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Star-Shapedness and K-Orbits in Complex Semisimple Lie Algebras

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Abstract. Given a complex semisimple Lie algebra $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ ($\mathfrak{k}$ is a compact real form of $\mathfrak{g}$), let $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be the orthogonal projection (with respect to the Killing form) onto the Cartan subalgebra $\mathfrak{h} := \mathfrak{t} + i\mathfrak{t}$, where $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{k}$. Given $x \in \mathfrak{g}$, we consider $\pi(\mathrm{Ad}(K)x)$, where K is the analytic subgroup G corresponding to $\mathfrak{k}$, and show that it is star-shaped. The result extends a result of Tsing. We also consider the generalized numerical range $f(\mathrm{Ad}(K)x)$, where f is a linear functional on $\mathfrak{g}$. We establish the star-shapedness of $f(\mathrm{Ad}(K)x)$ for simple Lie algebras of type B.

1 Introduction

Let $\mathfrak{gl}_n(\mathbb{C})$ denote the Lie algebra of all $n \times n$ complex matrices. Let $A \in \mathfrak{gl}_n(\mathbb{C})$. Consider the set

$$\mathcal{W}(A) := \{\mathrm{diag}(UAU^{-1}) : U \in \mathrm{U}(n)\},$$

where $\mathrm{U}(n)$ denotes the unitary group. It is the image of the projection of the orbit

$$O(A) := \{UAU^{-1} : U \in \mathrm{U}(n)\}$$

onto the set of diagonal matrices. One may replace $\mathrm{U}(n)$ by $\mathrm{SU}(n)$ in the definition of $\mathcal{W}(A)$ and $O(A)$. If $A \in \mathbb{C}_{n\times n}$ is Hermitian with eigenvalues $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, then the Schur–Horn theorem [7, 14] [13, pp. 218–220] asserts that $\mathcal{W}(A) = \mathrm{conv}\, S_n\lambda$, where $\mathrm{conv}\, S_n\lambda$ is the convex hull of the orbit of λ under the action of the symmetric group S_n. For general $A \in \mathbb{C}_{n\times n}$, $\mathcal{W}(A)$ is not convex [1,2]. Nonconvexity naturally prompted the question whether certain weaker geometric results are at least true. The following interesting result is due to Tsing [17].

Theorem 1.1 (Tsing [17]) *Let $A \in \mathbb{C}_{n\times n}$. Then $\mathcal{W}(A)$ is star-shaped with respect to the star center $\frac{1}{n}(\mathrm{tr}\, A)(1, \dots, 1)$.*

The above results can be reduced to the case $\mathrm{tr}\, A = 0$, *i.e.,* the (noncompact) simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ [6, pp. 186–187]. We may write $A = \hat{A} + \frac{\mathrm{tr}\, A}{n} I_n$, where $\hat{A} := A - \frac{\mathrm{tr}\, A}{n} I_n$ has zero trace. Then

$$\mathcal{W}(A) = \mathcal{W}(\hat{A}) + \frac{\mathrm{tr}\, A}{n}(1, \dots, 1).$$

Taking the diagonal part of $A \in \mathfrak{sl}_n(\mathbb{C})$ amounts to the orthogonal projection

$$\pi\colon \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{h},$$

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where $\mathfrak{h} \subseteq \mathfrak{sl}_n(\mathbb{C})$ denotes the set of diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$. Notice that

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} + \sum_{i\neq j} \mathbb{C}E_{ij} \tag{1.1}$$

is an orthogonal sum of $\mathfrak{h}$ and $\sum_{i\neq j} \mathbb{C}E_{ij}$ ($E_{ij} \in \mathfrak{sl}_n(\mathbb{C})$ denotes the matrix with 1 at the (i, j) position and zero elsewhere) with respect to the nondegenerate symmetric bilinear form $B(X,Y) = \operatorname{tr} XY$ or the inner product $\langle X,Y\rangle = \operatorname{tr} X^*Y$. We will see that the orthogonal sum (1.1) will be replaced by the root space decomposition and the bilinear form will be replaced by the Killing form when we consider (noncompact) complex semisimple Lie algebras $\mathfrak{g}$. The point is that we still have orthogonal projection π when we consider $\mathfrak{g}$.

After introducing some preliminary material in Section 2, we will extend Tsing's result in the context of semisimple Lie algebras in Section 3. Theorem 3.1 is the main result of the section and it answers a conjecture of Tam [15] ([16, Conjecture 2.11]) affirmatively.

For $A, C \in \mathfrak{gl}_n(\mathbb{C})$, the C-numerical range of A [9, pp. 77–88] is defined to be the following subset of $\mathbb{C}$:

$$W_C(A) := \{\operatorname{tr} CU^*AU : U \in \mathrm{U}(n)\}.$$

Since $\mathfrak{gl}_n(\mathbb{C})$ and its dual are isomorphic via the inner product $\langle X,Y\rangle = \operatorname{tr} X^*Y$ on $\mathfrak{gl}_n(\mathbb{C})$, all linear functionals are of the form

$$f_C(\,\cdot\,) = \operatorname{tr} C(\,\cdot\,) \tag{1.2}$$

for some $C \in \mathfrak{gl}_n(\mathbb{C})$. So $W_C(A)$ is the image $f_C(O(A))$ and vice versa. The following result asserts that $W_C(A)$ is star-shaped.

Theorem 1.2 (Cheung and Tsing [4]) *If $C \in \mathfrak{gl}_n(\mathbb{C})$, $W_C(A)$ is star-shaped with respect to* $(\operatorname{tr} A)(\operatorname{tr} C)/n$.

