AN ALGORITHM FOR A MINIMUM COVER OF AN ABSTRACT COMPLEX

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Introduction. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite set of m points and $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$ be a class of n subsets of X. Such a system of points and sets is called a *complex* $(X, \mathfrak{A})$. If every set of the class $\mathfrak{A}$ contains two points, the complex is a graph with m points $x_1, x_2, \ldots, x_m$ and n edges $A_1, A_2, \ldots, A_n$. A complex $(X, \mathfrak{A})$ in which every set has the same number of points is called *regular*. For any subclass $\mathfrak{A}_1$ of $\mathfrak{A}$, $\mathfrak{A}_1(x)$ denotes the class consisting of the sets which belong to $\mathfrak{A}_1$ and contain the point x. For a subset A, $\mathfrak{A}_1(A)$ denotes the class consisting of those sets which belong to $\mathfrak{A}_1$ and contain at least one point of A. $|A|$ and $|\mathfrak{A}_1|$ denote respectively the number of points in A and the number of sets in the class $\mathfrak{A}_1$. Let $\mathbf{c}$ be an m-vector of positive integers $c(x_1), c(x_2), \ldots, c(x_m)$. A subclass $\mathfrak{A}_1$ is called a $\mathbf{c}$-*cover* if for every point x, $|\mathfrak{A}_1(x)| \geqslant c(x)$. A subclass $\mathfrak{A}_1$ is a $\mathbf{c}$-matching if for every point x, $|\mathfrak{A}_1(x)| \leqslant c(x)$. Covers with a minimum number of sets and matchings with a maximum number of sets are respectively called *minimum covers* and *maximum matchings*. For a minimum $\mathbf{c}$-cover $\mathfrak{A}_1$, let X_1 be the set of those point x for which $|\mathfrak{A}_1(x)| = c(x)$. Following Fulkerson and Ryser **(2)** $|X_1|$ is called the $\mathbf{c}$-*height* of $\mathfrak{A}_1$. The minimum possible $\mathbf{c}$-height of a minimum $\mathbf{c}$-*cover* is called the $\mathbf{c}$-*height of the complex* $(X, \mathfrak{A})$. Similarly for a $\mathbf{c}$-matching $\mathfrak{A}_1$, let X_2 denote the set of those points x for which $|\mathfrak{A}_1(x)| < c(x)$. $|X_2|$ is called the $\mathbf{c}$-*depth* of the $\mathbf{c}$-matching $\mathfrak{A}_1$. The maximum possible $\mathbf{c}$-depth of a $\mathbf{c}$-matching is called the $\mathbf{c}$-depth *of the complex*.

To a complex $(X, \mathfrak{A})$ we can associate an incidence matrix $A = ((a_{ij}))$ with m rows and n columns, where $a_{ij} = 1$ if $x_i \in A_j$, and $a_{ij} = 0$, otherwise, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Fulkerson and Ryser **(2)** consider the minimum cover problem in terms of the matrix A. A set of ϵ columns of the matrix A is called an α-*set of representatives* if in the submatrix of A consisting of the m rows and the ϵ columns, each row sum is not less than α. Let $\epsilon(\alpha)$ be the minimum number of columns of A that form an α-set of representatives. The number $\epsilon(\alpha)$ is called the α-*width* of the matrix A. Obviously $\epsilon(\alpha)$ is the cardinality of the minimum $\mathbf{c}$-cover of the complex $(X, \mathfrak{A})$, where $\mathbf{c} = (\alpha, \alpha, \ldots, \alpha)$. Fulkerson and Ryser comment: "Very little is known concerning good computational methods for determining widths and heights of (0, 1)-matrices. Efficient algorithms in this domain would be of great interest."

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Petersen **(4)** introduced the concept of alternating path, which led to algorithms for minimum cover and maximum matching of graphs. Berge **(1)** first gave an inductive proof of the maximum matching theorem for graphs. Norman and Rabin **(3)** gave an ingenious proof of the maximum matching and minimum cover theorems for graphs. In this paper the idea of alternating path is extended to general complexes, and certain theorems are proved about minimum covers and maximum matchings of general complexes. The proofs of this theorem follow the line of proof of Norman and Rabin **(3)**. Section 1 proves the main theorem about the minimum cover of a complex and gives the corresponding algorithm. A "level transformation" is defined. By applying this transformation on a particular minimum cover all minimum covers can be obtained. In § 2 we prove a theorem about the minimum cover of a regular complex. This theorem gives a shorter algorithm for the minimum cover of a regular complex. In § 3 is proved a theorem about the height of a minimum cover which gives an algorithm for the height of a complex. A "parallel transformation" is defined. By applying this transformation on a minimum cover with minimum height the family of all minimum covers with minimum heights can be obtained. Section 4 gives a simpler algorithm for the height of a regular complex. Section 5 establishes a close relationship between a matching problem and a cover problem. It is shown that a maximum **c**-matching can be obtained from a certain minimum cover. Also the **c**-depth of a complex can be determined from the **d**-height of the complex for a certain vector **d**.

Extending the ideas of the present paper, it is possible to get an algorithm for any integer programming problem. This will be developed in a subsequent paper. There are many important practical applications of the minimum cover and maximum matching algorithm. For some special applications it is possible to obtain an algorithm sharper than the general algorithm. These applications also will be discussed in the subsequent paper.

1. Minimum c-cover. Let $\mathfrak{A}_1$ and $\mathfrak{B}_1$ be two classes of sets. Consider a finite sequence C of distinct sets $A_1, B_1, A_2, B_2, \ldots,$ where $A_i \in \mathfrak{A}_1$ and $B_i \in \mathfrak{B}_1$, $i = 1, 2, \ldots.$ Let H_i be the union of the sets $A_1, B_1, \ldots, A_{i-1}, B_{i-1}, A_i$, and let G_i be the union of the sets $A_1, B_1, \ldots, A_i, B_i$, $i = 1, 2, \ldots.$ $\mathfrak{D}_i$ and $\mathfrak{F}_i$ denote, respectively, the classes consisting of the sets $A_1, A_2, \ldots, A_i$ and $B_1, B_2, \ldots, B_i$, $i = 1, 2, \ldots.$ $\mathfrak{F}_0$ denotes the null class. Let

$$D_i = \{x | x \in H_i, \quad |\mathfrak{A}_1(x)| + |\mathfrak{F}_{i-1}(x)| - |\mathfrak{D}_i(x)| < c(x)\}$$

and

$$F_i = \{x | x \in G_i, \quad |\mathfrak{D}_i(x)| < |\mathfrak{F}_i(x)|\}, \qquad i = 1, 2, \ldots.$$

We shall use the above notations throughout §§ 1 and 2. Let $\mathfrak{A}_1$ be a **c**-cover and $\mathfrak{B}_1 = \mathfrak{A} - \mathfrak{A}_1$. A sequence $C = \{A_1, B_1, A_2, B_2, \ldots\}$ is an *alternating chain* if

