Canad. Math. Bull. Vol. **57** (2), 2014 pp. 344–356
http://dx.doi.org/10.4153/CMB-2013-038-8

On Localized Unstable K^1-groups and Applications to Self-homotopy Groups

Daisuke Kishimoto, Akira Kono, and Mitsunobu Tsutaya

Abstract. The method for computing the p-localization of the group $[X, \mathrm{U}(n)]$, by Hamanaka in 2004, is revised. As an application, an explicit description of the self-homotopy group of $\mathrm{Sp}(3)$ localized at $p \geq 5$ is given and the homotopy nilpotency of $\mathrm{Sp}(3)$ localized at $p \geq 5$ is determined.

1 Introduction

For a group-like space G, the pointed homotopy set $[X, G]$ can be regarded as a group by the pointwise multiplication of maps. Over the years, there have been considerable works on the group $[X, G]$ and its applications. Among these works, Hamanaka and the second author [HK] developed the computing method for the group $[X, \mathrm{U}(n)]$ when X is a $2n$-dimensional CW-complex. Notice that if $\dim X < 2n$, the group $[X, \mathrm{U}(n)]$ is naturally isomorphic to the K^1-group of X, and so the result deals with the first unstable case. In [Ha1], there was an attempt to deal with $[X, \mathrm{U}(n)]$ for $(2n+1)$-dimensional X; however things become quite complicated compared to the $2n$-dimensional case. So we may not expect to get complete information on $[X, \mathrm{U}(n)]$ for higher dimensional X, and then we should reduce information. One way of this reduction is the localization in the following sense. Let $\operatorname{nil} K$ and $\operatorname{cat} X$ denote the nilpotency class of a group K and the LS-category of a space X, respectively. Then G. Whitehead [W] proved the inequality

$$\operatorname{nil}[X, G] \leq \operatorname{cat} X. \tag{1.1}$$

This, in particular, implies that if X is a finite dimensional CW-complex, then $[X, G]$ is a nilpotent group for which we can consider localization at a prime in the sense of Hilton, Mislin and Roitberg [HMR]. In [Ha2], Hamanaka gave a p-local analogy of the computing method for $[X, \mathrm{U}(n)]$ with $(2n+2p-4)$-dimensional X, where p is an odd prime. This result was generalized in [KKT] for $(4n-4)$-dimensional X by assuming that X consists only of even cells. The aim of this paper is twofold. The first aim is to revise the above p-local analogy of Hamanaka [Ha2] by requiring a cohomological condition which is much weaker than the one in [KKT].

Let us now specialize X to G in the group $[X, G]$. The group $[G, G]$ is called the self-homotopy group of G, and is an interesting object reflecting homotopy theoretical properties of G. In particular, when G is a compact, connected Lie group, the

Received by the editors October 10, 2012; revised September 13, 2013.
Published electronically December 4, 2013.
AMS subject classification: **55P45**, 55P60,55Q05.
Keywords: Lie group, self-homotopy group, localization.

self-homotopy group of G has been studied extensively. However, an explicit description of the self-homotopy group of G is given only in the case that G is of rank ≤ 2 by a hard homotopy calculation. See [MO], for example. The second aim of this paper is to give a description of the self-homotopy group of Sp(3) localized at primes greater than 3. As its consequence, we will observe a relation between nilpotency of the localized self-homotopy group of Sp(3) and the homotopy nilpotency of localized Sp(3).

2 Revised Computing Method for Unstable K^1-groups

Let us first recall a property of the localization between groups and spaces. Let X be a finite dimensional CW-complex and let G be a group-like space. Then by (1.1), the group $[X, G]$ is nilpotent. In [HMR], the localization of nilpotent groups at the prime p is defined and then we have the p-localization $[X, G]_{(p)}$. On the other hand, since G is simple, the p-localization of G, denoted by $G_{(p)}$, is also defined in [HMR]. These are related by the natural isomorphism of groups

$$[X, G]_{(p)} \cong [X, G_{(p)}].$$

If X is further a nilpotent space, we also have the p-localization $X_{(p)}$ and the natural isomorphism of groups

$$[X, G_{(p)}] \cong [X_{(p)}, G_{(p)}].$$

We will use these two natural isomorphisms implicitly.

Put $W_n = \mathrm{U}(\infty)/\mathrm{U}(n)$ and let $\pi\colon \mathrm{U}(\infty) \to W_n$ be the projection. Our basic idea for calculating the group $[X, \mathrm{U}(n)]$ is basically the same as in [HK] and [Ha2]. Consider the homotopy fiber sequence

$$\Omega\mathrm{U}(\infty) \xrightarrow{\Omega\pi} \Omega W_n \xrightarrow{\delta} \mathrm{U}(n) \xrightarrow{j} \mathrm{U}(\infty),$$

in which all arrows are loop maps. Then there is an exact sequence of groups

$$\widetilde{K}^0(X) \xrightarrow{\Omega\pi_* \circ \beta} [X, \Omega W_n] \xrightarrow{\delta_*} [X, \mathrm{U}(n)] \xrightarrow{j_*} \widetilde{K}^{-1}(X), \tag{2.1}$$

where $\beta\colon \widetilde{K}^0(X) \xrightarrow{\cong} \widetilde{K}^{-2}(X)$ is the Bott periodicity. When X is finite dimensional, we can localize this exact sequence at the prime p. Using this exact sequence, we calculate the localized unstable K^1-group $[X, \mathrm{U}(n)]_{(p)}$ by identifying $[X, \Omega W_n]_{(p)}$ with the cohomology of X.