Let V^* denote the dual space of the linear space V. The main idea of the proof of Cheung and Tsing [4] is to show that

$$\mathbf{S}(A) := \{B \in \mathfrak{gl}_n(\mathbb{C}) : f(O(B)) \subseteq f(O(A)) \text{ for all } f \in \mathfrak{gl}_n(\mathbb{C})^*\}$$

is star-shaped with respect to $\frac{\operatorname{tr} A}{n} I$.

The study of $W_C(A)$ can be reduced to $A, C \in \mathfrak{sl}_n(\mathbb{C})$; that is, A and C have zero trace, since

$$f_C(A) = f_{\hat{C}}(\hat{A}) + (\operatorname{tr} C)(\operatorname{tr} A)/n,$$

where $\hat{A} = A - \frac{\operatorname{tr} A}{n} I_n$ and $\hat{C} = C - \frac{\operatorname{tr} C}{n} I_n$, so that

$$f_C(O(A)) = f_{\hat{C}}(O(\hat{A})) + (\operatorname{tr} C)(\operatorname{tr} A)/n.$$

The notion of C-numerical range is extended in the context of (noncompact) complex semisimple Lie algebra $\mathfrak{g}$ [5]. In Section 4, namely in Theorem 4.8, we show that if $\mathfrak{g}$ is of type B or D, then the star-shapedness result is valid. It provides more support for a conjecture of Tam [15] (see [16, Conjecture 2.10]).

2 Preliminaries

Let $\mathfrak{g}$ be a (noncompact) complex semisimple Lie algebra and let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}$ [6, p. 181]. Let G be a connected complex Lie group with Lie algebra $\mathfrak{g}$. It has a finite center [12, p. 375] so that K (the analytic group of $\mathfrak{k}$) is compact [6, p. 253]. As a real K-module, $\mathfrak{g}$ is just the direct sum of two copies of the adjoint module $\mathfrak{k}$ of K: $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$, *i.e.,* Cartan decomposition of $\mathfrak{g}$ [6, p. 185]. Denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Given $x \in \mathfrak{g}$, consider the orbit of x under the adjoint action of K

$$K \cdot x := \{\mathrm{Ad}(k)x : k \in K\}.$$

We will write $k \cdot x$ for $\mathrm{Ad}(k)x$. The orbit $K \cdot x$ depends on $\mathrm{Ad}_G K$, which is the analytic subgroup of the adjoint group [6, p. 126] $\mathrm{Int}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g})$ corresponding to $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Thus $K \cdot x$ is independent of the choice of G. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. The complexification $\mathfrak{h} := \mathfrak{t} + i\mathfrak{t}$ (direct sum) is a Cartan subalgebra of $\mathfrak{g}$ [6, p. 162]. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ (direct sum) be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ [6, p. 162], where Δ denotes the set of all roots. Denote by $B(\cdot\,,\cdot)$ the Killing form of $\mathfrak{g}$ [6, p. 131]. Notice that $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ [6, p. 166] whenever $\alpha + \beta \neq 0$ ($\mathfrak{g}_0 = \mathfrak{h}$) so that

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta^+} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

is an orthogonal sum with respect to the Killing form. Thus we have the orthogonal projection $\pi\colon \mathfrak{g} \to \mathfrak{h}$ under $B(\cdot\,,\cdot)$. For $x \in \mathfrak{g}$, we consider $\pi(K \cdot x)$, *i.e.,* the projection of $K \cdot x$ onto $\mathfrak{h}$. When $x \in \mathfrak{k}$, $K \cdot x \subseteq \mathfrak{k}$ so that $\pi(K \cdot x) \subseteq \mathfrak{t}$.

Let θ be the Cartan involution of $\mathfrak{g}$ if $\mathfrak{g}$ is viewed as a real Lie algebra, *i.e.,* $\theta\colon \mathfrak{g} \to \mathfrak{g}$ such that $x + y \mapsto x - y$ if $x \in \mathfrak{k}$ and $y \in i\mathfrak{k}$. In other words, $\mathfrak{k}$ is the $+1$ eigenspace of θ and $i\mathfrak{k}$ is the -1 eigenspace of θ.

3 Projection of K-Orbit onto Cartan Subalgebra

The main result in this section is Theorem 3.1, conjectured by Tam [15] (see [16, Conjecture 2.11]).

Theorem 3.1 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\pi\colon \mathfrak{g} \to \mathfrak{h}$ be the orthogonal projection onto the Cartan subalgebra $\mathfrak{h}$ with respect to the Killing form of $\mathfrak{g}$. If $x \in \mathfrak{g}$, then $\pi(K \cdot x) \subseteq \mathfrak{h}$ is star-shaped with star center 0.*

When $x \in \mathfrak{k}$, the projection $\pi(K \cdot x) \subseteq \mathfrak{t}$ is indeed equal to $\mathrm{conv}\, W x_{\mathfrak{t}}$, where $x_{\mathfrak{t}} \in K \cdot x \cap \mathfrak{t}$ and W is the Weyl group, *i.e.,* $W = N(T)/T$, a result due to Kostant [11]. It extends the Schur–Horn result.

The following lemma enables us to pick any model of $\mathfrak{g}$ to work with in order to show the star-shapedness of $\pi(K \cdot x)$, $x \in \mathfrak{g}$.