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$$(1.1)\qquad \begin{cases} A_1 \in \mathfrak{A}_1, \\ B_i \in (\mathfrak{B}_1 - \mathfrak{F}_{i-1})(D_i), \\ A_{i+1} \in (\mathfrak{A}_1 - \mathfrak{D}_i)(F_i), \end{cases} \qquad i = 1, 2, \ldots .$$

where by previous definition $(\mathfrak{B}_1 - \mathfrak{F}_{i-1})(D_i)$ denotes the subclass of those sets of $(\mathfrak{B}_1 - \mathfrak{F}_{i-1})$ which have elements in common with D_i. An alternating chain $C = \{A_1, B_1, \ldots, A_p, B_p, A_{p+1}\}$ is called a *reducible chain* if D_{p+1} is a null set. A single set A_1 is a reducible chain if D_1 is a null set, that is, if for every point x of A_1, $|\mathfrak{A}_1(x)| > c(x)$. The set A_1 of the reducible chain is called the *leader* of the chain. A **c**-cover $\mathfrak{A}_1$ is an *irreducible* **c**-*cover* if there exists no reducible chain with respect to $\mathfrak{A}_1$.

It is easy to see that if $\mathfrak{A}_1$ is a **c**-cover and $C = \{A_1, B_1, \ldots, A_p, B_p, A_{p+1}\}$ is a reducible chain with respect to $\mathfrak{A}_1$, then $\mathfrak{A}_2 = \mathfrak{A}_1 - \mathfrak{D}_{p+1} \cup \mathfrak{F}_p$ is a **c**-cover with cardinality less than that of $\mathfrak{A}_1$.

Let $\mathfrak{B}_1$ be a **c**-cover and $\mathfrak{A}_1 = \mathfrak{A} - \mathfrak{B}_1$. Consider a sequence C of sets $A_1, B_1, A_2, B_2, \ldots, A_p, B_p$ where $A_i \in \mathfrak{A}_1$ and $B_i \in \mathfrak{B}_1$, $i = 1, 2, \ldots p$. Let

$$D_i' = \{x | x \in H_i, \quad |\mathfrak{F}_{i-1}(x)| < |\mathfrak{D}_i(x)|\}, \qquad i = 1, 2, \ldots .$$

Such a sequence is called a *level chain* if

$$B_i \in (\mathfrak{B}_1 - \mathfrak{F}_{i-1})(D_i'),$$
$$A_{i+1} \in (\mathfrak{A}_1 - \mathfrak{D}_i)(F_i),$$

and

$$|\mathfrak{B}_1(x)| + |\mathfrak{D}_p(x)| - |\mathfrak{F}_p(x)| \geqslant c(x)$$

for every x in H_p and $i = 1, 2, \ldots,$ where $(\mathfrak{B}_1 - \mathfrak{F}_{i-1})(D_i')$ by previous definition denotes the subclass of sets of $(\mathfrak{B}_1 - \mathfrak{F}_{i-1})$ which have elements in common with D_i',

$$\mathfrak{B}_2 = \mathfrak{B}_1 - \mathfrak{F}_p \cup \mathfrak{D}_p.$$

Obviously $\mathfrak{B}_2$ is also a **c**-cover. Since the cardinality of $\mathfrak{B}_2$ is the same as that of $\mathfrak{B}_1$, the above transformation is called a "level transformation." Let $L(\mathfrak{B}_1)$ be the family of all classes which can be obtained from $\mathfrak{B}_1$ by applying the level transformation a finite number of times. For two covers $\mathfrak{A}_1$ and $\mathfrak{A}_0$, $d(\mathfrak{A}_1, \mathfrak{A}_0)$ denotes the number of sets of the class $\mathfrak{A}_1$ which do not belong to $\mathfrak{A}_0$. Let $\mathfrak{B}_1$ be a cover in the family $L(\mathfrak{A}_0)$ such that

$$d(\mathfrak{A}_1, \mathfrak{B}_1) \leqslant d(\mathfrak{A}_1, \mathfrak{A}'), \qquad \mathfrak{A}' \in L(\mathfrak{A}_0).$$

Then $\mathfrak{B}_1$ is said to be a cover of the family $L(\mathfrak{A}_0)$ *nearest to* $\mathfrak{A}_1$.

LEMMA 1. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_0$ be two* **c**-*covers. Let $\mathfrak{B}_1$ be a cover in the family $L(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. If there exists a set A_1 in $\mathfrak{A}_1 - \mathfrak{B}_1$, then there exists a reducible chain with respect to $\mathfrak{A}_1$*

$$C = \{A_1, B_1, \ldots, A_p, B_p, A_{p+1}\}$$

where $A_{i+1} \in \mathfrak{A}_1 - \mathfrak{B}_1$ and $B_i \in \mathfrak{B}_1 - \mathfrak{A}_1$, $i = 1, 2, \ldots, p$.

Proof. Assume that there exists no such reducible chain (with respect to $\mathfrak{A}_1$) with A_1 as the leader. Consider $(\mathfrak{B}_1 - \mathfrak{F}_0)(D_1)$. Since there is no reducible chain with A_1 as leader, D_1 is not empty. If possible, suppose

$$(1.1) \qquad (\mathfrak{B}_1 - \mathfrak{F}_0)(D_1) \subset \mathfrak{A}_1 - \mathfrak{D}_1.$$

Since $\mathfrak{B}_1$ is a **c**-cover, it follows from (1.1) that for every x in D_1

$$(1.2) \qquad |\mathfrak{A}_1(x)| - |\mathfrak{D}_1(x)| + |\mathfrak{F}_0(x)| \geqslant c(x).$$

This contradicts the definition of D_1. Hence (1.1) is not true and we can choose B_1 where

$$B_1 \in (\mathfrak{B}_1 - \mathfrak{F}_0)(D_1) - (\mathfrak{A}_1 - \mathfrak{D}_1).$$

Next we consider $(\mathfrak{A}_1 - \mathfrak{D}_1)(F_1)$. If possible, suppose that

$$(1.3) \qquad (\mathfrak{A}_1 - \mathfrak{D}_1)(F_1) \subset \mathfrak{B}_1 - \mathfrak{F}_1.$$

Since $\mathfrak{A}_1$ is a **c**-cover, from (1.3) it follows that for every x in F_1

$$(1.4) \qquad |\mathfrak{B}_1(x)| - |\mathfrak{F}_1(x)| + |\mathfrak{D}_1(x)| \geqslant c(x).$$

From (1.4) we can easily see that $C = \{A_1, B_1\}$ is a level chain with respect to $\mathfrak{B}_1$ and

$$\mathfrak{B}_2 = \mathfrak{B}_1 - \mathfrak{F}_1 \cup \mathfrak{D}_1 \in L(\mathfrak{A}_0).$$

Also we have

$$(1.5) \qquad d(\mathfrak{A}_1, \mathfrak{B}_2) < d(\mathfrak{A}_1, \mathfrak{B}_1),$$

which contradicts the assumption that $\mathfrak{B}_1$ is nearest to $\mathfrak{A}_1$. Hence (1.3) is not true and we can choose A_2 where

$$A_2 \in (\mathfrak{A}_1 - \mathfrak{D}_1)(F_1) - (\mathfrak{B}_1 - \mathfrak{F}_1).$$

Using the same arguments, by induction we can show that there exists an infinite sequence of distinct sets, which is a contradiction. Hence there must exist a reducible chain with A_1 as the leader with the required property.

