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The Commutant of an Abstract Backward Shift

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Abstract. A bounded linear operator T on a Banach space X is an abstract backward shift if the nullspace of T is one dimensional, and the union of the null spaces of T^k for all $k \geq 1$ is dense in X. In this paper it is shown that the commutant of an abstract backward shift is an integral domain. This result is used to derive properties of operators in the commutant.

Introduction

Let X be a linear space, and $T\colon X \to X$, a linear operator on X. We use the notation $N(T)$ and $R(T)$ to denote the null space of T and the range of T, respectively. Also, let $N_\infty(T) = \bigcup_{n=1}^{\infty} N(T^n)$. When X is a Banach space, $B(X)$ denotes the algebra of all the bounded linear operators on X.

The theory of weighted shifts and weighted backward shifts is an important part of operator theory and some other areas; see [1], [2], [3], [4], [5], [7] and [8] for example. These operators are usually defined on Hilbert spaces or Banach spaces which have a Schauder basis. J. Holub in his paper [5] introduced "basis free" concepts of shift and backward shift. His definitions make sense when the underlying space is an arbitrary separable Banach space. Two of the requirements made by Holub in his definition of a backward shift on an infinite dimensional Banach space X are:

(1) $\dim\big(N(T)\big) = 1$;
(2) $N_\infty(T)$ is dense in X.

He also requires an isometry condition on T which will play no role in this paper.

Definition An operator $T \in B(X)$ is an abstract backward shift if T satisfies (1) and (2) above. We use the convenient notation, ABS, to stand for "abstract backward shift".

It follows from work of S. Grabiner in [4, Theorem 1.1], that for every separable Banach space X, there exists an ABS on X. In contrast, Holub shows in [5] that backward shifts as defined in [5] do not exist on some classical Banach space, $C[0,1]$ for example.

In this paper we study properties of operators in the commutant of an ABS. Assume that T is an ABS, and let $\{T\}'$ denote the commutant of T. Using some purely algebraic results established in Section 1, we show in Section 2 that $\{T\}'$ is an integral domain [Theorem 5]. Also, we prove that operators $S \in \{T\}'$ have some interesting special properties; for example, S has connected spectrum, and when $S \neq 0$, then $R(S)$ is dense in X [Theorem 7].

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1 Some Algebraic Preliminaries

The purely algebraic results in this section form the basis of the proof of our main theorems. Throughout this section, X is an arbitrary linear space, and $T\colon X \to X$ is a linear operator on X. Let $\operatorname{nul}(T)$ denote the nullity of T, that is, $\operatorname{nul}(T) = \dim\bigl(N(T)\bigr)$. Also, let $\alpha(T)$ denote the ascent of T [9, pp. 289–290]. In particular, when T is an ABS, then $\operatorname{nul}(T) = 1$, and $\alpha(T) = \infty$.

Elements of the scalar field are denoted by lower case Greek letters. For $E \subseteq X$, let $\langle E\rangle$ denote the linear span of E in X.

Proposition 1 *Assume T has $\alpha(T) = \infty$ and $\operatorname{nul}(T) = 1$. Suppose $N(T) = \langle x_0\rangle$. Then there exists an infinite linearly independent sequence $\{x_0, x_1, x_2, \dots\}$ with $N(T^m) = \langle x_0, x_1, \dots, x_{m-1}\rangle$ for all $m \geq 1$, and $T(x_k) = x_{k-1}$ for $k \geq 1$.*

Proof $N(T) = \langle x_0\rangle$ by hypothesis. Choose $x \in N(T^2)$ such that $x \notin N(T)$. Then $T(Tx) = 0$, so that $Tx = \lambda x_0$ for some $\lambda \neq 0$. Set $x_1 = \lambda^{-1}x$. Clearly, $\{x_0, x_1\}$ is a l.i. subset of $N(T^2)$. If $y \in N(T^2)$, then $Ty = \mu x_0 = T(\mu x_1)$ for some μ. Thus, $T(y - \mu x_1) = 0$, $y - \mu x_1 = \delta x_0$ for some δ. Therefore, $y \in \langle x_0, x_1\rangle$.

Now suppose $m \geq 2$, and $\{x_0, x_1, \dots, x_{m-1}\}$ is a l.i. set such that $\langle x_0, x_1, \dots, x_{m-1}\rangle = N(T^m)$, and $T(x_k) = x_{k-1}$ for $1 \leq k \leq m-1$. Choose $x \in N(T^{m+1})$ such that $x \notin N(T^m)$. Then $\{x_0, x_1, \dots, x_{m-1}, x\}$ is a l.i. set. For some $\lambda \neq 0$, $T^m x = \lambda x_0$. Take any $y \in N(T^{m+1})$. Then $T^m y = \mu x_0$ for some μ. Thus, $y - (\mu/\lambda)x \in N(T^m)$, so $y \in \langle x_0, x_1, \dots, x_{m-1}, x\rangle$. This proves that $N(T^{m+1})$ is $(m+1)$-dimensional.