We set some notation. Let $\bar{x}_{2i-1} \in H^{2i-1}\big(\mathrm{U}(n); \mathbb{Z}_{(p)}\big)$ be the suspension of the universal Chern class $c_i \in H^{2i}\big(B\mathrm{U}(n); \mathbb{Z}_{(p)}\big)$, where n can be ∞. Then we know that

$$H^*\big(\mathrm{U}(n); \mathbb{Z}_{(p)}\big) = \Lambda(\bar{x}_1, \bar{x}_3, \dots, \bar{x}_{2n-1}),$$

and hence by a standard spectral sequence argument, we see that the cohomology of W_n is given as

$$H^*(W_n; \mathbb{Z}_{(p)}) = \Lambda(x_{2n+1}, x_{2n+3}, \dots), \quad \pi^*(x_{2i-1}) = \bar{x}_{2i-1}. \tag{2.2}$$

Since $\mathcal{P}^1\rho(\bar{x}_{2i-1}) = i\rho(\bar{x}_{2i+2p-3})$, we have

$$\mathcal{P}^1\rho(x_{2i-1}) = i\rho(x_{2i+2p-3}). \tag{2.3}$$

Let $\rho\colon K(\mathbb{Z}_{(p)}, 2i+1) \to K(\mathbb{Z}/p, 2i+1)$ denote the mod p reduction, and let $\beta\colon K(\mathbb{Z}/p, 2i+1) \to K(\mathbb{Z}_{(p)}, 2i+2)$ be the Bockstein operation associated with the short exact sequence

$$0 \to \mathbb{Z}_{(p)} \xrightarrow{\times p} \mathbb{Z}_{(p)} \to \mathbb{Z}/p \to 0.$$

Define F_{2i+1} to be the homotopy fiber of the map

$$\beta\mathcal{P}^1\rho\colon K(\mathbb{Z}_{(p)}, 2i+1) \to K(\mathbb{Z}_{(p)}, 2i+2p).$$

Then it follows that

$$\begin{gathered} H^*(F_{2i+1};\mathbb{Z}_{(p)}) = \Lambda(u_{2i+1}, u_{2i+2p-1}), \\ \mathcal{P}^1\rho(u_{2i+1}) = \rho(u_{2i+2p-1}) \quad \text{for } * \le 2i+4p-4 \end{gathered} \tag{2.4}$$

where $u_{2i+1} \in H^{2i+1}(F_{2i+1};\mathbb{Z}_{(p)})$ is identified with the inclusion $F_{2i+1} \to K(\mathbb{Z}_{(p)}, 2i+1)$ of the homotopy fiber. We notice that since $\beta\mathcal{P}^1\rho$ is a stable operation, F_{2i+1} is an infinite loop space. Let

$$g\colon W_n \to \prod_{i=n}^{n+p-2} K(\mathbb{Z}_{(p)}, 2i+1)$$

be the product of x_{2i+1} for $n \le i \le n+p-2$. Put

$$F = \prod_{i=n}^{n+p-2} F_{2i+1}.$$

Then for $\beta\mathcal{P}^1\rho\big(H^*(W_n;\mathbb{Z}_{(p)})\big) = 0$, g lifts to $\tilde{g}\colon W_n \to F$. Moreover, it follows from (2.2), (2.3), and (2.4) that the induced map $\tilde{g}^*\colon H^*(F;\mathbb{Z}_{(p)}) \to H^*(W_n;\mathbb{Z}_{(p)})$ is an isomorphism for $* \le 2(\lceil \frac{n}{p} \rceil + 1)p - 2$, implying that $\tilde{g}_{(p)}$ is a $\big(2(\lceil \frac{n}{p} \rceil + 1)p - 3\big)$-equivalence, where $\lceil x \rceil$ is the minimum integer greater than or equal to x. Thus we obtain the following.

Lemma 2.1 *Let X be a CW-complex of dimension at most $2(\lceil \frac{n}{p} \rceil + 1)p - 4$. Then the map*

$$(\Omega\tilde{g}_{(p)})_*\colon [X, \Omega W_{n(p)}] \to [X, \Omega F]$$

is an isomorphism of abelian groups.

Using Lemma 2.1, we identify the group $[X, \Omega W_{n(p)}]$ with a certain subset of the cohomology of X. Let $a_{2i} \in H^{2i}(\Omega W_n;\mathbb{Z}_{(p)})$ be the suspension of $x_{2i+1} \in H^{2i+1}(W_n;\mathbb{Z}_{(p)})$. We define a map

$$\Phi\colon [X, \Omega W_{n(p)}] \to \bigoplus_{i=n}^{(\lceil \frac{n}{p} \rceil + 1)p-2} H^{2i}(X;\mathbb{Z}_{(p)}), \quad \alpha \mapsto \bigoplus_{i=n}^{(\lceil \frac{n}{p} \rceil + 1)p-2} \alpha^*(a_{2i}).$$

https://doi.org/10.4153/CMB-2013-038-8 Published online by Cambridge University Press

Lemma 2.2 *Let X be a CW-complex of dimension at most $2(\lceil \frac{n}{p} \rceil + 1)p - 4$. If $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module, the map Φ is a monomorphism. Moreover, in dimension $\leq 2n + 2p - 4$, Φ is an isomorphism.*

Proof Since a_{2i} is a suspension of x_{2i+1}, it is a loop map, implying that Φ is a homomorphism. By definition, there is a homotopy fiber sequence

$$F \xrightarrow{\prod_{i=n}^{n+p-2} u_{2i+1}} \prod_{i=n}^{n+p-2} K(\mathbb{Z}_{(p)}, 2i+1) \xrightarrow{\beta\mathcal{P}^1\rho} \prod_{i=n}^{n+p-2} K(\mathbb{Z}_{(p)}, 2i+2p).$$

Then since $\beta\mathcal{P}^1\rho\big(H^*(X;\mathbb{Z}_{(p)})\big) = 0$, from the associated homotopy exact sequence we get a short exact sequence of groups

$$0 \to \bigoplus_{i=n}^{n+p-2} H^{2i+2p-2}(X;\mathbb{Z}_{(p)}) \to [X, \Omega F] \to \bigoplus_{i=n}^{n+p-2} H^{2i}(X;\mathbb{Z}_{(p)}) \to 0. \tag{2.5}$$