THEOREM 1. *A* **c**-*cover* $\mathfrak{A}_1$ *of a complex* $(X, \mathfrak{A})$ *is a minimum* **c**-*cover if and only if* $\mathfrak{A}_1$ *is irreducible.*

Proof. Necessity is obvious. To prove sufficiency, assume that $\mathfrak{A}_1$ is irreducible. Let $\mathfrak{A}_0$ be a minimum **c**-cover. Let $\mathfrak{B}_1$ be a cover in the family $L(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. Since $\mathfrak{B}_1$ is also a minimum **c**-cover, it is sufficient to show that $\mathfrak{A}_1 \subset \mathfrak{B}_1$. If possible, suppose there exists a set A_1 contained in $\mathfrak{A}_1 - \mathfrak{B}_1$. Then by Lemma 1, there is a reducible chain with respect to $\mathfrak{A}_1$, which is a contradiction. Hence $\mathfrak{A}_1 - \mathfrak{B}_1$ is empty. This completes the proof of the theorem.

THEOREM 2. *If* $\mathfrak{A}_1$ *is a minimum* **c**-*cover, then any other minimum* **c**-*cover* $\mathfrak{A}_2$ *belongs to the family* $L(\mathfrak{A}_1)$.

Proof. Let $\mathfrak{B}_1$ be a cover in the family $L(\mathfrak{A}_1)$ nearest to $\mathfrak{A}_2$. Since $\mathfrak{A}_2$ is a minimum **c**-cover, it is irreducible. Now by the same arguments as used in the proof of Theorem 1, we can show that $\mathfrak{B}_1 = \mathfrak{A}_2$.

Theorem 1 gives the following algorithm for the minimum cover of a complex. We start with a **c**-cover $\mathfrak{A}_1$. If there is no reducible chain with respect to $\mathfrak{A}_1$, $\mathfrak{A}_1$ is a minimum **c**-cover. So the algorithm consists in looking for reducible chains. Whenever a reducible chain is obtained, we get a new **c**-cover, whose cardinality is one less than that of the original **c**-cover. In this way, finally we get a **c**-cover that is irreducible. To test whether there is a reducible chain with respect to a **c**-cover $\mathfrak{A}_1$ with a given set A_1 of $\mathfrak{A}_1$ as the leader, we can proceed as follows. Let $\mathfrak{B}_1 = \mathfrak{A} - \mathfrak{A}_1$. We start with A_1 and test whether A_1 is a reducible chain with respect to $\mathfrak{A}_1$. If not, if possible we choose $B_1 \in \mathfrak{B}_1(D_1)$. Next, if possible, we find

$$A_2 \in (\mathfrak{A}_1 - \mathfrak{D}_1)(F_1)$$

and test whether D_2 is a null set. If D_2 is a null set, $\mathfrak{D}_2 \cup \mathfrak{F}_1$ is a reducible chain with respect to $\mathfrak{A}_1$ and the test is completed. If not, if possible, we find

$$B_2 \in (\mathfrak{B}_1 - \mathfrak{F}_1)(D_2).$$

In this manner, we proceed to build a chain until we get a reducible chain or until no further addition of sets to the chain satisfying the conditions (1.1) is possible. For instance the chain will terminate after the selection of A_1, $B_1, \ldots, A_p, B_p$ if $(\mathfrak{A}_1 - \mathfrak{D}_p)(F_p)$ is a null class. The chain will terminate after the selection of $A_1, B_1, \ldots, A_p, B_p, A_{p+1}$ if $\mathfrak{D}_{p+1} \cup \mathfrak{F}_p$ is a reducible chain with respect to $\mathfrak{A}_1$ or $(\mathfrak{B}_1 - \mathfrak{F}_p)(D_{p+1})$ is a null class. At the terminating stage, we have a maximal chain. By varying the choice of the sets $B_1, A_2, \ldots$, we can get all possible maximal chains and find out whether there is a reducible chain with respect to $\mathfrak{A}_1$ with A_1 as the leader. Theorem 2 gives an algorithm for finding all minimum **c**-covers starting from a given minimum **c**-cover $\mathfrak{A}_1$. The algorithm consists in looking for level chains. First we find all possible level chains with respect to $\mathfrak{A}_1$ and find minimum **c**-covers $\mathfrak{A}_2, \mathfrak{A}_3, \ldots, \mathfrak{A}_k$. Next we find level chains with respect to each of $\mathfrak{A}_2, \mathfrak{A}_3, \ldots, \mathfrak{A}_k$ and find new minimum **c**-covers $\mathfrak{A}_{k+1}, \ldots, \mathfrak{A}_k$, and so on. Finally we shall reach a stage when we do not find any more minimum **c**-covers.

2. Minimum covers of regular complexes.

Let $\mathfrak{A}_1$ be a **c**-cover of a regular complex $(X, \mathfrak{A})$. A set A of $\mathfrak{A}_1$ is called a *heavy set* if for at least one point x of A, $|\mathfrak{A}_1(x)| > c(x)$. A reducible chain whose leader is a heavy set is called a *heavy reducible chain.* In this section we consider only regular complexes.

LEMMA 2. *A reducible chain with respect to a* **c***-cover* $\mathfrak{A}_1$ *contains at least one heavy set.*

Proof. Suppose $C = \{A_1, B_1, \ldots, A_p, B_p, A_{p+1}\}$ is a reducible chain with respect to $\mathfrak{A}_1$. Let H_{p+1} be the union of the sets in the chain C. Let H' be the union of the sets $A_1, A_2, \ldots, A_{p+1}$ and $H^* = H_{p+1} - H'$. If possible, suppose there is no point x in H' for which $|\mathfrak{A}_1(x)| > c(x)$. Let q be the number of points in every set of $\mathfrak{A}$. Then it follows easily that

$$(2.1) \qquad \sum_{x \in H_{p+1}} |\mathfrak{D}_{p+1}(x)| = (p+1)q > pq = \sum_{x \in H_{p+1}} |\mathfrak{F}_p(x)|.$$

Let $\mathfrak{A}' = \mathfrak{A}_1 - \mathfrak{D}_{p+1}$. Since C is a reducible chain, $\mathfrak{A}_2 = \mathfrak{A}' \cup \mathfrak{F}_p$ is a $\mathbf{c}$-cover. For every point x in H', $|\mathfrak{A}_1(x)| = |\mathfrak{A}(x)| + |\mathfrak{D}_{p+1}(x)| = c(x)$. Also $|\mathfrak{A}_2(x)| = |\mathfrak{A}'(x)| + |\mathfrak{F}_p(x)|$. Hence for x in H'

$$(2.2) \qquad |\mathfrak{A}'(x)| + |\mathfrak{D}_{p+1}(x)| - c(x) \leqslant |\mathfrak{A}'(x)| + |\mathfrak{F}_p(x)| - c(x).$$