Let $S\colon N(T^{m+1}) \to N(T^m)$ be defined by $Sw = Tw$, $w \in N(T^{m+1})$. We have $\dim\bigl(N(S)\bigr) + \dim\bigl(R(S)\bigr) = m+1$. Since $\dim\bigl(N(S)\bigr) = 1$, it follows that $\dim\bigl(R(S)\bigr) = m$, so $R(S)$ is all of $N(T^m)$. Choose $x_m \in N(T^{m+1})$ such that $T(x_m) = x_{m-1}$. Note that $x_m \notin N(T^m)$ since $T^m(x_m) = x_0$. Therefore, $N(T^{m+1}) = \langle x_0, x_1, \dots, x_m\rangle$.

As Proposition 1 shows, when $\alpha(T) = \infty$ and $\operatorname{nul}(T) = 1$, then there is a sequence which has span $N_\infty(T)$, and on which T acts like a backward shift. We use this in what follows to derive information about $\{T\}'$.

Proposition 2 *Let T and $\{x_0, x_1, x_2, \dots\}$ be as in Proposition 1. Assume $S \in \{T\}'$, so $S\bigl(N(T^m)\bigr) \subseteq N(T^m)$ for $m \geq 1$. For each $m \geq 0$, $S(x_m) = \lambda_m x_0 + z_m$ where $z_m \in \langle x_1, \dots, x_m\rangle$. Set $p_m(T) = \sum_0^m \lambda_k T^k$. Then for $m \geq 0$, $S = p_m(T)$ on $N(T^{m+1})$.*

Proof The statement is true for $m = 0$, since $S(x_0) = \lambda_0 x_0$ (the definition of λ_0), and $p_0(T) = \lambda_0 I$ (the definition of $p_0(T)$. Now suppose the statement is true for some $m \geq 0$. We prove $S = p_{m+1}(T)$ on $N(T^{m+2})$. Note that $p_{m+1}(T) = p_m(T) + \lambda_{m+1}T^{m+1}$. Also, $N(T^{m+2}) = \langle x_0, x_1, \dots, x_{m+1}\rangle$. Now for $0 \leq j \leq m$, $T^{m+1}x_j = 0$. Thus,

$$Sx_j = p_m(T)x_j = p_{m+1}(T)x_j \quad \text{for } 0 \leq j \leq m.$$

We have $Sx_{m+1} = \lambda_{m+1}x_0 + z_{m+1}$ where $z_{m+1} \in \langle x_1, \dots, x_{m+1}\rangle$. Now $p_m(T)Tx_{m+1} = p_m(T)x_m = Sx_m = STx_{m+1} = TSx_{m+1} = T(\lambda_{m+1}x_0 + z_{m+1}) = T(z_{m+1})$. Therefore, $T\bigl(z_{m+1} - p_m(T)x_{m+1}\bigr) = 0$. Thus, for some μ, $z_{m+1} - p_m(T)x_{m+1} = \mu x_0$. Now $z_{m+1} \in \langle x_1, \dots, x_{m+1}\rangle$ and $p_m(T)x_{m+1} \in \langle x_1, \dots, x_{m+1}\rangle$, so $\mu = 0$. This proves $z_{m+1} = p_m(T)x_{m+1}$. Therefore $Sx_{m+1} = \lambda_{m+1}x_0 + p_m(T)x_{m+1} = \lambda_{m+1}T^{m+1}x_{m+1} + p_m(T)x_{m+1} = p_{m+1}(T)x_{m+1}$. This completes the proof that $S = p_{m+1}(T)$ on $N(T^{m+2}) = \langle x_0, x_1, \dots, x_{m+1}\rangle$.

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Corollary 3 *Assume T has $\alpha(T) = \infty$, and $\operatorname{nul}(T) = 1$. If n is an integer, $n \geq 2$, then there does not exist a linear operator S on X such that $S^n = T$.*

Proof We adopt the notation of Proposition 2. Suppose $S^n = T$ for some integer $n \geq 2$. Then $S \in \{T\}'$. Therefore $S(x_0) = \lambda_0 x_0$, and it follows that $0 = T(x_0) = S^n(x_0) = \lambda_0^n x_0$. Thus, $\lambda_0 = 0$. Furthermore, $S(x_1) = p_1(T)(x_1) = \lambda_1 x_0$. Then $x_0 = T(x_1) = S^n(x_1) = S^{n-1}(\lambda_1 x_0) = 0$, a contradiction.