Lemma 3.2 *Suppose $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' + i\mathfrak{k}'$ (Cartan decompositions) are isomorphic complex semisimple Lie algebras. Let $\mathfrak{t}$ and $\mathfrak{t}'$ be maximal abelian subalgebras of $\mathfrak{k}$ and $\mathfrak{k}'$, respectively. Set $\mathfrak{h} := \mathfrak{t} + i\mathfrak{t}$ and $\mathfrak{h}' := \mathfrak{t}' + i\mathfrak{t}'$. Let $\pi\colon \mathfrak{g} \to \mathfrak{h}$ and $\pi'\colon \mathfrak{g}' \to \mathfrak{h}'$*

be the orthogonal projections with respect to the Killing forms of $\mathfrak{g}$ *and* $\mathfrak{g}'$ *respectively. If* $x \in \mathfrak{g}$, *then there is an isomorphism* $\psi \colon \mathfrak{g} \to \mathfrak{g}'$ *such that* $\psi(\mathfrak{t}) = \mathfrak{t}'$ *and*

$$\pi'(K' \cdot \psi(x)) = \psi(\pi(K \cdot x)),$$

where K *and* K' *are the analytic groups corresponding to* $\mathfrak{k}$ *and* $\mathfrak{k}'$ *respectively.*

Proof Notice that $\mathfrak{k}$ and $\mathfrak{k}'$ are compact real forms of $\mathfrak{g}$ and $\mathfrak{g}'$. For any isomorphism $\gamma \colon \mathfrak{g} \to \mathfrak{g}'$, $\mathfrak{g}' = \gamma(\mathfrak{k}) + i\gamma(\mathfrak{k})$ is a Cartan decomposition for $\mathfrak{g}'$ and $\gamma(\mathfrak{k})$ is compact. Hence there is $\sigma \in \operatorname{Int}(\mathfrak{g}')$ so that $\sigma(\gamma(\mathfrak{k})) = \mathfrak{k}'$ [6, p. 183]. Clearly $\sigma(\gamma(\mathfrak{t}))$ is a maximal abelian subalgebra of $\mathfrak{k}'$. The maximal abelian subalgebras of the compact $\mathfrak{k}'$ are conjugate via $\operatorname{Ad}(k')$ for some $k' \in K'$ [6, p. 248]. So we have an isomorphism

$$\psi := \operatorname{Ad}(k') \circ \sigma \circ \gamma \colon \mathfrak{g} \to \mathfrak{g}'$$

such that $\psi(\mathfrak{t}) = \mathfrak{t}'$. Notice that $\psi(\mathfrak{k}) = \mathfrak{k}'$, since $\sigma(\gamma(\mathfrak{k})) = \mathfrak{k}'$ and $\operatorname{Ad}(k') \in \operatorname{Aut}(\mathfrak{k}')$. Clearly $\psi(\mathfrak{h}) = \mathfrak{h}'$. Since $\psi \colon \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism, $\operatorname{ad}(\psi(X)) = \psi \circ \operatorname{ad} X \circ \psi^{-1}$ for all $X \in \mathfrak{g}$ so that the Killing forms [6, p. 131] $B(\cdot\,,\cdot)$ of $\mathfrak{g}$ and $B'(\cdot\,,\cdot)$ of $\mathfrak{g}'$ are related by $B(x, y) = B'(\psi(x), \psi(y))$ for all $x, y \in \mathfrak{g}$. So $\psi(\mathfrak{h}^\perp) = \mathfrak{h}'^\perp$. Thus

$$\psi \circ \pi = \pi' \circ \psi. \tag{3.1}$$

Since $K \cdot x$ is independent of the choice of G (the analytic group of $\mathfrak{g}$), we may assume that G is simply connected [18, p. 101]. To the isomorphism $\psi \colon \mathfrak{g} \to \mathfrak{g}'$ there exists a unique isomorphism $\varphi \colon G \to G'$ (G' is the analytic group of $\mathfrak{g}'$) such that $d\varphi_e = \psi$ [18, p. 101]. Since $\psi(\mathfrak{k}) = \mathfrak{k}'$, we have $\varphi(K) = K'$. Now for all $t \in \mathbb{R}$, $k \in K$, $x \in \mathfrak{g}$ using [6, Lemma 1.12, p. 110] and [6, p. 127]

$$e^{t d\varphi_e(\operatorname{Ad}(k)x)} = \varphi(e^{t\operatorname{Ad}(k)x}) = \varphi(k)\varphi(e^{tx})\varphi(k)^{-1} = e^{t\operatorname{Ad}(\varphi(k))d\varphi_e(x)}.$$

Taking derivative yields $\psi(\operatorname{Ad}(k)x) = \operatorname{Ad}(\varphi(k))\psi(x)$, so that

$$\psi(K \cdot x) = K' \cdot \psi(x). \tag{3.2}$$

By (3.1) and (3.2)

$$\pi'(K' \cdot \psi(x)) = \pi'(\psi(K \cdot x)) = \psi(\pi(K \cdot x)). \qquad \blacksquare$$

By Lemma 3.2, Theorem 1.1 can be stated as follows.

Theorem 3.3 *Theorem* 3.1 *is true for simple* $\mathfrak{g}$ *of type* A.

A connected Lie group is called *almost simple* [3, p. 355] if its Lie algebra is simple, and the quotient of a direct product of Lie groups by a discrete central subgroup is called an *almost direct product*. Corresponding to the Cartan involution $\theta \colon \mathfrak{g} \to \mathfrak{g}$, there is [6, p. 253] an involutive analytic automorphism $\Theta \colon G \to G$ (called the global Cartan involution [12, p. 305] and many authors write θ for Θ) such that $d\Theta_e = \theta$. A subgroup $S \subseteq G$ is said to be θ-stable if S is stable under Θ.

The following lemma enables us to deduce Theorem 3.1 from its validity for simple Lie algebras of type A.