For x in H^*, $|\mathfrak{D}_{p+1}(x)| = 0$. Hence for $x \in H^*$,

$$(2.3) \qquad |\mathfrak{A}'(x)| + |\mathfrak{D}_{p+1}(x)| - c(x) \leqslant |\mathfrak{A}'(x)| + |\mathfrak{F}_p(x)| - c(x).$$

From (2.2) and (2.3), we get

$$(2.4) \qquad \sum_{x \in H_{p+1}} (|\mathfrak{A}'(x)| + |\mathfrak{D}_{p+1}(x)| - c(x)) \leqslant \sum_{x \in H_{p+1}} (|\mathfrak{A}'(x)| + |\mathfrak{F}_p(x)| - c(x)).$$

The inequality (2.4) contradicts (2.1). Hence there must exist a point x in H' for which $|\mathfrak{A}_1(x)| > c(x)$. In other words there is a heavy set in the reducible chain.

LEMMA 3. *Let $\mathfrak{A}_1$ be any $\mathbf{c}$-cover and $\mathfrak{A}_0$ be a minimum $\mathbf{c}$-cover. Let $\mathfrak{B}_1$ be a $\mathbf{c}$-cover in the family $L(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. If all heavy sets of $\mathfrak{A}_1$ are contained in $\mathfrak{B}_1$, then $\mathfrak{A}_1$ is a minimum $\mathbf{c}$-cover.*

Proof. It is sufficient to show that $\mathfrak{A}_1 \subset \mathfrak{B}_1$. If not, suppose $\mathfrak{A}_1 - \mathfrak{B}_1$ contains a set A_1. Then by Lemma 1, there exists a reducible chain $C = \{A_1, B_1, \ldots, A_p, B_p, A_{p+1}\}$ where $A_{i+1} \in \mathfrak{A}_1 - \mathfrak{B}_1$ and $B_i \in \mathfrak{B}_1 - \mathfrak{A}_1$, $i = 1, 2, \ldots, p$. By Lemma 2 there is a heavy set A in the reducible chain C. This contradicts our assumption. Hence the lemma follows.

THEOREM 3. *If a $\mathbf{c}$-cover $\mathfrak{A}_1$ is such that there is no heavy reducible chain with respect to $\mathfrak{A}_1$, then $\mathfrak{A}_1$ is a minimum $\mathbf{c}$-cover.*

Proof. Let $\mathfrak{A}_0$ be a minimum $\mathbf{c}$-cover and $\mathfrak{B}_1$ be a $\mathbf{c}$-cover in the family $L(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. By Lemma 3, it is sufficient to show that all heavy sets of $\mathfrak{A}_1$ are contained in $\mathfrak{B}_1$. If not, suppose $\mathfrak{A}_1 - \mathfrak{B}_1$ contains a heavy set A_1. Then by Lemma 1 there exists a reducible chain with respect to $\mathfrak{A}_1$:

$$C = \{A_1, B_1, \ldots, A_p, B_p, A_{p+1}\}.$$

By definition, C is a heavy reducible chain. This contradicts our hypothesis. Hence the theorem is true.

Theorem 3 gives a shorter algorithm for minimum covers of regular complexes. It shows that in the case of regular complexes it is sufficient to look for heavy reducible chains.

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3. Height of a complex. Let $\mathfrak{A}_1$ and $\mathfrak{B}_1$ be two subclasses of $\mathfrak{A}$. Consider a sequence E of sets $E_1, E_2, \ldots, E_{2p-1}, E_{2p}$ belonging to $\mathfrak{A}$. Let

$$P_i = \bigcup_{j=1}^{i} E_j, \qquad i = 1, 2, \ldots, 2p.$$

Let $\mathfrak{R}_i$ and $\mathfrak{S}_i$ denote, respectively, the classes consisting of those sets of $\mathfrak{A}_1$ and $\mathfrak{B}_1$ which occur among $E_1, E_2, \ldots, E_i$ $(i = 1, 2, \ldots, 2p)$. Let

$$\text{(3.1)} \qquad \begin{cases} R_i = \{x \,|x \in P_i, \quad |\mathfrak{R}_i(x)| > |\mathfrak{S}_i(x)|\} \\ S_i = \{x \,|x \in P_i, \quad |\mathfrak{R}_i(x)| < |\mathfrak{S}_i(x)|\} \end{cases} \quad i = 1, 2, \ldots, 2p.$$

We shall use the above notation throughout §§ 3 and 4. Now we assume that $\mathfrak{A}_1$ is a **c**-cover and $\mathfrak{B}_1 = \mathfrak{A} - \mathfrak{A}_1$. A sequence E is called a *properly connected sequence* (with respect to $\mathfrak{A}_1$) if the following four conditions are satisfied for $k = 1, 2, \ldots, p$.

(i) If $E_{2k} \in \mathfrak{B}_1$ and $(\mathfrak{A}_1 - \mathfrak{R}_{2k})(S_{2k})$ is not empty,

$$E_{2k+1} \in (\mathfrak{A}_1 - \mathfrak{R}_{2k})(S_{2k}),$$

and

$$E_{2k+2} \in (\mathfrak{B}_1 - \mathfrak{S}_{2k+1})(R_{2k+1}).$$

(ii) If $E_{2k} \in \mathfrak{B}_1$ and $(\mathfrak{A}_1 - \mathfrak{R}_{2k})(S_{2k})$ is empty,

$$E_{2k+1} \in (\mathfrak{B}_1 - \mathfrak{S}_{2k})(R_{2k}),$$

and

$$E_{2k+2} \in (\mathfrak{A}_1 - \mathfrak{R}_{2k+1})(S_{2k+1}).$$

(iii) If $E_{2k} \in \mathfrak{A}_1$ and $(\mathfrak{B}_1 - \mathfrak{S}_{2k})(R_{2k})$ is not empty,

$$E_{2k+1} \in (\mathfrak{B}_1 - \mathfrak{S}_{2k})(R_{2k}),$$

and

$$E_{2k+2} \in (\mathfrak{A}_1 - \mathfrak{R}_{2k+1})(S_{2k+1}).$$

(iv) If $E_{2k} \in \mathfrak{A}_1$ and $(\mathfrak{B}_1 - \mathfrak{S}_{2k})(R_{2k})$ is empty,

$$E_{2k+1} \in (\mathfrak{A}_1 - \mathfrak{R}_{2k})(S_{2k}),$$

and

$$E_{2k+2} \in (\mathfrak{B}_1 - \mathfrak{S}_{2k+1})(R_{2k+1}).$$

A properly connected sequence E is called an *exchange sequence* if $\mathfrak{A}_2 = \mathfrak{A}_1 - \mathfrak{R}_{2p} \cup \mathfrak{S}_{2p}$ is a **c**-cover. An exchange sequence E (with respect to $\mathfrak{A}_1$) is called a *low sequence* (with respect to $\mathfrak{A}_1$) if $E_1 \in \mathfrak{A}_1$ and $E_2 \in \mathfrak{B}_1(R_1)$ and

$$\text{(3.2)} \qquad |\{x|\,|\mathfrak{A}_1(x)| = c(x)\}| > |\{x|\,|\mathfrak{A}_2(x)| = c(x)\}|.$$