Since $[X, \Omega F]$ is an abelian group and $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module, the group $[X, \Omega F]$ is also a free $\mathbb{Z}_{(p)}$-module. Then we obtain that the rationalization $r\colon \Omega F \to \Omega F_{(0)}$ induces a monomorphism

$$r_*\colon [X, \Omega F] \to [X, \Omega F_{(0)}] \cong [X, \Omega F]_{(0)}.$$

Put

$$\widehat{\Phi} = \prod_{i=n}^{n+2p-3} \Omega u_{2i+1}\colon \Omega F \to \prod_{i=n}^{n+2p-3} K(\mathbb{Z}_{(p)}, 2i).$$

It is obvious that the rationalization $\widehat{\Phi}_{(0)}$ is a homotopy equivalence and that the composite

$$[X, \Omega W_n]_{(p)} \xrightarrow[\cong]{(\Omega\tilde{g}_{(p)})_*} [X, \Omega F] \xrightarrow{\widehat{\Phi}_*} \bigoplus_{i=n}^{n+2p-3} H^{2i}(X;\mathbb{Z}_{(p)}) \xrightarrow{\text{proj}} \bigoplus_{i=n}^{(\lceil \frac{n}{p} \rceil+1)p-2} H^{2i}(X;\mathbb{Z}_{(p)})$$

is equal to the map Φ. There is a commutative diagram

$$\begin{array}{ccc}
[X, \Omega F] & \xrightarrow{r_*} & [X, \Omega F]_{(0)} \\
\Big\downarrow \widehat{\Phi}_* & & \cong \Big\downarrow (\widehat{\Phi}_{(0)})_* \\
\bigoplus_{i=n}^{n+2p-3} H^{2i}(X;\mathbb{Z}_{(p)}) & \longrightarrow & \bigoplus_{i=n}^{n+2p-3} H^{2i}(X;\mathbb{Q})
\end{array}$$

in which the bottom arrow is induced from the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Q}$. Since r_* is injective and $\widehat{\Phi}_{(0)}$ is an isomorphism as above, we obtain that the left vertical arrow is injective. Thus we have proved the first assertion. The second assertion follows from the last arrow in the exact sequence (2.5). ■

https://doi.org/10.4153/CMB-2013-038-8 Published online by Cambridge University Press

We have identified $[X, \Omega W_n]_{(p)}$ with $\operatorname{Im}\Phi \subset H^*(X;\mathbb{Z}_{(p)})$ when $\dim X \leq 2(\lceil \frac{n}{p} \rceil + 1)p - 4$ and $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module. We next analyze the p-localization of the exact sequence (2.1) together with this identification.

Theorem 2.3 *Let X be a CW-complex such that $\dim X \leq 2(\lceil \frac{n}{p} \rceil + 1)p - 4$ and that $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module. Then there is an exact sequence of groups*

$$\widetilde{K}^0(X)_{(p)} \xrightarrow{\Theta} \operatorname{Im}\Phi \xrightarrow{\bar{\delta}} [X, \mathrm{U}(n)]_{(p)} \xrightarrow{(j_*)_{(p)}} \widetilde{K}^1(X)_{(p)}$$

satisfying, for $\xi \in \widetilde{K}^0(X)_{(p)}$,

$$\Theta(\xi) = \bigoplus_{i=n}^{(\lceil \frac{n}{p} \rceil+1)p-2} i!\,\mathrm{ch}_i(\xi),$$

where $\mathrm{ch}_i(\xi)$ denotes the $2i$-dimensional part of the Chern character $\mathrm{ch}(\xi)$.

Proof For the cohomology suspension σ, we have

$$\big((\Omega\pi_*)_{(p)} \circ \beta_{(p)}\big)^*(a_{2k}) = \beta_{(p)}^*\big(\sigma^2(c_{k+1})\big) = k!\,\mathrm{ch}_k\,.$$

Then if we put $\Theta = \Phi \circ (\Omega\pi_*)_{(p)} \circ \beta_{(p)}$ and $\bar{\delta} = (\delta_*)_{(p)} \circ \Phi^{-1}$, we obtain the desired exact sequence by (2.1) and Lemma 2.2. ∎

Let us further investigate the exact sequence in Theorem 2.3. To this end, we recall a result in [HK]. Let $\gamma\colon \mathrm{U}(n) \wedge \mathrm{U}(n) \to \mathrm{U}(n)$ be the reduced commutator map. Since $\mathrm{U}(\infty)$ is homotopy abelian, the composite $j \circ \gamma$ is null homotopic, implying there is a lift $\widetilde{\gamma}\colon \mathrm{U}(n) \wedge \mathrm{U}(n) \to \Omega W_n$ of γ through the map $\delta\colon \Omega W_n \to \mathrm{U}(n)$. We can choose a lift $\widetilde{\gamma}$ that behaves well in cohomology as follows.

Lemma 2.4 (Hamanaka and Kono [HK]) *There is a lift $\widetilde{\gamma}$ satisfying*

$$\widetilde{\gamma}^*(a_{2k}) = \sum_{\substack{i+j-1=k\\ 1\leq i,j\leq n}} x_{2i-1} \otimes x_{2j-1}.$$

The commutators in the group $[X, \mathrm{U}(n)]_{(p)}$ are described explicitly in the exact sequence of Theorem 2.3 as follows.

Theorem 2.5 *Let X be a CW-complex such that $\dim X \leq 2(\lceil \frac{n}{p} \rceil + 1)p - 4$ and that $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module. Then for $\lambda, \mu \in [X, \mathrm{U}(n)]_{(p)}$, the commutator $[\lambda, \mu]$ is equal to*

$$\bar{\delta}\Big(\bigoplus_{k=n}^{(\lceil \frac{n}{p} \rceil+1)p-2} \sum_{\substack{i+j-1=k\\ 1\leq i,j\leq n}} \lambda^*(x_{2i-1}) \cup \mu^*(x_{2j-1})\Big).$$

https://doi.org/10.4153/CMB-2013-038-8 Published online by Cambridge University Press

Proof By definition, we have

$$[\lambda, \mu] = \gamma_{(p)} \circ (\lambda \wedge \mu) \circ \overline{\Delta}$$

for the reduced diagonal map $\overline{\Delta}\colon X \to X \wedge X$. Then the result follows from Theorem 2.3 and Lemma 2.4. ■

Combining Theorems 2.3 and 2.5, we can thus compute the group $[X, \mathrm{U}(n)]_{(p)}$ when X satisfies the conditions in Theorem 2.3. In particular, we have the following corollaries.