Proposition 4 *Let T and $\{x_0, x_1, x_2, \dots\}$ be as in Proposition 1. If $S \in \{T\}'$, then either $S\big(N_\infty(T)\big) = \{0\}$ or $N_\infty(T) \subseteq R(S)$.*

Proof Suppose $S\big(N_\infty(T)\big) \neq \{0\}$. Let m be the smallest non-negative integer such that $Sx_m \neq 0$. Then since in this case $p_{m-1}(T)x_m = 0$, $Sx_m = \lambda_m x_0 + p_{m-1}(T)x_m = \lambda_m x_0 \neq 0$. Thus, $\lambda_m \neq 0$ and $x_0 \in R(S)$. Assume that $x_k \in R(S)$ for $0 \leq k \leq q$. Now for $j \geq m$, $p_j(T) = \sum_m^j \lambda_k T^k$. Set $j = q+m$. Then $Sx_{j+1} = \lambda_{j+1}x_0 + p_j(T)x_{j+1} = \lambda_{j+1}x_0 + \lambda_m x_{j-m+1} + \{$a linear combination of x_k's where $k \leq q\}$. This proves $x_{j-m+1} = x_{q+1} \in R(S)$. Thus by induction, $N_\infty(T) = \operatorname{span}\{x_k : 0 \leq k\} \subseteq R(S)$.

2 Applications to Operators on a Banach Space

Now we look at the situation where X is a complex Banach space and T is an ABS on X.

Theorem 5 *Assume that T is an ABS. Then $\{T\}'$ is an integral domain and a maximal commutative subalgebra of $B(X)$.*

Proof Assume $S, R \in \{T\}'$. By Proposition 2 for all $n \geq 1$, both S and R are equal to polynomials in T on $N(T^n)$. Thus, $SR - RS = 0$ on $N(T^n)$. Since $N_\infty(T) = \bigcup_1^\infty N(T^n)$ is dense, we have $SR = RS$.

Now suppose that $S, R \in \{T\}'$ and $RS = 0$. If $S \neq 0$, then by Proposition 4, $N_\infty(T) \subseteq R(S)$. Then $R\big(N_\infty(T)\big) = \{0\}$, so $R = 0$. This proves that $\{T\}'$ is an integral domain. That this algebra is a maximal commutative subalgebra of $B(X)$ is immediate.

The next result is known; it follows from the work of S. Grabiner in [3]. A new proof is provided by Theorem 5 and the result that there exists an ABS on every separable Banach space, also due to Grabiner [4, Theorem 1.1].

Corollary 6 *For every separable Banach space, $B(X)$ contains a maximal commutative subalgebra which is an integral domain.*

When T is an ABS, the operators in $\{T\}'$ have a number of interesting properties. We note several of them in the next theorem.

Theorem 7 *Assume $T \in B(X)$ is an ABS. An operator $S \in \{T\}'$ has the properties:*

(1) $\sigma(S)$ is connected;
(2) if $S \neq 0$, then $R(S)$ is dense in X;
(3) if S is a Fredholm operator, then $R(S) = X$. In this case, S has a right inverse in $B(X)$;

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(4) let $N(T) = \langle x_0 \rangle$*. If* $S(x_0) = 0$*, then for any* $w \in X$*,*

$$\|S^k(w)\| = o(\|S^k\|).$$

Proof By Theorem 5, $\{T\}'$ is an integral domain. Thus, the only projections in $\{T\}'$ are 0 and the identity. If $\sigma(S)$ were disconnected, then there would be a proper projection in $\{T\}'$. This proves (1).

(2) is a direct application of Proposition 4.

It follows from (2) that when S is a Fredholm operator, then $R(S) = X$. Furthermore, in this case S has a right inverse by [6, Theorem 5.4, p. 89]. Therefore (3) holds.

Now assume that $S(x_0) = 0$. First we show that $S^n\big(N(T^n)\big) = \{0\}$. This is true when $n = 1$ by hypothesis. Suppose that $S^m\big(N(T^m)\big) = \{0\}$. Assume that $y \in N(T^{m+1})$. Then $T(S^m y) = S^m(Ty) = 0$. Therefore $S^m y = \lambda x_0$ for some λ. Thus, $S^{m+1} y = 0$. This proves by induction that $S^n\big(N(T^n)\big) = \{0\}$ for all n.

Fix $w \in X$. Let $d_k = \inf\{\|w - y\| : y \in N(T^k)\}$ for all $k \geq 1$, and note that $d_k \to 0$ as $k \to \infty$. For each k choose $y_k \in N(T^k)$ such that $\|w - y_k\| < 2d_k$. Then $\|S^k w\| = \|S^k(w - y_k)\| \leq 2d_k\|S^k\|$. This proves 4.

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