An exchange sequence E (with respect to $\mathfrak{A}_1$) is called a *parallel sequence* (with respect to $\mathfrak{A}_1$) if $E_1 \in \mathfrak{B}_1$ and $E_2 \in \mathfrak{A}_1(S_1)$ and

$$\text{(3.3)} \qquad |\{x|\,|\mathfrak{A}_1(x)| = c(x)\}| = |\{x|\,|\mathfrak{A}_2(x)| = c(x)\}|.$$

Obviously if $\mathfrak{A}_1$ is a **c**-cover and E is a low sequence with respect to $\mathfrak{A}_1$, $\mathfrak{A}_2$ is a **c**-cover with height less than that of $\mathfrak{A}_1$. Similarly if E is a parallel sequence, $\mathfrak{A}_2$ is a **c**-cover with height equal to that of $\mathfrak{A}_1$. In this case, for convenience, we say that $\mathfrak{A}_2$ is obtained from $\mathfrak{A}_1$ by applying a *parallel transformation*. Let $M(\mathfrak{A}_1)$ denote the family of all **c**-covers which can be obtained from $\mathfrak{A}_1$ by applying a parallel transformation a finite number of times.

LEMMA 4. *Suppose $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are two minimum* **c***-covers, $\mathfrak{B}_1$ having minimum height. Let $E = \{E_1, E_2, \ldots, E_{2p}\}$ be a properly connected sequence with respect to both $\mathfrak{A}_1$ and $\mathfrak{B}_1$ where $E_1 \in \mathfrak{A}_1 - \mathfrak{B}_1$ and $E_2 \in \mathfrak{B}_1 - \mathfrak{A}_1$ and each E_i belongs to either $\mathfrak{A}_1 - \mathfrak{B}_1$ or $\mathfrak{B}_1 - \mathfrak{A}_1$. If*

$$\left\{\begin{array}{l}(\mathfrak{A}_1 - \mathfrak{R}_{2p})(S_{2p}) \subset (\mathfrak{B}_1 - \mathfrak{S}_{2p})(S_{2p}),\\ (\mathfrak{B}_1 - \mathfrak{S}_{2p})(R_{2p}) \subset (\mathfrak{A}_1 - \mathfrak{R}_{2p})(R_{2p}),\end{array}\right. \tag{3.4}$$

then either E is a low sequence with respect to $\mathfrak{A}_1$ or E is a parallel sequence with respect to $\mathfrak{B}_1$.

Proof. First we show that E is an exchange sequence with respect to both $\mathfrak{A}_1$ and $\mathfrak{B}_1$. For this we have to show that both $\mathfrak{A}_2 = \mathfrak{A}_1 - \mathfrak{R}_{2p} \cup \mathfrak{S}_{2p}$ and $\mathfrak{B}_2 = \mathfrak{B}_1 - \mathfrak{S}_{2p} \cup \mathfrak{R}_{2p}$ are **c**-covers. To show that $\mathfrak{A}_2$ is a **c**-cover, it is sufficient to show that for every $x \in R_{2p}$, $|\mathfrak{A}_2(x)| \geqslant c(x)$. Using (3.4) and the fact that $\mathfrak{B}_1$ is a **c**-cover, we get for $x \in R_{2p}$

$$\begin{aligned}|\mathfrak{A}_2(x)| &= |(\mathfrak{A}_1 - \mathfrak{R}_{2p})(x)| + |\mathfrak{S}_{2p}(x)|\\ &\geqslant |(\mathfrak{B}_1 - \mathfrak{S}_{2p})(x)| + |\mathfrak{S}_{2p}(x)|\\ &= |\mathfrak{B}_1(x)| \geqslant c(x).\end{aligned}$$

Similarly we can show that $\mathfrak{B}_2$ is a **c**-cover. From (3.4), we can easily obtain that

$$\left\{\begin{array}{l}\{x\,|x \in R_{2p}, |\mathfrak{A}_2(x)| = c(x)\} \subset \{x\,|x \in R_{2p}, |\mathfrak{B}_1(x)| = c(x)\},\\ \{x\,|x \in S_{2p}, |\mathfrak{B}_2(x)| = c(x)\} \subset \{x\,|x \in S_{2p}, |\mathfrak{A}_1(x)| = c(x)\}.\end{array}\right. \tag{3.5}$$

If E is not a low sequence with respect to $\mathfrak{A}_1$, we have

$$|\{x|\ |\mathfrak{A}_1(x)| > c(x)\}| \geqslant |\{x|\ |\mathfrak{A}_2(x)| > c(x)\}|. \tag{3.6}$$

If x does not belong to $R_{2p} \cup S_{2p}$, $|\mathfrak{A}_1(x)| = \mathfrak{A}_2(x)|$. Hence from (3.6), it follows that

$$\begin{aligned}&|\{x|\ x \in R_{2p}, |\mathfrak{A}_1(x)| > c(x)\}| - |\{x|\ x \in R_{2p}, |\mathfrak{A}_2(x)| > c(x)\}|\\ &\qquad \geqslant |\{x|\ x \in S_{2p}, |\mathfrak{A}_2(x)| > c(x)\}| - |\{x|\ x \in S_{2p}, |\mathfrak{A}_1(x)| > c(x)\}|.\end{aligned} \tag{3.7}$$

It is easy to check that

$$\left\{\begin{array}{l}\{x|\ x \in R_{2p}, |\mathfrak{A}_2(x)| = c(x)\}\\ \qquad = \{x|\ x \in R_{2p}, |\mathfrak{A}_1(x)| > c(x)\} - \{x|\ x \in R_{2p}, |\mathfrak{A}_2(x)| > c(x)\},\\ \{x|\ x \in S_{2p}, |\mathfrak{A}_1(x)| = c(x)\}\\ \qquad = \{x|\ x \in S_{2p}, |\mathfrak{A}_2(x)| > c(x)\} - \{x|\ x \in S_{2p}, |\mathfrak{A}_1(x)| > c(x)\}.\end{array}\right. \tag{3.8}$$

Also we have

$$(3.9)\quad \begin{cases} \{x|\ x \in R_{2p}, |\mathfrak{A}_1(x)| > c(x)\} \supset \{x\ |x \in R_{2p}, |\mathfrak{A}_2(x)| > c(x)\}, \\ \{x|\ x \in S_{2p}, |\mathfrak{A}_2(x)| > c(x)\} \supset \{x\ |x \in S_{2p}, |\mathfrak{A}_1(x)| > c(x)\}. \end{cases}$$