Corollary 2.6 *Let X be a CW-complex such that $\dim X \leq 2(\lceil \frac{n}{p} \rceil + 1)p - 4$ and that $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module. Then the short exact sequence*

$$0 \to \operatorname{Coker}\Theta \to [X, \mathrm{U}(n)]_{(p)} \to \operatorname{Im}(j_*)_{(p)} \to 0$$

induced from Theorem 2.3 is central.

Proof All we have to show is that for maps $\lambda\colon X \to \mathrm{U}(n)_{(p)}$ and $\mu\colon X \to (\Omega W_n)_{(p)}$, the commutator $[\lambda, \delta_{(p)} \circ \mu]$ in $[X, \mathrm{U}(n)]_{(p)}$ is trivial. This follows from Theorem 2.5 together with the fact that $\delta^*_{(p)}(x_{2i-1}) = 0$. ■

Corollary 2.7 *Let X be a CW-complex such that $\dim X \leq 2(\lceil \frac{n}{p} \rceil + 1)p - 4$ and that $H^*(X;\mathbb{Z}_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module. If $H^{2i-1}(X;\mathbb{Z}_{(p)}) = 0$ for $i \leq n$, the group $[X, \mathrm{U}(n)]_{(p)}$ is abelian.*

Proof By assumption, any map $X \to \mathrm{U}(n)_{(p)}$ is trivial in cohomology. Then the proof is completed by Theorem 2.5. ■

3 The Self-homotopy Group of $\mathrm{Sp}(3)$

In this section, we give an explicit description of the self-homotopy group of $\mathrm{Sp}(3)$ localized at the prime $p > 3$ as an application of the results in the previous section. By Corollary 2.6, there is a central extension

$$0 \to \operatorname{Coker}\Theta \to [\mathrm{Sp}(3), \mathrm{U}(6)]_{(p)} \to \operatorname{Im}(j_*)_{(p)} \to 0 \tag{3.1}$$

for $p > 3$. Then we first determine $\operatorname{Coker}\Theta$ and $\operatorname{Im}(j_*)_{(p)}$. We next determine the commutators in $[\mathrm{Sp}(3), \mathrm{U}(6)]_{(p)}$ using Theorem 2.5 to get an explicit description of the group $[\mathrm{Sp}(3), \mathrm{U}(6)]_{(p)}$. From this description, we will deduce a description of the self-homotopy group of $\mathrm{Sp}(3)$ localized at $p > 3$.

We set notation. Let $y_{4i-1} \in H^{4i-1}\big(\mathrm{Sp}(3);\mathbb{Z}\big)$ be the suspension of the universal symplectic Pontrjagin class $q_i \in H^{4i}\big(B\mathrm{Sp}(3);\mathbb{Z}\big)$ for $i = 1, 2, 3$. Then we have

$$H^*\big(\mathrm{Sp}(3);\mathbb{Z}\big) = \Lambda(y_3, y_7, y_{11}),$$

and for the natural inclusion $\mathbf{c}\colon \mathrm{Sp}(3) \to \mathrm{U}(6)$, there holds

$$\mathbf{c}^*(x_{2i-1}) = \begin{cases} -y_{2i-1} & i = 2, 6, \\ y_{2i-1} & i = 4, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

https://doi.org/10.4153/CMB-2013-038-8 Published online by Cambridge University Press

3.1 The Case $p = 5$

Recall from [MNT] that there is a mod 5 decomposition of Sp(3) as

$$\mathrm{Sp}(3)_{(5)} \simeq B \times S^7_{(5)}, \tag{3.3}$$

where B is the 5-localization of a certain S^3-bundle over S^{11}. Using this decomposition, we define three self-maps of $\mathrm{Sp}(3)_{(5)}$. Let λ_1 and λ_2 be the composites

$$\mathrm{Sp}(3)_{(5)} \xrightarrow{\text{proj}} B \xrightarrow{\text{incl}} \mathrm{Sp}(3)_{(5)} \quad \text{and} \quad \mathrm{Sp}(3)_{(5)} \xrightarrow{\text{proj}} S^7_{(5)} \xrightarrow{\text{incl}} \mathrm{Sp}(3)_{(5)},$$

respectively. Let λ_3 be the composite

$$\mathrm{Sp}(3)_{(5)} \xrightarrow{\text{proj}} B \xrightarrow{\text{proj}} S^{11}_{(5)} \to \mathrm{Sp}(3)_{(5)},$$

where the last arrow is the 5-localization of a generator of $\pi_{11}\big(\mathrm{Sp}(3)\big) \cong \mathbb{Z}$.

We determine the image of $(j_*)_{(5)} \colon [\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)} \to \widetilde{K}^{-1}\big(\mathrm{Sp}(3)\big)_{(5)}$. Let $\overline{\lambda}_i$ for $i = 1, 2, 3$ be the composite

$$\mathrm{Sp}(3)_{(5)} \xrightarrow{\lambda_i} \mathrm{Sp}(3)_{(5)} \xrightarrow{\mathbf{c}_{(5)}} \mathrm{U}(6)_{(5)} \xrightarrow{j_{(5)}} \mathrm{U}(\infty)_{(5)}.$$

Then by definition, the image of $(j_*)_{(5)} \colon [\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)} \to \widetilde{K}^{-1}\big(\mathrm{Sp}(3)\big)_{(5)}$ includes the submodule generated by $\overline{\lambda}_i$ for $i = 1, 2, 3$. Let $\overline{\lambda}_4$ be the composite

$$\mathrm{Sp}(3)_{(5)} \to S^{21}_{(5)} \to \mathrm{U}(\infty)_{(5)},$$

where the first arrow is the pinch map onto the top cell and the second arrow is the 5-localization of a generator of $\pi_{21}\big(\mathrm{U}(\infty)\big) \cong \mathbb{Z}$. By definition of $\overline{\lambda}_i$ together with (3.2), we can calculate the Chern character of $\overline{\lambda}_i$ as

$$\begin{aligned} \mathrm{ch}(\overline{\lambda}_1) &= -\Sigma y_3 - \tfrac{1}{5!}\Sigma y_{11}, & \mathrm{ch}(\overline{\lambda}_2) &= \tfrac{1}{3!}\Sigma y_7, \\ \mathrm{ch}(\overline{\lambda}_3) &= \Sigma y_{11}, & \mathrm{ch}(\overline{\lambda}_4) &= \Sigma y_3 y_7 y_{11}, \end{aligned} \tag{3.4}$$

implying that $\widetilde{K}^{-1}\big(\mathrm{Sp}(3)\big)_{(5)}$ is a free $\mathbb{Z}_{(5)}$-module with a basis $\{\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3, \overline{\lambda}_4\}$. Since the projection $\pi \colon \mathrm{U}(\infty) \to W_6$ induces an injection in π_{21}, we obtain the following.

