^\infty \xi^{\mu-1}\Gamma_{\mu-1}(\alpha, x; \xi)d\xi \qquad \mu>0, \tag{38}$$

where $\Gamma(\alpha, x)$ is the classical incomplete gamma function.

THEOREM 3.4.

$$\gamma_\nu(\alpha+\mu, x; y) = 2^{1-\mu}[\Gamma(\mu)]^{-1} y^{\nu+1} \int_y^\infty (\xi^2 - y^2)^{\mu-1} \gamma_{\nu+\nu}(\alpha, x; \xi) d\xi$$
$$(\mu > 0, \nu > -1, y \geq 0). \tag{39}$$

PROOF. Let

$$f(t) = t^{-\alpha-1} e^{-1/t} H\left(t - \frac{1}{x}\right) \qquad x > 0. \tag{40}$$

Then, following the steps of Theorem 3.3, we get the proof of (39). In particular, the substitution $\nu = -1$ in (39) leads to

$$\gamma(\alpha+\mu, x; y) = \frac{2^{1-\mu}}{\Gamma(\mu)} \int_y^\infty \xi^{1-\mu} (\xi^2 - y^2)^{\mu-1} \gamma_{\mu-1}(\alpha, x; \xi) d\xi$$
$$(x > 0, \mu > 0, y \geq 0). \tag{41}$$

4. Differential representations

The properties of the K–transforms and the relations (11) – (14) could be exploited to prove the differential representations of the generalized incomplete gamma functions and their extensions. In this section we prove these representations.

THEOREM 4.1.

$$\gamma_\nu(\alpha - m, x; y) = y^\nu \left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^m \left[y^{m-\nu} \gamma_{\nu-m}(\alpha, x; y)\right]$$
$$(x > 0, m = 0, 1, 2, 3, \ldots,). \tag{42}$$

PROOF. Let $f(t) = t^{-\alpha-1} e^{-1/t} H(t - 1/x)$, $x > 0$. Then, according to (12)

$$g(y; \nu) = R_\nu\{f(t); y\} = \left(\frac{\pi}{2}\right)^{1/2} \gamma_{\nu-\frac{1}{2}}(\alpha, x; y) m, \tag{43}$$

$$R_\nu\{t^m f(t); y\} = \left(\frac{\pi}{2}\right)^{1/2} \gamma_{\nu-\frac{1}{2}}(\alpha - m, x; y). \tag{44}$$

However, according to [9, p. 125(4)],

$$R_\nu\{t^m f(t); y\} = y^{\nu+\frac{1}{2}} \left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^m \left\{y^{m-\nu-\frac{1}{2}} g(y; \nu - m)\right\}. \tag{45}$$

Therefore, from (43) – (45), we get

$$\gamma_{\nu-\frac{1}{2}}(\alpha-m,x;y)=y^{\nu+\frac{1}{2}}\left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^{m}\left\{y^{m-\nu-\frac{1}{2}}\gamma_{\nu-m-\frac{1}{2}}(\alpha,x;y)\right\}. \qquad (46)$$

Replacing ν by $\nu+\frac{1}{2}$ in (46) completes the proof of (42). In particular the substitution $\nu=m$ in (42) leads to

$$\gamma_{m}(\alpha-m,x;y)=y^{m}\left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^{m}\{\gamma(\alpha,x;y)\} \quad (m=0,1,2,3,\dots,). \quad (47)$$

THEOREM 4.2.

$$\Gamma_{\nu}(\alpha-m,x;y)=y^{\nu}\left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^{m}\left\{y^{m-\nu}\Gamma_{\nu-m}(\alpha,x;y)\right\} \quad (m=0,1,2,3,\dots,).(48)$$

PROOF. If we take $f(t)=t^{-\alpha-1}e^{-1/t}H(1/x-t)H(t)$ and follow the steps of the proof of Theorem (4.1), we get the proof of (48). In particular the substitution $\nu=m$ in (48) leads to

$$\Gamma_{m}(\alpha-m,x;y)=y^{m}\left(-\frac{1}{y}\frac{\partial}{\partial y}\right)^{m}\{\Gamma(\alpha,x;y)\} \quad (m=0,1,2,3,\dots,). \quad (49)$$

5. Functional recurrence relations

THEOREM 5.1.

$$\gamma_{\nu}(\alpha+1,x;y)=\frac{y}{2\nu+1}\left[\gamma_{\nu+1}(\alpha,x;y)-\gamma_{\nu-1}(\alpha,x;y)\right]. \qquad (50)$$

PROOF. Let $f(t)=t^{-\alpha-1}e^{-1/t}H(t-1/x)$, $x>0$. Then, according to (12),

$$R_{\nu}\left\{t^{-1}f(t);y\right\}=\left(\frac{\pi}{2}\right)^{1/2}\gamma_{\nu-\frac{1}{2}}(\alpha+1,x;y)=g(y;\nu). \qquad (51)$$

However, according to [9, p. 125(5)],

$$R_{\nu}\left\{t^{-1}f(t);y\right\}=\frac{y}{2\nu}\left[g(y,\nu+1)-g(y;\nu-1)\right]. \qquad (52)$$

From (51) – (52), we get

$$\gamma_{\nu-\frac{1}{2}}(\alpha+1,x;y)=\frac{y}{2\nu}\left[\gamma_{\nu+\frac{1}{2}}(\alpha,x;y)-\gamma_{\nu-\frac{3}{2}}(\alpha,x;y)\right]. \qquad (53)$$

Replacing ν by $\nu+\frac{1}{2}$ in (53) completes the proof.