Lemma 3.4 (Đoković and Tam [5]) *Let $H \subseteq G$ be the analytic subgroup corresponding to $\mathfrak{h}$. There exists a closed connected θ-stable complex semisimple Lie subgroup S of G containing H and S is an almost direct product of θ-stable almost simple subgroups S_i ($i = 1, \dots, m$) of type A.*

Proof of Theorem 3.1 Let $S = S_1 S_2 \cdots S_m$ be as in Lemma 3.4. The Lie algebra $\mathfrak{s}$ of S is a direct sum of its simple ideals $\mathfrak{s}_i$, where $\mathfrak{s}_i$ is the Lie algebra of S_i, $i = 1, \dots, m$ [10, p. 22]. Since Cartan subalgebras of the semisimple $\mathfrak{g}$ are precisely the nilpotent subalgebra that equals its normalizer in $\mathfrak{g}$ [10, p. 80] and $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{g}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}$. Likewise $\mathfrak{h}_i = \mathfrak{s}_i \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}_i$ and $\mathfrak{h} = \sum_{i=1}^m \mathfrak{h}_i$ (direct sum). Denote by $\mathfrak{q}$ the sum of the root spaces $\mathfrak{g}_\alpha$ that are not contained in $\mathfrak{s}$. Then $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{q}$, $\mathfrak{q} = \mathfrak{s}^\perp$, and $\mathfrak{q}$ is S-stable, *i.e.*, $\mathfrak{q}$ is stable under the adjoint action of S (as the Killing form is invariant under the adjoint action of S [6, p. 131]). The subgroup $K_i := K \cap S_i$ is a maximal compact subgroup of S_i. Denote by $\pi_i \colon \mathfrak{s}_i \to \mathfrak{h}_i$ the orthogonal projection, $i = 1, \dots, m$.

Each $x \in \mathfrak{g}$ can be decomposed uniquely as $x = \sum_{i=1}^m x_i + x'$, where $x_i \in \mathfrak{s}_i$ and $x' \in \mathfrak{q}$. Since each S_i is of type A, Theorem 3.3 implies that for each $0 \le \lambda \le 1$, there exists $k_i \in K_i \subseteq K \cap S$ such that

$$\lambda \pi_i(x_i) = \pi_i(k_i \cdot x_i) \in \pi_i(K_i \cdot x_i).$$

Set $k := k_1 k_2 \cdots k_m \in K \cap S$. Since $\mathfrak{q}$ is S-stable, we have $k \cdot x' \in \mathfrak{q}$. So

$$\pi(k \cdot x) = \pi\Big(\sum_{i=1}^m k_i \cdot x_i + k \cdot x'\Big) = \pi\Big(\sum_{i=1}^m k_i \cdot x_i\Big) = \sum_{i=1}^m \pi_i(k_i \cdot x_i) = \lambda \pi(x).$$

Hence $\lambda \pi(x) \in \pi(K \cdot x)$. ∎

Example 3.5 Let $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$ ($n \ge 2$), $K = \mathrm{SO}(n)$, $\mathfrak{h} = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \mathbb{C}(E_{2i-1,2i} - E_{2i,2i-1})$ [6, pp. 187–189]. So the projection $\pi \colon \mathfrak{g} \to \mathfrak{h}$ is given by

$$\pi(A) = \begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & a_{34} \\ -a_{34} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & a_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor} \\ -a_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor} & 0 \end{pmatrix}$$

and may be identified with

$$\pi(A) = \left(a_{12}, a_{34}, \dots, a_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor}\right).$$

Then the set $\{\pi(OAO^t) : O \in \mathrm{SO}(n)\}$ is star-shaped with respect to $0 \in \mathbb{C}^{\lfloor \frac{n}{2} \rfloor}$ by Theorem 3.1.

The following is a matrix approach to Example 3.5. Notice that $\mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) + \mathfrak{sl}_2(\mathbb{C})$ [6, p. 465]. So a star-shapedness result holds for $\mathfrak{so}_3(\mathbb{C})$ and $\mathfrak{so}_4(\mathbb{C})$ by Theorem 3.3 and Lemma 3.2. Hence for each $A \in \mathfrak{so}_4(\mathbb{C})$,

$$\pi(\mathrm{SO}(4) \cdot A) = \{\pi(OAO^t) : O \in \mathrm{SO}(4)\}$$

is star-shaped with respect to $0 \in \mathbb{C}^2$.

We now identify $\mathfrak{h}$ with $\mathbb{C}^{\lfloor \frac{n}{2} \rfloor}$ naturally. Suppose that $n \geq 5$ and $A \in \mathfrak{so}_n(\mathbb{C})$. If $(x_{12}, x_{34}, \dots, x_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor}) \in \pi(\mathrm{SO}(n) \cdot A)$, we want to show that for each $0 \leq \lambda \leq 1$,

$$\lambda\big(x_{12}, x_{34}, \dots, x_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor}\big) \in \pi\big(\mathrm{SO}(n) \cdot A\big).$$

We may assume that

$$(x_{12}, x_{34}, \dots, x_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor}) = (a_{12}, a_{34}, \dots, a_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor}).$$

Now consider $A[2i-1, 2i \mid 2j-1, 2j]$, which denotes the 4×4 submatrix of A lying in rows and columns indexed by $2i-1, 2i, 2j-1, 2j$ ($1 \leq 2i < 2j \leq n$ if n is even, and $1 \leq 2i < 2j < n$ if n is odd). Let $0 \leq \xi \leq 1$. For any two admissible $i < j$, there exists $O \in \mathrm{SO}(n)$ with

$$O[2i-1, 2i \mid 2j-1, 2j] \in \mathrm{SO}(4), \quad O(2i-1, 2i \mid 2j-1, 2j) = I_{n-4},$$

where $O(2i-1, 2i \mid 2j-1, 2j)$ denotes the $(n-4) \times (n-4)$ submatrix of O complementary to $O[2i-1, 2i|2j-1, 2j]$, so that

$$\begin{aligned}(a_{12}, \dots, \xi a_{2i-1,2i}, \dots, \xi a_{2j-1,2j}, \dots, a_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor}) &= \pi(OAO^t) \\ &\in \pi(\mathrm{SO}(n) \cdot A).\end{aligned}$$