Lemma 3.1 *The image of $(j_*)_{(5)} \colon [\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)} \to \widetilde{K}^{-1}\big(\mathrm{Sp}(3)\big)_{(5)}$ is a free $\mathbb{Z}_{(5)}$-module with a basis $\{\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3\}$.*

We next determine $\operatorname{Coker}\Theta$ in Theorem 2.3 for the group $[\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)}$. By definition, $\operatorname{Im}\Phi$ is a subgroup of $H^{14}\big(\mathrm{Sp}(3); \mathbb{Z}_{(5)}\big) \oplus H^{18}\big(\mathrm{Sp}(3); \mathbb{Z}_{(5)}\big)$, and then by Lemma 2.2, we get

$$\operatorname{Im}\Phi = H^{14}\big(\mathrm{Sp}(3); \mathbb{Z}_{(5)}\big) \oplus H^{18}\big(\mathrm{Sp}(3); \mathbb{Z}_{(5)}\big) = \langle y_3 y_{11}, y_7 y_{11} \rangle.$$

https://doi.org/10.4153/CMB-2013-038-8 Published online by Cambridge University Press

Put $\rho_1 = \beta^{-1}(\overline{\lambda}_1\overline{\lambda}_2)$ and $\rho_2 = \beta^{-1}(\overline{\lambda}_2\overline{\lambda}_3)$. We also put ρ_3 to be the composite

$$\mathrm{Sp}(3)_{(5)} \xrightarrow{\text{proj}} B \to S^{14}_{(5)} \to B\mathrm{U}(\infty),$$

where the second arrow pinches onto the top cell and the last arrow is the 5-localization of a generator of $\pi_{14}\big(B\mathrm{U}(\infty)\big) \cong \mathbb{Z}$. Then it follows from (3.4) that

$$\mathrm{ch}(\rho_1) = -y_3y_7 + \tfrac{1}{5!}y_7y_{11}, \quad \mathrm{ch}(\rho_2) = y_7y_{11}, \quad \mathrm{ch}(\rho_3) = y_3y_{11},$$

implying that $\widetilde{K}^0\big(\mathrm{Sp}(3)\big)_{(5)}$ is a free $\mathbb{Z}_{(5)}$-module with a basis $\{\rho_1, \rho_2, \rho_3\}$. Thus by Theorem 2.3, we obtain the following.

Lemma 3.2 *The cokernel of* $\Theta\colon \widetilde{K}^0\big(\mathrm{Sp}(3)\big)_{(5)} \to \mathrm{Im}\,\Phi$ *is* $\langle y_3y_{11}\rangle/\langle 5y_3y_{11}\rangle \cong \mathbb{Z}/5$.

Put $\lambda_i' = \mathbf{c}_{(5)} \circ \lambda_i$ for $i = 1, 2, 3$. We finally calculate the commutators of λ_i' in the group $[\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)}$. By Theorem 2.5, the commutators $[\lambda_1', \lambda_2']$, $[\lambda_2', \lambda_3']$, $[\lambda_1', \lambda_3']$ are given as

$$\bar{\delta}(x_7x_{11}) = 0, \quad \bar{\delta}(x_7x_{11}) = 0, \quad \bar{\delta}(x_3x_{11}),$$

respectively. Combining the above calculation, we obtain the following.

Theorem 3.3 *There is a central extension*

$$0 \to \mathbb{Z}/5 \to [\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)} \xrightarrow{\theta} (\mathbb{Z}_{(5)})^3 \to 0$$

determined by the following.

(i) θ *sends* $\lambda_1', \lambda_2', \lambda_3'$ *to* $(1,0,0), (0,1,0), (0,0,1)$, *respectively.*
(ii) *The kernel of* θ *is generated by* $[\lambda_1', \lambda_3']$.
(iii) $[\lambda_1', \lambda_2'] = [\lambda_2', \lambda_3'] = 1$.

Remark 3.4. Since $[\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)}$ is 5-local, it follows from the Hall–Witt formula together with the relation $\big[\lambda_i', [\lambda_j', \lambda_k']\big] = 1$ that

$$[\lambda_i^{\frac{1}{m}}, \lambda_j^{\frac{1}{n}}] = [\lambda_i, \lambda_j]^{\frac{1}{mn}}$$

for any m, n with $5 \nmid mn$. Similar equalities hold in what follows.

Recall that the inclusion $\mathbf{c}\colon \mathrm{Sp}(3) \to \mathrm{U}(6)$ yields the mod 5 decomposition

$$\mathrm{U}(6)_{(5)} \simeq \mathrm{Sp}(3)_{(5)} \times S^1_{(5)} \times S^5_{(5)} \times S^9_{(5)}.$$

As in [T], we know that $\pi_i(S^1)_{(5)} = \pi_i(S^5)_{(5)} = \pi_i(S^9)_{(5)} = 0$ for $i = 3, 7, 10, 11, 14, 18, 21$. Then since $\mathrm{Sp}(3)$ consists of cells in dimension $3, 7, 10, 11, 14, 18, 21$, we get that the homotopy set $[\mathrm{Sp}(3)_{(5)}, S^1_{(5)} \times S^5_{(5)} \times S^9_{(5)}]$ is trivial, implying that the induced map

$$(\mathbf{c}_*)_{(5)}\colon [\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(5)} \to [\mathrm{Sp}(3), \mathrm{U}(6)]_{(5)}$$

is an isomorphism of groups. Therefore we obtain the following.

https://doi.org/10.4153/CMB-2013-038-8 Published online by Cambridge University Press

Corollary 3.5 *There is a central extension*

$$0 \to \mathbb{Z}/5 \to [\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(5)} \xrightarrow{\theta} (\mathbb{Z}_{(5)})^3 \to 0$$

determined by the following.