From (3.6), (3.7), (3.8), and (3.9), we get that if E is not a low sequence with respect to $\mathfrak{A}_1$,

$$(3.10)\quad |\{x|\ x \in R_{2p}, |\mathfrak{A}_2(x)| = c(x)\}| \geqslant |\{x|\ x \in S_{2p}, |\mathfrak{A}_1(x)| = c(x)\}|.$$

Similarly using the fact that $\mathfrak{B}_1$ is a **c**-cover with least height, we can prove that if E is not a level sequence with respect to $\mathfrak{B}_1$

$$(3.11)\quad |\{x|\ x \in S_{2p}, |\mathfrak{B}_2(x)| = c(x)\}| > |\{x|\ x \in R_{2p}, |\mathfrak{B}_1(x)| = c(x)\}|.$$

Suppose the lemma is not true, then from (3.10) and (3.11) we have

$$(3.12)\quad \begin{aligned} &|\{x|\ x \in R_{2p}, |\mathfrak{A}_2(x)| = c(x)\}| + |\{x|\ x \in S_{2p}, |\mathfrak{B}_2(x)| = c(x)\}| \\ &\qquad > |\{x|\ x \in S_{2p}, |\mathfrak{A}_1(x)| = c(x)\}| + |\{x|\ x \in R_{2p}, |\mathfrak{B}_1(x)| = c(x)\}|, \end{aligned}$$

which contradicts (3.5). Hence the lemma is true.

LEMMA 5. *Let $\mathfrak{A}_1$ be a minimum* **c**-*cover and $\mathfrak{A}_0$ be a minimum* **c**-*cover with least height. Let $\mathfrak{B}_1$ be a* **c**-*cover in the family $M(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. If $\mathfrak{A}_1 - \mathfrak{B}_1$ contains a set E_1, there exists a low sequence $E = \{E_1, E_2, \ldots, E_{2p}\}$ with respect to $\mathfrak{A}_1$, where any set in the sequence E belonging to $\mathfrak{A}_1$ belongs to $\mathfrak{A}_1 - \mathfrak{B}_1$.*

Proof. If possible, suppose there is no such low sequence with respect to $\mathfrak{A}_1$ with E_1 as the leader. Consider $\mathfrak{B}_1(R_1)$. If $\mathfrak{B}_1(R_1) \subset \mathfrak{A}_1 - \mathfrak{R}_1$, we can show that $\mathfrak{A}_1 - \mathfrak{R}_1$ is a **c**-cover. This contradicts the assumption that $\mathfrak{A}_1$ is a minimum **c**-cover. So we can choose E_2 where

$$E_2 \in \mathfrak{B}_1(R_1) - (\mathfrak{A}_1 - \mathfrak{R}_1)(R_1).$$

Next we consider $(\mathfrak{A}_1 - \mathfrak{R}_2)(S_2)$. There are two possible cases.

Case 1. In this case

$$(\mathfrak{A}_1 - \mathfrak{R}_2)(S_2) \not\subset (\mathfrak{B}_1 - \mathfrak{S}_2)(S_2).$$

So we choose E_3 where

$$(3.13)\quad E_3 \in (\mathfrak{A}_1 - \mathfrak{R}_2)(S_2) - (\mathfrak{B}_1 - \mathfrak{S}_2)(S_2).$$

Then consider $(\mathfrak{B}_1 - \mathfrak{S}_3)(R_3)$. If possible, suppose

$$(3.14)\quad (\mathfrak{B}_1 - \mathfrak{S}_3)(R_3) \subset (\mathfrak{A}_1 - \mathfrak{R}_3)(R_3).$$

In this case we can show that $\mathfrak{A}_1 - \mathfrak{R}_3 \cup \mathfrak{S}_3$ is a **c**-cover with cardinality less than that of $\mathfrak{A}_1$, which is a contradiction. Hence (3.14) is not true and we can choose

$$(3.15)\quad E_4 \in (\mathfrak{B}_1 - \mathfrak{S}_3)(R_3) - (\mathfrak{A}_1 - \mathfrak{R}_3)(R_3).$$

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Case 2. In this case

$$(\mathfrak{A}_1 - \mathfrak{R}_2)(S_2) \subset (\mathfrak{B}_1 - \mathfrak{S}_2)(S_2). \tag{3.16}$$

Now we consider $(\mathfrak{B}_1 - \mathfrak{S}_2)(R_2)$. If possible, suppose

$$(\mathfrak{B}_1 - \mathfrak{S}_2)(R_2) \subset (\mathfrak{A}_1 - \mathfrak{R}_2)(R_2). \tag{3.17}$$

Then it is easily checked that the sequence $E = \{E_1, E_2\}$ satisfies the conditions of Lemma 4. Hence either E is a low sequence with respect to $\mathfrak{A}_1$ or E is a parallel sequence with respect to $\mathfrak{B}_1$. Since by our assumption there is no low sequence with respect to $\mathfrak{A}_1$, E must be a parallel sequence with respect to $\mathfrak{B}_1$. Then $\mathfrak{B}_2 = \mathfrak{B}_1 - \mathfrak{S}_2 \cup \mathfrak{R}_2$ belongs to the family $M(\mathfrak{A}_0)$ and is nearer to $\mathfrak{A}_1$. This contradicts the assumption that $\mathfrak{B}_1$ is a **c**-cover in the family $M(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. Hence (3.17) is not true and we can choose

$$E_3 \in (\mathfrak{B}_1 - \mathfrak{S}_2)(R_2) - (\mathfrak{A}_1 - \mathfrak{R}_2)(R_2). \tag{3.18}$$

Next consider $(\mathfrak{A}_1 - \mathfrak{R}_3)(S_3)$. If possible, suppose

$$(\mathfrak{A}_1 - \mathfrak{R}_3)(S_3) \subset (\mathfrak{B}_1 - \mathfrak{S}_3)(S_3). \tag{3.19}$$

From (3.19), we can show that $\mathfrak{A}_2 = \mathfrak{A}_1 - \mathfrak{R}_3 \cup \mathfrak{S}_3$ is a **c**-cover with cardinality less than that of $\mathfrak{A}_1$. This contradicts the assumption that $\mathfrak{A}_1$ is a minimum **c**-cover. Hence (3.19) is not true and we can choose

$$E_4 \in (\mathfrak{A}_1 - \mathfrak{R}_3)(S_3) - (\mathfrak{B}_1 - \mathfrak{S}_3)(S_3).$$

Using the arguments given above, by induction, we can show that if there is no low sequence with respect to $\mathfrak{A}_1$ with E_1 as the leader there exists an infinite sequence of distinct sets, which is a contradiction. This completes the proof of the lemma.

THEOREM 4. *A minimum* **c**-*cover* $\mathfrak{A}_1$ *has minimum height if and only if there is no low sequence with respect to* $\mathfrak{A}_1$.