By choosing such a matrix $O \in \mathrm{SO}(n)$ for every admissible pair $i < j$ and multiplying these matrices we get a matrix $\hat{O} \in \mathrm{SO}(n)$ such that

$$\xi^{\lfloor \frac{n}{2} \rfloor -1}\big(a_{12}, a_{34}, \dots, a_{2\lfloor \frac{n}{2} \rfloor -1, 2\lfloor \frac{n}{2} \rfloor}\big) = \pi(\hat{O}A\hat{O}^t) \in \pi\big(\mathrm{SO}(n) \cdot A\big).$$

So for each $0 \leq \lambda \leq 1$, set ξ such that $\xi^{\lfloor \frac{n}{2} \rfloor -1} = \lambda$. Hence $\pi(\mathrm{SO}(n) \cdot A)$ is star-shaped.

Example 3.6 Let [6, pp. 189–190]

$$\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}) := \{X = \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^t \end{pmatrix} : X_1, X_2, X_3 \in \mathfrak{gl}_n(\mathbb{C}), X_2, X_3 \text{ symmetric}\},$$

and $\mathfrak{k} = \mathfrak{sp}(n)$, *i.e.*,

$$K = \mathrm{Sp}(n) := \mathrm{Sp}_n(\mathbb{C}) \cap U(2n) = \{g = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} : A, B \in \mathfrak{gl}_n(\mathbb{C})\} \cap U(2n).$$

By [6, p. 189] we may take

$$\mathfrak{h} = \{\mathrm{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n) : x_1, \dots, x_n \in \mathbb{C}\}.$$

So $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is simply taking diagonal part. By Theorem 3.1, given $A \in \mathfrak{sp}_n(\mathbb{C})$, the set

$$\pi(\mathrm{Sp}(n) \cdot A) = \{\pi(UAU^{-1}) : U \in \mathrm{Sp}(n)\}$$

is star-shaped with respect to $0 \in \mathfrak{h}$.

The following is a matrix approach to Example 3.6. Since $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sp}_1(\mathbb{C})$ [6, p. 465], a star-shapedness result is true for $n = 1$ case. By considering the 2×2 submatrix $A[i, n+i] \in \mathfrak{sp}_n(\mathbb{C})$ of A, $i = 1, \ldots, n$, where $U \in \mathrm{Sp}(n)$ and $U[i, n+i] \in \mathrm{Sp}(1)$ and $U(i, n+i) = I_{2n-2}$, for each $0 \le \lambda \le 1$,

$$(x_1, \ldots, \lambda x_i, \ldots, x_n, -x_1, \ldots, -\lambda x_i, \ldots, -x_n) \in \pi(\mathrm{Sp}(n) \cdot A)$$

if $(x_1, \ldots, x_n, -x_1, \ldots, -x_n) \in \pi(\mathrm{Sp}(n) \cdot A)$. So

$$\lambda(x_1, \ldots, x_n, -x_1, \ldots, -x_n) \in \pi(\mathrm{Sp}(n) \cdot A)$$

as i runs through $1, \ldots, n$.

4 Linear Functional on K-Orbit

In [5, 15] the notion of C-numerical range is extended to (noncompact) complex semisimple Lie algebras $\mathfrak{g}$. Let $\mathfrak{g}^*$ denote the dual of $\mathfrak{g}$. For any $f \in \mathfrak{g}^*$, $x \in \mathfrak{g}$, consider the range $f(K \cdot x)$. When $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ with $K = \mathrm{SU}(n)$, the range $f(K \cdot x)$ is reduced to $f_C(O(A))$, where f_C is given in (1.2).

Motivated by the approach of Cheung and Tsing in [4], we introduce the set

$$\mathbf{S}(x) := \{z \in \mathfrak{g} : f(K \cdot z) \subseteq f(K \cdot x)\} \subseteq \mathfrak{g}.$$

Obviously $\mathbf{S}(x)$ is K-invariant and

$$K \cdot x \subseteq \mathbf{S}(x) \subseteq \operatorname{conv} K \cdot x \subseteq \operatorname{conv} \mathbf{S}(x)$$

so that

$$\operatorname{conv} K \cdot x = \operatorname{conv} \mathbf{S}(x).$$

For $x, y \in \mathfrak{g}$, we write $x \le y$ if $x \in \mathbf{S}(y)$, or equivalently $f(x) \in f(K \cdot y)$ for all $f \in \mathfrak{g}^*$ [5]. The relation $\le$ defines a partial order on $\mathfrak{g}$ and extends the partial order of Cheung and Tsing [4] for $\mathfrak{g}$. The partial order depends on the choice of K and is strongly K-invariant in the sense that $x \le y$ implies that $a \cdot x \le b \cdot y$ for $a, b \in K$, and so it induces a partial order on the orbit space $\mathfrak{g}/K$ [5].

In general $K \cdot x \ne \mathbf{S}(x)$, for example, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $K = \mathrm{SU}(2)$,

$$x = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \in \mathfrak{su}(2), \quad a, b, c \in \mathbb{R}.$$

Then

$$K \cdot x = \left\{ \begin{pmatrix} z_1 & z_2 + iz_3 \\ z_2 - iz_3 & -z_1 \end{pmatrix} : z_1, z_2, z_3 \in \mathbb{R}^2,\ z_1^2 + z_2^2 + z_3^2 = a^2 + b^2 + c^2 \right\},$$

which is viewed as the sphere in $\mathbb{R}^3$ centered at the origin and of radius $r = (a^2 + b^2 + c^2)^{1/2}$, but $\mathbf{S}(x) = \operatorname{conv} K \cdot x$ is the ball that is clearly the convex hull of the sphere.