(i) *θ sends $\lambda_1, \lambda_2, \lambda_3$ to $(1,0,0), (0,1,0), (0,0,1)$, respectively.*
(ii) *The kernel of θ is generated by $[\lambda_1, \lambda_3]$.*
(iii) $[\lambda_1, \lambda_2] = [\lambda_2, \lambda_3] = 1$.

3.2 The Case $p > 5$

Recall that for $p > 5$, there is a homotopy equivalence

$$\mathrm{Sp}(3)_{(p)} \simeq S^3_{(p)} \times S^7_{(p)} \times S^{11}_{(p)}.$$

We then define the self-map λ_i of $\mathrm{Sp}(3)$ for $i = 1, 2, 3$ as the composite

$$\mathrm{Sp}(3)_{(p)} \xrightarrow{\text{proj}} S^{4i-1}_{(p)} \xrightarrow{\text{incl}} \mathrm{Sp}(3)_{(p)}.$$

Put $\lambda'_i = \mathbf{c}_{(p)} \circ \lambda_i$ and $\overline{\lambda}_i = j_{(p)} \circ \mathbf{c}_{(p)} \circ \lambda_i \in \widetilde{K}^{-1}\big(\mathrm{Sp}(3)\big)_{(p)}$ for $i = 1, 2, 3$. Quite analogously to the above calculation, we see that the image of $(j_*)_{(p)} \colon [\mathrm{Sp}(3), \mathrm{U}(6)]_{(p)} \to \widetilde{K}^{-1}\big(\mathrm{Sp}(3)\big)_{(p)}$ is $\langle \overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3 \rangle$.

We also have

$$\mathrm{Im}\,\Phi = H^{14}\big(\mathrm{Sp}(3); \mathbb{Z}_{(5)}\big) \oplus H^{18}\big(\mathrm{Sp}(3); \mathbb{Z}_{(5)}\big) = \langle y_3 y_{11}, y_7 y_{11} \rangle$$

as above. Put $\rho_1 = \beta^{-1}(\overline{\lambda}_1 \overline{\lambda}_2)$, $\rho_2 = \beta^{-1}(\overline{\lambda}_2 \overline{\lambda}_3)$ and $\rho_3 = \beta^{-1}(\overline{\lambda}_1 \overline{\lambda}_3)$. Then $\widetilde{K}^0\big(\mathrm{Sp}(3)\big)_{(p)}$ is $\langle \rho_1, \rho_2, \rho_3 \rangle$, and hence we see from Theorem 2.3 that the image of Θ is generated by

$$7 y_3 y_{11} \quad \text{and} \quad 7 y_7 y_{11}.$$

Thus $\mathrm{Coker}\,\Theta$ is trivial for $p > 7$ and

$$\mathrm{Coker}\,\Theta = \langle x_3 x_{11} \rangle / \langle 7 x_3 x_{11} \rangle \oplus \langle x_7 x_{11} \rangle / \langle 7 x_7 x_{11} \rangle \cong \mathbb{Z}/7 \oplus \mathbb{Z}/7$$

for $p = 7$.

By Theorem 2.5, the commutators $[\lambda'_1, \lambda'_2], [\lambda'_2, \lambda'_3], [\lambda'_1, \lambda'_3]$ are given as

$$\bar{\delta}(x_3 x_7) = 0, \quad \bar{\delta}(x_7 x_{11}), \quad \bar{\delta}(x_3 x_{11}),$$

respectively. Thus since there is a central extension (3.1), we have established the following.

Theorem 3.6 *For $p > 5$, we have the following description of $[\mathrm{Sp}(3), \mathrm{U}(6)]_{(p)}$.*

(i) *For $p = 7$, there is a central extension*

$$0 \to (\mathbb{Z}/7)^2 \to [\mathrm{Sp}(3), \mathrm{U}(6)]_{(7)} \xrightarrow{\theta} (\mathbb{Z}_{(7)})^3 \to 0$$

determined by the following.

(a) *θ sends $\lambda'_1, \lambda'_2, \lambda'_3$ to* $(1,0,0), (0,1,0), (0,0,1)$.
(b) *The kernel of θ is generated by* $[\lambda'_1, \lambda'_3]$, $[\lambda'_2, \lambda'_3]$.
(c) $[\lambda'_1, \lambda'_2] = 1$.

(ii) *If $p > 7$, $[\mathrm{Sp}(3), \mathrm{U}(6)]_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module with a basis $\{\lambda'_1, \lambda'_2, \lambda'_3\}$.*

As in the above case $p = 5$, we see that the inclusion $\mathbf{c}$ induces an isomorphism of groups

$$(\mathbf{c}_*)_{(p)} \colon [\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(p)} \xrightarrow{\cong} [\mathrm{Sp}(3), \mathrm{U}(6)]_{(p)},$$

and thus we obtain the following.

Corollary 3.7 *For $p > 5$, we have the following description of* $[\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(p)}$.

(i) *For $p = 7$, there is a central extension*

$$0 \to (\mathbb{Z}/7)^2 \to [\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(7)} \xrightarrow{\theta} (\mathbb{Z}_{(7)})^3 \to 0$$

determined by the following.

(a) *θ sends $\lambda_1, \lambda_2, \lambda_3$ to* $(1,0,0), (0,1,0), (0,0,1)$.
(b) *The kernel of θ is generated by* $[\lambda_1, \lambda_3], [\lambda_2, \lambda_3]$.
(c) $[\lambda_1, \lambda_2] = 1$.