THEOREM 5.2.

$$\Gamma_{\nu}(\alpha+1,x;y)=\frac{y}{2\nu+1}\left[\Gamma_{\nu+1}(\alpha,x;y)-\Gamma_{\nu-1}(\alpha,x;y)\right] \quad (x\geq 0,\ y>0). \quad (54)$$

PROOF. This is similar to the proof of Theorem 5.1.

6. $C^{(\nu)}(\alpha, x; y)$ and $S^{(\nu)}(\alpha, x; y)$ functions

The Hankel and K–transforms are related to each other via [9, p. 121]

$$H_\nu\{f(t); y\} = \frac{1}{\pi}\left[e^{\frac{i}{2}(\nu+\frac{1}{2})\pi} R_\nu\{f(t); iy\} + e^{-\frac{i}{2}(\nu+\frac{1}{2})\pi} R_\nu\{f(t); -iy\}\right]. \quad (55)$$

Taking $f(t)$ as defined by (18), replacing ν by $\nu + \frac{1}{2}$ in (55) and using (15), we get

$$H_{\nu+\frac{1}{2}}\left\{t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x} - t\right)H(t); y\right\} = \frac{1}{\sqrt{2\pi}}\left[e^{\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha, x; iy) + e^{-\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha, x; -iy)\right]. \quad (56)$$

Substituting $x = 0$ in (56) and using the relation

$$H(\infty - t) = 1, \quad (57)$$

we get

$$H_{\nu+\frac{1}{2}}\left\{t^{-\alpha-1}e^{-1/t}; y\right\} = \frac{1}{\sqrt{2\pi}}\left[e^{\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha, 0; iy) + e^{-\frac{i}{2}(\nu+1)}\Gamma_\nu(\alpha, 0; -iy)\right]. \quad (58)$$

According to [9, p. 30(15)],

$$H_{\nu+\frac{1}{2}}\left\{t^{-3/2}e^{-1/t}; y\right\} = 2\sqrt{y}J_{\nu+\frac{1}{2}}\left(\sqrt{2y}\right)K_{\nu+\frac{1}{2}}\left(\sqrt{2y}\right). \quad (59)$$

Substituting $\alpha = \frac{1}{2}$ in (58) and using (59), we get an interesting relation

$$e^{\frac{i}{2}(\nu+1)}\Gamma_\nu(1/2, 0; iy) + e^{-\frac{i}{2}(\nu+1)}\Gamma_\nu(1/2, 0; -iy) = 2\sqrt{2\pi y}\, J_{\nu+\frac{1}{2}}\left(\sqrt{2y}\right)K_{\nu+\frac{1}{2}}\left(\sqrt{2y}\right). \quad (60)$$

In particular, for $\nu = -1$ in (60) and using $\Gamma_{-1}(\alpha, x; b) = \Gamma(\alpha, x; b)$, we get

$$\Gamma(1/2, 0; iy) + \Gamma(1/2, 0; -iy) = 2\sqrt{2\pi y}\, J_{-1/2}\left(\sqrt{2y}\right)K_{-1/2}\left(\sqrt{2y}\right) = 2\sqrt{\pi}\, e^{-\sqrt{2y}}\cos\left(\sqrt{2y}\right), \quad (61)$$

which can be verified directly from (2). Similarly, the substitution $\nu = 0$ in (60) leads to

$$e^{\frac{i}{2}\pi}\Gamma(1/2, 0; iy) + e^{-\frac{i}{2}\pi}\Gamma(1/2, 0; -iy) = 2\sqrt{\pi}\, e^{-\sqrt{2y}}\sin\left(\sqrt{2y}\right). \quad (62)$$

Therefore, it seems natural to introduce a new pair of functions defined by

$$C^{(\nu)}(\alpha, x; y) = \frac{1}{2}\left[e^{i(\nu+1)\pi/2}\Gamma_\nu(\alpha, x; iy) + e^{-i(\nu+1)\pi/2}\Gamma_\nu(\alpha, x; -iy)\right], \tag{63}$$

$$S^{(\nu)}(\alpha, x; y) = \frac{1}{2i}\left[e^{i(\nu+1)\pi/2}\Gamma_\nu(\alpha, x; iy) - e^{-i(\nu+1)\pi/2}\Gamma_\nu(\alpha, x; -iy)\right]. \tag{64}$$

The identities (60) – (62) can now be written as

$$C^{(\nu)}(1/2, 0; y) = 2\sqrt{2\pi y}\, J_{\nu+\frac{1}{2}}\left(\sqrt{2y}\right) K_{\nu+\frac{1}{2}}\left(\sqrt{2y}\right),$$

$$C^{(-1)}(1/2, 0; y) = 2e^{-\sqrt{2y}}\cos\left(\sqrt{2y}\right),$$

$$C^{(0)}(1/2, 0; y) = 2e^{-\sqrt{2y}}\sin\left(\sqrt{2y}\right).$$

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