The following proposition enables us to work with $\mathbf{S}(x)$ if we want show that $f(K \cdot x)$ is star-shaped for all $x \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$, namely Conjecture 4.2.

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Proposition 4.1 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. If $x \in \mathfrak{g}$, then $\mathbf{S}(x)$ is star-shaped with respect to the zero matrix 0 if and only if $f(K \cdot x)$ is star-shaped with respect to the origin in $\mathbb{C}$ for all $f \in \mathfrak{g}^*$.*

Proof Notice that $f(K \cdot x) = f(\mathbf{S}(x))$ by the definition of $\mathbf{S}(x)$, since $x \in \mathbf{S}(x)$ for $f \in \mathfrak{g}^*$. So the star-shapedness of $\mathbf{S}(x)$ implies the star-shapedness of $f(K \cdot x)$.

Conversely, suppose that $f(K \cdot x)$ is star-shaped with respect to 0 for all $f \in \mathfrak{g}^*$. If $y \in \mathbf{S}(x)$, then for all $0 \le \alpha \le 1$,

$$f(K \cdot \alpha y) = \alpha f(K \cdot y) \subseteq \alpha f(K \cdot x) \subseteq f(K \cdot x). \qquad \blacksquare$$

The following is a possible extension of Theorem 1.2.

Conjecture 4.2 (Tam [15]) *For $x \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$, the set $f(K \cdot x)$ is star-shaped with respect to the origin, or equivalently, for $x \in \mathfrak{g}$ and $t \in [0,1]$, $tx \le x$ holds.*

It is known ([4, 5]) that if the simple components of $\mathfrak{g}$ are of type A, D, E_6, or E_7, then Conjecture 4.2 is valid (see [16, Conjecture 2.10]). So, among the four classical complex simple Lie algebras, the unknown cases are types B and C. We will prove that the conjecture is true for type B. Indeed our approach works for type D as well.

The following lemma allows us to work with any model of $\mathfrak{g}$, similar to Lemma 3.2.

Lemma 4.3 *Suppose $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' + i\mathfrak{k}'$ (Cartan decompositions) are isomorphic complex semisimple Lie algebras. If $x, y \in \mathfrak{g}$, then there is an isomorphism $\psi\colon \mathfrak{g} \to \mathfrak{g}'$ such that $\psi(\mathfrak{k}) = \mathfrak{k}'$ and*

$$B'_{\theta'}(K' \cdot \psi(x), \psi(y)) = B_\theta(K \cdot x, y).$$

Proof As in Lemma 3.2 we have (3.2), *i.e.*, $\psi(K \cdot x) = K' \cdot \psi(x)$. Since $\operatorname{ad}(\psi(x)) = \psi \circ \operatorname{ad} x \circ \psi^{-1}$,

$$B'(\psi(x), \psi(y)) = B(x, y), \quad x, y \in \mathfrak{g}$$

is the Killing form of $\mathfrak{g}'$. Moreover, $\theta' = \psi \circ \theta \circ \psi^{-1}$. Thus

$$B'_{\theta'}(\psi(x), \psi(y)) = -B'(\psi(x), \theta' \circ \psi(y)) = -B(x, \theta y) = B_\theta(x, y). \tag{4.1}$$

By (3.2) and (4.1),

$$B'_{\theta'}\big(K' \cdot \psi(x), \psi(y)\big) = B'_{\theta'}\big(\psi(K \cdot x), \psi(y)\big) = B_\theta(K \cdot x, y). \qquad \blacksquare$$

By Lemma 4.3, in order to prove Conjecture 4.2 for simple complex Lie algebra $\mathfrak{g}$ of type B or D, we can choose $\mathfrak{so}_n(\mathbb{C})$, the algebra of $n \times n$ complex skew symmetric matrices as the model and set $K = \mathrm{SO}(n)$. Given $A, C \in \mathfrak{so}_n(\mathbb{C})$, in the forthcoming discussion define

$$O(A) := \{O^t A O : O \in \mathrm{SO}(n)\}$$
$$f_C(O(A)) := \{\operatorname{tr} C O^t A O : O \in \mathrm{SO}(n)\},$$

and we want to show that it is star-shaped with respect to the origin. When $n = 2$, $f_C(O(A))$ is a singleton set, that is, if

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix},$$

then $f_C(O(A)) = \{-2ac\}$. Notice that $\mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) + \mathfrak{sl}_2(\mathbb{C})$ and the corresponding $f_C(O(A))$ are star-shaped with respect to 0 by Lemma 4.3 and Theorem 1.2. So it suffices to consider the simple Lie algebra $\mathfrak{so}_n(\mathbb{C})$, $n \geq 5$. Nevertheless we will consider $n \geq 3$.

The tool we use is an analog of $\mathbf{S}(A)$ of Cheung and Tsing:

$$\mathbf{S}_{SO(n)}(A) := \{B \in \mathfrak{so}_n(\mathbb{C}) : f_C(O(B)) \subseteq f_C(O(A)) \text{ for all } C \in \mathfrak{so}_n(\mathbb{C})\}.$$

We remark that $\mathbf{S}_{SO(n)}(A)$ is invariant under special orthogonal similarity, that is, $B \in \mathbf{S}_{SO(n)}(A)$ if and only if $O^t BO \in \mathbf{S}_{SO(n)}(A)$ for each $O \in \mathrm{SO}(n)$.