(ii) *If $p > 7$, $[\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module with a basis $\{\lambda_1, \lambda_2, \lambda_3\}$.*

3.3 Nilpotency

For a group K, let $\gamma\colon K \times K \to K$ be the commutator map and put $\gamma_0 = 1$ and $\gamma_n = \gamma \circ (\gamma_{n-1} \times 1)\colon K^{n+1} \to K$ for $n > 1$. Recall that the group K is called nilpotent of class n if γ_n is the constant map but γ_{n-1} is not. We denote the nilpotency class of K by $\mathrm{nil}\, K$. There is a homotopy analog of nilpotency of groups as follows. For a group-like space G, let $\gamma\colon G \times G \to G$ be the commutator map. Put $\gamma_0 = 1$ and $\gamma_n = \gamma \circ (1 \times \gamma_{n-1})$ for $n > 1$. G is called homotopy nilpotent of class n if $\gamma_n \simeq *$ and $\gamma_{n-1} \not\simeq *$. We denote the homotopy nilpotency class of G by $\mathrm{honil}\, G$. Of course, not every group-like space is homotopy nilpotent, but it is proved in [Ho] that if a group-like space G is a finite complex with torsion free homology, G is homotopy nilpotent. Then in particular, compact Lie groups with torsion free homology are known to be homotopy nilpotent. We are now interested in the homotopy nilpotency class of localized Lie groups. The homotopy nilpotency class of Lie groups localized at large primes were determined in [KK] and [K]. However, it is quite hard to determine the homotopy nilpotency class, in general.

There is the canonical relation between the nilpotency class of the self homotopy group of a group-like space G and the homotopy nilpotency class of G as

$$\mathrm{nil}[G, G] \leq \mathrm{honil}\, G. \tag{3.5}$$

https://doi.org/10.4153/CMB-2013-038-8 Published online by Cambridge University Press

Let us observe this inequality for $G = \mathrm{Sp}(3)_{(p)}$ with $p \geq 5$.

By Corollary 3.5 and 3.7, we have the following.

Corollary 3.8

$$\mathrm{nil}[\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(p)} = \begin{cases} 2 & p = 5, 7, \\ 1 & p > 7. \end{cases}$$

When G is a Lie group and $G_{(p)}$ has the homotopy type of a product of p-localized spheres, the homotopy nilpotency class of $G_{(p)}$ was determined by Kaji and the first author [KK]. In particular, we have the following.

Proposition 3.9 (Kaji and Kishimoto [KK])

$$\mathrm{honil}\,\mathrm{Sp}(3)_{(p)} = \begin{cases} 3 & p = 7, \\ 2 & p = 11, \\ 1 & p > 11. \end{cases}$$

The only missing case in Proposition 3.9 compared to Corollary 3.8 is $p = 5$. So we compute the homotopy nilpotency class of $\mathrm{Sp}(3)_{(5)}$. The main idea of computing the homotopy nilpotency class of a group-like space G in [KK] and [K] is to reduce the computation to the iterated Samelson products of the inclusions $X_i \to G$ when G has the homotopy type of a product of spaces $X_1, \dots, X_n$. We now have the mod 5 decomposition of $\mathrm{Sp}(3)_{(5)}$ as in (3.3). Let μ_1, μ_2 denote the inclusions $B \to \mathrm{Sp}(3)_{(5)}$ and $S^7_{(5)} \to \mathrm{Sp}(3)_{(5)}$, respectively. Then we have the following reduction.

Lemma 3.10 (Kaji and Kishimoto [KK]) $\mathrm{honil}\,\mathrm{Sp}(3)_{(5)} \leq 2$ *if and only if the* 2*-fold Samelson product* $\langle \mu_i, \langle \mu_j, \mu_k \rangle \rangle$ *is trivial for* $(i, j, k) = (1,1,1), (1,1,2), (1,2,2), (2,1,1), (2,1,2), (2,2,2)$.

Theorem 3.11 $\mathrm{honil}\,\mathrm{Sp}(3)_{(5)} = 2$.

Proof We first check that all the 2-fold Samelson products in Lemma 3.10 are trivial.

By definition, the composite

$$\mathrm{Sp}(3)_{(5)} \xrightarrow{\Delta} \mathrm{Sp}(3)_{(5)} \times \mathrm{Sp}(3)_{(5)} \xrightarrow{\pi_1 \times \pi_2} B \times S^7_{(5)} = \mathrm{Sp}(3)_{(5)}$$

is homotopic to the identity map, where Δ is the diagonal map and π_1, π_2 are the projections $\mathrm{Sp}(3)_{(5)} \to B$ and $\mathrm{Sp}(3)_{(5)} \to S^7_{(5)}$, respectively. Then the commutator $[\lambda_1, \lambda_2]$ in the self-homotopy group of $\mathrm{Sp}(3)_{(5)}$ is identified with the composite

$$\mathrm{Sp}(3)_{(5)} = B \times S^7_{(5)} \xrightarrow{\mu_1 \times \mu_2} \mathrm{Sp}(3)_{(5)} \times \mathrm{Sp}(3)_{(5)} \xrightarrow{\gamma} \mathrm{Sp}(3)_{(5)},$$

where γ is the commutator map. Since the pinch map $X \times X \to X \wedge X$ induces an injection $[X \wedge X, G] \to [X \times X, G]$ for a loop space G, triviality of $[\lambda_1, \lambda_2]$ in Corollary 3.5 implies that of the Samelson product $\langle \mu_1, \mu_2 \rangle$. Thus we obtain that the 2-fold Samelson product $\langle \mu_i, \langle \mu_j, \mu_k \rangle \rangle$ for $(i, j, k) = (1,1,2), (2,1,2)$ is trivial.

As in [K], we know $\pi_{14}(B) = 0$, implying that the Samelson product $\langle \mu_1, \mu_2 \rangle$ can be compressed into $S^7_{(5)} \subset \mathrm{Sp}(3)_{(5)}$. Then we obtain that $\langle \mu_1, \langle \mu_2, \mu_2 \rangle \rangle$ is trivial by the above triviality of $\langle \mu_1, \mu_2 \rangle$.