Proof. Necessity is obvious. To prove sufficiency, assume that there is no low sequence with respect to the minimum cover $\mathfrak{A}_1$. Let $\mathfrak{A}_0$ be a minimum cover with minimum height. Let $\mathfrak{B}_1$ be the cover in the family $M(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. It is sufficient to show that $\mathfrak{A}_1 \subset \mathfrak{B}_1$. If possible, suppose $\mathfrak{A}_1 - \mathfrak{B}_1$ contains a set E_1. Then by Lemma 5, there is a low sequence with respect to $\mathfrak{A}_1$. This contradicts our hypothesis. Hence $\mathfrak{A}_1 \subset \mathfrak{B}_1$.

THEOREM 5. *Let* $\mathfrak{A}_0$ *be a minimum cover with minimum height. Then any other minimum cover* $\mathfrak{A}_1$ *belongs to the family* $M(\mathfrak{A}_0)$.

The proof follows from Lemma 5 and the fact that there is no low sequence with respect to $\mathfrak{A}_1$.

Theorem 4 gives an algorithm for the height of a complex. We start with a minimum **c**-cover $\mathfrak{A}_1$ and look for low sequences with respect to $\mathfrak{A}_1$. Whenever we get a low sequence, we get a new minimum cover with height less than that of the original minimum cover. In this way we finally get a minimum cover with respect to which there are no low sequences. To test whether there is any low sequence with respect to $\mathfrak{A}_1$ with a set E_1 of $\mathfrak{A}_1$ as the leader, we can proceed as follows. If possible, we choose E_2 belonging to $\mathfrak{B}_1(R_1)$, where $\mathfrak{B}_1 = \mathfrak{A} - \mathfrak{A}_1$. After choosing E_2, we examine whether $E = \{E_1, E_2\}$ is a low sequence. For this it is necessary to compute

$$|\mathfrak{A}_2(x)| = |\mathfrak{A}_1(x)| - |\mathfrak{R}_2(x)| + |\mathfrak{S}_2(x)|$$

for $x \in R_2$. For E to be an exchange sequence, it is necessary that for $x \in R_2$, $|\mathfrak{A}_2(x)| \geqslant c(x)$. If E is an exchange sequence, E will be a low sequence with respect to $\mathfrak{A}_1$ if and only if

$$|\{x|\, x \in R_2, |\mathfrak{A}_2(x)| = c(x)\}| < |\{x|\, x \in S_2, |\mathfrak{A}_1(x)| = c(x)\}|.$$

If E is a low sequence, the test terminates. If not, we consider $(\mathfrak{A}_1 - \mathfrak{R}_2)(S_2)$. There will be two possible cases: (i) $(\mathfrak{A}_1 - \mathfrak{R}_2)(S_2)$ is not empty and (ii) $(\mathfrak{A}_1 - \mathfrak{R}_2)(S_2)$ is empty. If (i) is the case, we choose E_3 belonging to the class $(\mathfrak{A}_1 - \mathfrak{R}_2)(S_2)$. Next, if possible, we choose E_4 from the class $(\mathfrak{B}_1 - \mathfrak{S}_3)(R_3)$. If (ii) is the case, if possible we choose E_3 from the class $(\mathfrak{B}_1 - \mathfrak{S}_2)(R_2)$. Next, if possible, we choose E_4 from the class $(\mathfrak{A}_1 - \mathfrak{R}_3)(S_3)$. After choosing E_4, we examine whether $E = \{E_1, E_2, E_3, E_4\}$ is a low sequence or not. In this manner we proceed to build a sequence $E = \{E_1, E_2, \ldots, E_{2p}\}$. The sequence terminates at an even stage if we get a low sequence or if at some stage (say the $2p$-th stage) both $(\mathfrak{A}_1 - \mathfrak{R}_{2p})(S_{2p})$ and $(\mathfrak{B}_1 - \mathfrak{S}_{2p})(R_{2p})$ are empty classes. By varying the choices of the sets $E_2, E_3, E_4, \ldots,$ we can find out whether there is a low sequence with E_2 as the leader or not. Theorem 5 gives an algorithm for finding all the minimum covers with minimum height.

4. Height of a regular complex. Let $\mathfrak{A}_1$ be a minimum **c**-cover of a regular complex $(X, \mathfrak{A})$: A set E_2 of $\mathfrak{A}_1$ is called a *loaded set* if for some point x in E_2, $|\mathfrak{A}_1(x)| > c(x) + 1$. A low sequence $E = \{E_1, E_2, \ldots, E_{2p}\}$ with respect to $\mathfrak{A}_1$ whose leader E_1 is a loaded set is called a *loaded low sequence.* In this section we consider only regular complexes.

LEMMA 6. *A low sequence with respect to a minimum* **c**-*cover* $\mathfrak{A}_1$ *contains at least one loaded set.*

Proof. Let $E = \{E_1, E_2, \ldots, E_{2p}\}$ be a low sequence with respect to $\mathfrak{A}_1$. Let P_{2p} be the union of the sets $E_1, E_2, \ldots, E_{2p}$. Let P' be the union of those sets of $\mathfrak{A}_1$ that occur in the sequence E and $P^* = P_{2p} - P'$. Let q be the number of points in each set of the complex $(X, \mathfrak{A})$. It follows easily that

$$\sum_{x \in P_{2p}} |\mathfrak{R}_{2p}(x)| = \sum_{x \in P_{2p}} |\mathfrak{S}_{2p}(x)| = pq. \tag{4.1}$$

If possible, suppose that for no point x in P', $|\mathfrak{A}_1(x)| > c(x) + 1$.

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Let

$$\mathfrak{A}' = \mathfrak{A}_1 - \mathfrak{R}_{2p}, \qquad \mathfrak{A}_2 = \mathfrak{A}' \cup \mathfrak{S}_{2p}.$$

Since E is a low sequence with respect to $\mathfrak{A}_1$, we have

$$(4.2) \qquad |\{x|\, x \in P_{2p}, |\mathfrak{A}_2(x)| > c(x)\}| > |\{x|\, x \in P_{2p}, |\mathfrak{A}_1(x)| > c(x)\}|.$$

Let

$$(4.3) \qquad \begin{aligned} &\{x|\, x \in P_{2p},\ |\mathfrak{A}_1(x)| > c(x)\} = G_1, \\ &\{x|\, x \in P_{2p},\ |\mathfrak{A}_2(x)| > c(x)\} = G_2, \\ &\{x|\, x \in P',\ |\mathfrak{A}_1(x)| - c(x) = 1\} = G_3, \\ &\{x|\, x \in P^*,\ |\mathfrak{A}_1(x) - c(x) \geqslant 1\} = G_4, \\ &\{x|\, x \in P',\ |\mathfrak{A}_2(x)| - c(x) \geqslant 1\} = G_5, \\ &\{x|\, x \in P^* - G_4,\ |\mathfrak{A}_2(x)| - c(x) \geqslant 1\} = G_6. \end{aligned}$$