The following lemma can be readily verified.

Lemma 4.4 *Let $X, Y \in \mathfrak{so}_n(\mathbb{C})$ ($n \geq 3$) be in the partitioned forms*

$$X = \begin{pmatrix} 0 & x_{12} & X_{13} \\ -x_{12} & 0 & X_{23} \\ -X_{13}^t & -X_{23}^t & X_{33} \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & y_{12} & Y_{13} \\ -y_{12} & 0 & Y_{23} \\ -Y_{13}^t & -Y_{23}^t & Y_{33} \end{pmatrix},$$

where $x_{12}, y_{12} \in \mathbb{C}$, $X_{33}, Y_{33} \in \mathfrak{so}_{n-2}(\mathbb{C})$. Let

$$R_2(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}.$$

Then

$$\begin{aligned} \operatorname{tr} XR_2^t(\theta)YR_2(\theta) &= -2x_{12}y_{12} + \operatorname{tr} X_{33}Y_{33} - 2\cos\theta\ \operatorname{tr}(X_{13}^tY_{13} + X_{23}^tY_{23}) \\ &\qquad + 2\sin\theta\ \operatorname{tr}(X_{23}^tY_{13} - X_{13}^tY_{23}) \\ &\in f_X(O(Y)), \end{aligned}$$

the locus of which, when θ runs from 0 to 2π, is an ellipse centered at $-2x_{12}y_{12}+\operatorname{tr} X_{33}Y_{33}$ with length of major axis determined by $\operatorname{tr}(X_{13}^tY_{13} + X_{23}^tY_{23})$ and $\operatorname{tr}(X_{23}^tY_{13} - X_{13}^tY_{23})$.

Moreover, if $X_{13}, X_{23}, Y_{13}, Y_{23}$ are real matrices, then the ellipse is degenerate (a point or a line segment).

Remark 4.5 If the row vectors Y_{13} and Y_{23} are all multiplied by a scalar $0 \leq \epsilon \leq 1$, the resulting ellipse, in particular the point $\operatorname{tr} XY(\epsilon)$ (see the notation $B(\epsilon)$ in Lemma 4.7 with $k = 1$ and $\ell = 2$), lies within the relative interior of the original ellipse.

Lemma 4.6 *Let $X \in \mathfrak{so}_n(\mathbb{C})$ $(n \geq 3)$ and $1 \leq k < \ell \leq n$. Then there exists $O \in \mathrm{SO}(n)$ such that the entries of O^tXO on the k-th and ℓ-th rows and columns are real, except $(O^tXO)_{k\ell}$ and $(O^tXO)_{\ell k}$.*

Proof Without loss of generality one may assume that $k = 1$ and $\ell = 2$. Since $X^t = -X$, we can write $X = X_1 + iX_2$ for real skew symmetric matrices X_1, X_2. By the well-known theorem [8, p. 107] on the normal form of real skew-symmetric matrices under orthogonal similarity, we find an $O \in \mathrm{SO}(n)$ such that

$$O^tX_2O = \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & x_{\lfloor \frac{n}{2} \rfloor} \\ -x_{\lfloor \frac{n}{2} \rfloor} & 0 \end{pmatrix} \oplus (0),$$

where the last 1×1 zero block is present if n is odd, and $x_1, \dots, x_{\lfloor \frac{n}{2} \rfloor} \in \mathbb{R}$. So O^tXO has the desired form. ■

Lemma 4.7 *Suppose $B = (b_{ij}) \in \mathbf{S}_{SO(n)}(A)$. Let $1 \leq k < \ell \leq n$, $0 \leq \epsilon \leq 1$, and $B(\epsilon) = (b_{ij}(\epsilon))$ be defined by*

$$b_{ij}(\epsilon) = \begin{cases} \epsilon b_{ij}, & \text{if exactly one of } i, j \text{ equals } k \text{ or } \ell, \\ b_{ij}, & \text{otherwise,} \end{cases}$$

that is, $B(\epsilon) \in \mathfrak{so}_n(\mathbb{C})$ is obtained from B by multiplying its entries on the k-th and the ℓ-th rows and columns by ϵ, except for its (k, ℓ)-th and (ℓ, k)-th entries. Then $B(\epsilon) \in \mathbf{S}_{SO(n)}(A)$.

Proof We assume without loss of generality that $k = 1$ and $\ell = 2$. Suppose $B = (b_{ij}) \in \mathbf{S}_{SO(n)}(A)$. Then, by definition, for all $U, V \in \mathrm{SO}(n)$, $C \in \mathfrak{so}_n(\mathbb{C})$, and $\theta \in \mathbb{R}$,

$$\begin{aligned} \xi(U, V, \theta) &:= \operatorname{tr} U^tCUR_2^t(\theta)V^tBVR_2(\theta) \\ &\in f_C(O(B)) \subseteq f_C(O(A)), \end{aligned} \tag{4.2}$$

where $R_2(\theta)$ is given in Lemma 4.4.