It is proved in [F] that there is a homotopy equivalence $B\mathrm{Spin}(7)_{(5)} \simeq B\mathrm{Sp}(3)_{(5)}$. Through this homotopy equivalence, we can identify the inclusion $\mu_1 \colon B \to \mathrm{Sp}(3)_{(5)}$ with the 5-localization of the natural inclusion $G_2 \to \mathrm{Spin}(5)$ of a Lie subgroup. On the other hand, $G_{2(5)}$ is shown to be homotopy commutative in [M]. Thus by naturality of Samelson products, we obtain that $\langle \mu_1, \mu_1 \rangle$ is trivial, implying that the 2-fold Samelson product $\langle \mu_i, \langle \mu_j, \mu_k \rangle \rangle$ for $(i, j, k) = (1, 1, 1), (2, 1, 1)$ is trivial.

By [T] and [K], we know $\pi_{21}(S^7)_{(5)} = 0$ and $\pi_{21}(B) = 0$, or equivalently, $\pi_{21}(\mathrm{Sp}(3))_{(5)} = 0$. Then $\langle \mu_2, \langle \mu_2, \mu_2 \rangle \rangle$ is trivial.

By the above calculation of Samelson products together with Lemma 3.10, we have established $\operatorname{honil} \mathrm{Sp}(3)_{(5)} \leq 2$. On the other hand, by Corollary 3.5 and (3.5), we have $\operatorname{honil} \mathrm{Sp}(3)_{(5)} \geq 2$. Thus the proof of the theorem is completed. ■

Let G be a compact, simply connected Lie group. As in [MNT], the homotopy type of the p-localization $G_{(p)}$ becomes simpler as p increases. Then the H-structure of $G_{(p)}$ might also get simpler as p increases. So one might expect that $\operatorname{honil} G_{(p)}$ is a monotonically decreasing function in p. This is true if $G_{(p)}$ has the homotopy type of a product of spheres by [M] and [KK]. But in [K], it was shown that this expectation is negative. Actually, it was shown that for $n = 9$ and $n \geq 13$, $\operatorname{honil} \mathrm{SU}(n)_{(p)}$ is not monotonically decreasing in p. By Proposition 3.9 and Theorem 3.11, we see that Sp(3) also yields a counterexample to the above expectation.

Corollary 3.12 $\operatorname{honil} \mathrm{Sp}(3)_{(p)}$ *is not monotonically decreasing in* p.

Differently from $\operatorname{honil} \mathrm{Sp}(3)_{(p)}$, $\operatorname{nil}[\mathrm{Sp}(3), \mathrm{Sp}(3)]_{(p)}$ is monotonically decreasing in p by Corollary 3.8. In fact, we do not have so far a counterexample to the assertion that $\operatorname{nil}[G, G]_{(p)}$ is monotonically decreasing in p when G is a simple Lie group. However, we dare to conjecture the following.

Conjecture 3.13 *There exists a simple Lie group G for which* $\operatorname{nil}[G, G]_{(p)}$ *is not monotonically decreasing in* p.

References

[F] E. M. Friedlander, *Exceptional isogenies and the classifying spaces of simple Lie groups*. Ann. Math. **101**(1975), 510–520. http://dx.doi.org/10.2307/1970938

[Ha1] H. Hamanaka, *On* $[X, \mathrm{U}(n)]$ *when* $\dim X$ *is* $2n + 1$. J. Math. Kyoto Univ. **44**(2004), 655–667.

[Ha2] ———, *On Samelson products in p-localized unitary groups.* Topology Appl. **154**(2007), 573–583. http://dx.doi.org/10.1016/j.topol.2006.07.011

[Ho] M. J. Hopkins, *Nilpotence and finite H-spaces*. Israel J. Math. **66**(1989), 238–246. http://dx.doi.org/10.1007/BF02765895

[HK] H. Hamanaka and A. Kono, *On* $[X, \mathrm{U}(n)]$ *when* $\dim X$ *is* $2n$. J. Math. Kyoto Univ. **43**(2003), 333–348.

[HMR] P. Hilton, G. Mislin, and J. Roitberg, *Localization of nilpotent groups and spaces.* In: North-Holland Mathematics Studies **15**, Notas de Matemática (Notes on Mathematics) vol. **55**, North-Holland Publishing Co./American Elsevier Publishing Co., Inc., Amsterdam, Oxford/New York, 1975.

[KK] S. Kaji and D. Kishimoto, *Homotopy nilpotency in p-regular loop spaces*. Math. Z. **264**(2010), 209–224. http://dx.doi.org/10.1007/s00209-008-0459-6

[K] D. Kishimoto, *Homotopy nilpotency in localized* SU(n). Homology, Homotopy Appl. **11**(2009), 61–79.
[KKT] D. Kishimoto, A. Kono, and M. Tsutaya, *On p-local homotopy types of gauge groups*. Preprint.
[M] C. McGibbon, *Homotopy commutativity in localized groups.* Amer. J. Math. **106**(1984), 665–687. http://dx.doi.org/10.2307/2374290
[MNT] M. Mimura, G. Nishida, and H. Toda, Mod *p decomposition of compact Lie groups.* Publ. Res. Inst. Math. Sci. **13**(1977/1978), 627–680. http://dx.doi.org/10.2977/prims/1195189602
[MO] M. Mimura and H. Ōshima, *Self Homotopy groups of Hopf spaces with at most three cells*. J. Math. Soc. Japan **51**(1999), 71–92. http://dx.doi.org/10.2969/jmsj/05110071
[T] H. Toda, *Composition methods in homotopy groups of spheres.* Ann. Math. Studies **46**, Princeton University Press, Princeton, NJ, 1962.
[W] G. W. Whitehead, *On mappings into group-like spaces*. Comment. Math. Helv. **28**(1954), 320–328. http://dx.doi.org/10.1007/BF02566938

Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan
e-mail: kishi@math.kyoto-u.ac.jp

Faculty of Science and Engineering, Doshisha University, Kyoto 610-0321, Japan
e-mail: akono@mail.doshisha.ac.jp tsutaya@math.kyoto-u.ac.jp