Since, by our assumption, $|\mathfrak{A}_1(x)| > c(x) + 1$ for no point $x \in P'$, it follows that G_1 is the union of the disjoint sets G_3 and G_4. Also it is easily checked that G_2 is the union of the disjoint sets G_4, G_5, and G_6. From (4.2), we have

$$(4.4) \qquad |G_5| + |G_6| > |G_3|.$$

Using (4.3) and (4.4), we get

$$\begin{aligned} (4.5) \quad \sum_{x \in P_{2p}} \{|\mathfrak{A}'(x)| + |\mathfrak{S}_{2p}(x)| - c(x)\} &= \sum_{x \in P_{2p}} \{|\mathfrak{A}_2(x)| - c(x)\} \\ &= \sum_{x \in G_4} \{|\mathfrak{A}_2(x)| - c(x)\} + \sum_{x \in G_5} \{|\mathfrak{A}_2(x)| - c(x)\} \\ &\qquad + \sum_{x \in G_6} \{|\mathfrak{A}_2(x)| - c(x)\} \\ &\geqslant \sum_{x \in G_4} \{|\mathfrak{A}_1(x)| - c(x)\} + |G_5| + |G_6| \\ &> \sum_{x \in G_4} \{|\mathfrak{A}_1(x)| - c(x)\} + |G_3| \\ &= \sum_{x \in G_4} \{|\mathfrak{A}_1(x)| - c(x)\} + \sum_{x \in G_3} \{|\mathfrak{A}_1(x)| - c(x)\} \\ &= \sum_{x \in P_{2p}} \{|\mathfrak{A}_1(x)| - c(x)\} \\ &= \sum_{x \in P_{2p}} \{|\mathfrak{A}'(x)| + |\mathfrak{R}_{2p}(x)| - c(x)\}. \end{aligned}$$

Obviously (4.5) contradicts (4.1). Hence the low sequence E must contain at least one loaded set.

LEMMA 7. *Let $\mathfrak{A}_1$ be a minimum* **c**-*cover and $\mathfrak{A}_0$ be a minimum* **c**-*cover with minimum height. Let $\mathfrak{B}_1$ be a minimum* **c**-*cover in the family $M(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. If all loaded sets of $\mathfrak{A}_1$ are contained in $\mathfrak{B}_1$, $\mathfrak{A}_1$ is a minimum* **c**-*cover with minimum height.*

Proof. It is sufficient to show that $\mathfrak{A}_1 \subset \mathfrak{B}_1$. If not, suppose $\mathfrak{A}_1 - \mathfrak{B}_1$ contains a set E_1. Then by Lemma 5 there exists a low sequence

$$E = \{E_1, E_2, \ldots, E_{2p}\}$$

with respect to $\mathfrak{A}_1$, where any set of the sequence E belonging to $\mathfrak{A}_1$ belongs to $\mathfrak{A}_1 - \mathfrak{B}_1$. By Lemma 6, the sequence E must contain at least one loaded set of $\mathfrak{A}_1$. This set also belongs to $\mathfrak{A}_1 - \mathfrak{B}_1$, which contradicts our hypothesis.

THEOREM 6. *A minimum* **c**-*cover* $\mathfrak{A}_1$ *of a regular complex* $(X, \mathfrak{A})$ *has minimum height if and only if there is no loaded low sequence with respect to* $\mathfrak{A}_1$.

Proof. Necessity is obvious. To prove sufficiency, assume that there is no loaded low sequence with respect to the minimum **c**-cover $\mathfrak{A}_1$. Let $\mathfrak{A}_0$ be a minimum **c**-cover with minimum height and $\mathfrak{B}_1$ be a minimum **c**-cover in the family $M(\mathfrak{A}_0)$ nearest to $\mathfrak{A}_1$. By Lemma 7 it is sufficient to show that all loaded sets of $\mathfrak{A}_1$ are contained in $\mathfrak{B}_1$. If not, suppose $\mathfrak{A}_1 - \mathfrak{B}_1$ contains a loaded set E_1. By Lemma 5, there exists a low sequence $E = \{E_1, E_2, \ldots, E_{2p}\}$ with respect to $\mathfrak{A}_1$. Since E_1 is a loaded set, E is a loaded low sequence with respect to $\mathfrak{A}_1$. This contradicts our hypothesis. Hence all loaded sets of $\mathfrak{A}_1$ are contained in $\mathfrak{B}_1$. This completes the proof of the theorem.

Theorem 6 gives a simpler algorithm for the height of a regular complex. For a regular complex $(X, \mathfrak{A})$, when searching for low sequences with respect to a minimum **c**-cover $\mathfrak{A}_1$, it is sufficient to start with loaded sets of $\mathfrak{A}_1$.

5. Relationship between maximum matching and minimum cover. Let $(X, \mathfrak{A})$ be a complex and $\mathbf{c} = (c(x_2), c(x_2), \ldots, c(x_m))$ be an m-vector of positive integers. Let $\mathbf{d} = (d(x_1), d(x_2), \ldots, d(x_m))$ be an associated m-vector, where $d(x_i) = |\mathfrak{A}(x_i)| - c(x_i)$, $i = 1, 2, \ldots, m$. Let $\mathfrak{A}_1$ be a **c**-matching of the complex.

THEOREM 7. *A* **c**-*matching* $\mathfrak{A}_1$ *is a maximum* **c**-*matching if and only if* $\mathfrak{B}_1 = \mathfrak{A} - \mathfrak{A}_1$ *is a minimum* **d**-*cover.*

Proof. It is easily checked that if $\mathfrak{B}_1$ is a **d**-cover, $\mathfrak{A}_1 = \mathfrak{A} - \mathfrak{B}_1$ is a **c**-matching and vice versa. If possible, suppose $\mathfrak{B}_1$ is a minimum **d**-cover and $\mathfrak{A}_1$ is not a maximum **c**-matching. Let $\mathfrak{A}_0$ be a maximum **c**-matching. Then $\mathfrak{B}_0 = \mathfrak{A} - \mathfrak{A}_0$ is a **d**-cover. Also we have

$$|\mathfrak{A}_1| + |\mathfrak{B}_1| = |\mathfrak{A}| = |\mathfrak{A}_0| + |\mathfrak{B}_0|.$$

Since $\mathfrak{A}_1$ is not a maximum **c**-matching, $|\mathfrak{A}_1| < |\mathfrak{A}_0|$. Hence $|\mathfrak{B}_1|$ must be greater than $|\mathfrak{B}_0|$, which contradicts the assumption that $|\mathfrak{B}_1|$ is a minimum **c**-cover. This completes the proof of the theorem.

Similarly it can be proved that if $\mathfrak{B}_1$ is a minimum **d**-cover with minimum height, $\mathfrak{A}_1 = \mathfrak{A} - \mathfrak{B}_1$ is a maximum **c**-matching with maximum depth.

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