Now choose $U \in \mathrm{SO}(n)$ arbitrarily. Partition $B, C \in \mathfrak{so}_n(\mathbb{C})$ into the block form of Lemma 4.4, that is, according to the partition $\{1\}, \{2\}, \{3, \dots, n\}$. Write B and C in the form $X_1 + iX_2$, where X_1, X_2 are real skew symmetric matrices. By Lemma 4.6 there exist $U_1, V_1 \in \mathrm{SO}(n)$ such that $U_1^tCU_1$ and $V_1^tBV_1$ have real $(1, 3)$ and $(2, 3)$ blocks ($(3, 1)$ and $(3, 2)$ blocks as well). Since $\mathrm{SO}(n)$ is path connected, we can choose two continuous functions $U(\cdot), V(\cdot)\colon [0, 1] \to \mathrm{SO}(n)$, such that

$$U(0) = U, \quad V(0) = I, \quad U(1) = U_1, \quad V(1) = V_1.$$

Hence $U(1)^tCU(1)$ and $V(1)^tBV(1)$ have real $(1, 3)$ and $(2, 3)$ blocks. By Lemma 4.4 and (4.2), for each $t \in [0, 1]$,

$$\mathbf{E}(t) := \{\xi(U(t), V(t), \theta) : \theta \in [0, 2\pi]\} \subseteq f_C(O(B)) \subseteq f_C(O(A)) \tag{4.3}$$

is an ellipse. Since both $U(t)$ and $V(t)$ are continuous, $\mathbf{E}(0)$ deforms continuously to $\mathbf{E}(1)$ as t runs from 0 to 1. By Lemma 4.3, the ellipse $\mathbf{E}(1)$ degenerates into a point or a line segment. Let $\zeta \in \mathbb{C}$ be any point in the interior of $\mathbf{E}(0)$. If $\zeta \in \mathbf{E}(1)$, then $\zeta \in f_C(O(B)) \subseteq f_C(O(A))$ by (4.3). If $\zeta \notin \mathbf{E}(1)$, then ζ must be swept across by some ellipse $\mathbf{E}(t)$, when $\mathbf{E}(0)$ deforms to the degenerated ellipse $\mathbf{E}(1)$ as t runs from 0 to 1, *i.e.*, $\zeta \in \mathbf{E}(t)$. Hence $\mathbf{E}(0)$ and its interior are contained in $f_C(O(B)) \subseteq f_C(O(A))$.

By Remark 4.5, the point $\operatorname{tr} U^tCUB(\epsilon) \in f_C(O(B(\epsilon)))$ is in the interior of the ellipse $\mathbf{E}(0)$ and hence is contained in $f_C(O(A))$. As this is true for any $U \in \mathrm{SO}(n)$, we conclude that $f_C(O(B(\epsilon))) \subseteq f_C(O(A))$ for all $C \in \mathfrak{so}_n(\mathbb{C})$, *i.e.*, $B(\epsilon) \in \mathbf{S}_{SO(n)}(A)$. ■

The $n = 2m$ case, *i.e.*, type D, of the following theorem is known [5].

Theorem 4.8 *If $\mathfrak{g}$ is a simple complex Lie algebra of type B or D, and $f \in \mathfrak{g}^*$, then $f(K \cdot x) \subseteq \mathbb{C}$ is star-shaped with respect to the origin. Equivalently, $\mathbf{S}(x)$ is star-shaped with respect to $0 \in \mathfrak{g}$. Hence, if $n \geq 3$ and $A, C \in \mathfrak{so}_n(\mathbb{C})$, then*

(i) *the set $\mathbf{S}_{SO(n)}(A)$ is star-shaped with respect to the zero matrix;*
(ii) *the set $f_C(O(A)) \subseteq \mathbb{C}$ is star-shaped with respect to the origin.*

Proof The equivalence follows from Proposition 4.1. By Lemma 4.3 we can choose any model to work with so that it suffices to show the first statement.

(i) Suppose $B = (b_{ij}) \in \mathbf{S}_{SO(n)}(A)$ and $\alpha \in [0, 1]$. Let $\epsilon \in [0, 1]$ be such that $\epsilon^{n-1} = \alpha$. Applying Lemma 4.7 repeatedly on B, with

$$(k, \ell) = (1, 2), (1, 3), \ldots, (1, n), (2, 3), (2, 4), \ldots, (2, n), (3, 4), \ldots, (n - 1, n),$$

we obtain $\alpha B \in \mathbf{S}_{SO(n)}(A)$ so that $\mathbf{S}_{SO(n)}(A)$ is star-shaped with respect to the zero matrix. ■

Corollary 4.9 *Let $n \geq 3$. For any $A, C \in \mathfrak{so}_n(\mathbb{C})$, the sets*

$$V_C(A) := \{\operatorname{tr} CO^tAO : O \in \mathrm{O}(n)\},$$
$$V_C^-(A) := \{\operatorname{tr} CO^tAO : O \in \mathrm{O}(n) \setminus \mathrm{SO}(n)\},$$

are star-shaped with respect to the origin. Equivalently, the sets

$$\mathbf{S_0}(A) := \{B \in \mathfrak{so}_n(\mathbb{C}) : V_C(B) \subseteq V_C(A) \text{ for all } C \in \mathfrak{so}_n(\mathbb{C})\},$$
$$\mathbf{S}^-(A) := \{B \in \mathfrak{so}_n(\mathbb{C}) : V_C^-(B) \subseteq V_C^-(A) \text{ for all } C \in \mathfrak{so}_n(\mathbb{C})\}$$

are star-shaped with respect to the zero matrix.

Proof Suppose $n \geq 3$. Note that $V_C(A) = f_C(O(A)) \cup V_C^-(A)$ and

$$V_C^-(A) = f_C((-1 \oplus I_{n-1})A(-1 \oplus I_{n-1})).$$

So the results follow from Theorem 4.8. ■

When $n = 2$, $V_C(A) = \{\pm 2ac\}$ and $V_C^-(A) = \{2ac\}$, where

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}.$$

Finally we remark that Conjecture 4.2 can be reduced to the simple cases. To summarize, the known cases are simple Lie algebras of type A [4], B (Theorem 4.8), D, E_6, and E_7 [5]; the unknown simple Lie algebras are of type C, G_2, F_4, and E_8.

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