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# A UNIFORM ASYMPTOTIC EXPANSION OF THE JACOBI POLYNOMIALS WITH ERROR BOUNDS

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**1. Introduction.** In a recent investigation of the asymptotic behavior of the Lebesgue constants for Jacobi polynomials, we encountered the problem of obtaining an asymptotic expansion for the Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta)$, as $n \to \infty$, which is uniformly valid for $\theta$ in $\left[0, \frac{\pi}{2}\right]$. The leading term of such an expansion is provided by the following well-known formula of "Hilb's type" [**13**, p. 197]:

$$\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(\cos\theta)$$

$$(1.1)\qquad = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}(\theta/\sin\theta)^{1/2}J_\alpha(N\theta) + \begin{cases} \theta^{1/2}O(n^{-3/2}) & \text{if } cn^{-1} \leqq \theta \leqq \pi - \epsilon \\ \theta^{\alpha+2}O(n^{\alpha}) & \text{if } 0 \leqq \theta \leqq cn^{-1}, \end{cases}$$

where $\alpha > -1$, $\beta$ real and $N = n + \frac{1}{2}(\alpha + \beta + 1)$; $c$ and $\epsilon$ are fixed positive numbers. Note that the remainder in (1.1) is always $\theta^{1/2}O(n^{-3/2})$. When $\alpha = \beta = 0$, the Jacobi polynomial reduces to the Legendre polynomial, and in this case, we have the following full expansion from a well-known paper of Szegö [**11**]:

$$(1.2)\qquad P_n(\cos\theta) = \sum_{\nu=0}^{m-1} a_\nu(\theta)\frac{J_\nu\left[\left(n+\frac{1}{2}\right)\theta\right]}{\left(n+\frac{1}{2}\right)^{\nu}} + O(n^{-m-1/2}),$$

for every $m \geqq 1$, where the coefficients $a_\nu(\theta)$ are analytic functions for $0 \leqq \theta < \pi$, and the $O$-term is uniform with respect to $\theta \in [0, \theta_0)$, $\theta_0 = 0.828\ldots\pi$. Thus it seems natural to suggest that a corresponding result exists for the more general polynomial $P_n^{(\alpha,\beta)}(\cos\theta)$. There is

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indeed such a result, and the construction of this result will be the purpose of the present article.

MAIN THEOREM. *For* $\alpha > -\frac{1}{2}$, $\alpha - \beta > -2m$ *and* $\alpha + \beta \geqq -1$, *we have*

$$(1.3)\quad P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+1)}{n!}\left(\sin\frac{\theta}{2}\right)^{-\alpha}\left(\cos\frac{\theta}{2}\right)^{-\beta}\left(\frac{\theta}{\sin\theta}\right)^{1/2} \times \left[\sum_{l=0}^{m-1} A_l(\theta)\frac{J_{\alpha+l}(N\theta)}{N^{\alpha+l}} + \theta^{\alpha}O(N^{-m})\right],$$

*where*

$$(1.4)\quad N = n + \frac{1}{2}(\alpha + \beta + 1),\ \theta \in (0, \pi).$$

*The coefficients* $A_l(\theta)$ *are analytic functions for* $0 \leqq \theta < \pi$, *and are explicitly given in* (3.15). *The O-term is uniform with respect to* $\theta \in [0, \pi - \epsilon]$, $\epsilon$ *being an arbitrary positive number.*

An expansion similar to (1.3) had been given earlier by Szegö in a not so well-known paper [**12**], and this paper was also not available to us until the recent appearance of his collected works [**3**]. In any case, our analysis differs from that of Szegö, and provides more information for the error term. For instance, we have the following useful consequence.

COROLLARY 1. *For* $\alpha > -1/2$, $\alpha - \beta > -4$ *and* $\alpha + \beta \geqq -1$, *we have*

$$(1.5)\quad \left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta}P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\left[\frac{J_\alpha(N\theta)}{N^\alpha} + A_1(\theta)\frac{J_{\alpha+1}(N\theta)}{N^{\alpha+1}} + \sigma_2\right],$$

*where*

$$(1.6)\quad A_1(\theta) = \left(\alpha^2 - \frac{1}{4}\right)\left(\frac{1 - \theta\cot\theta}{2\theta}\right) - \frac{\alpha^2 - \beta^2}{4}\tan\frac{\theta}{2},$$

*and*

$$(1.7)\quad |\sigma_2| \leqq \frac{E_2}{N^2}\theta^{2+\alpha},\quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

*The constant* $E_2$ *is given explicitly in* (4.27).

A similar two-term expansion has been given by Gatteschi [7, Eq. (22) ], who uses

$$\nu = \left[\left(n + \frac{\alpha + \beta + 1}{2}\right)^2 + \frac{1 - \alpha^2 - 3\beta^2}{12}\right]^{1/2}$$

as his asymptotic variable and shows that the remainder term is $O(n^{-3/2})$ for $\theta \in \left[0, \frac{\pi}{2}\right]$.

If $\alpha = \beta = 0$, then Corollary 1 reduces to

$$(1.8) \quad \left(\frac{\sin \theta}{\theta}\right)^{1/2} P_n(\cos \theta) = J_0\left\{\left(n + \frac{1}{2}\right)\theta\right\} + \frac{\theta \cot \theta - 1}{8\theta\left(n + \frac{1}{2}\right)} J_1\left\{\left(n + \frac{1}{2}\right)\theta\right\} + \sigma_2'$$

with

$$(1.9) \quad |\sigma_2'| < \frac{0.13}{\left(n + \frac{1}{2}\right)^2}\theta^2, \quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

This result should be compared with another result of Gatteschi [**6**], which states that

$$(1.10) \quad \left(\frac{\sin \theta}{\theta}\right)^{1/2} P_n(\cos \theta) = J_0\left\{\left(n + \frac{1}{2}\right)\theta\right\} - \frac{\theta}{24\left(n + \frac{1}{2}\right)} J_1\left\{\left(n + \frac{1}{2}\right)\theta\right\} + \sigma'$$

where

$$|\sigma'| < 0.03\theta^4 \qquad \text{if } 0 < \theta \leqq \frac{\pi}{2n}$$

$$|\sigma'| < 0.25\theta^{5/2} n^{-3/2} \text{ if } \frac{\pi}{2n} \leqq \theta \leqq \frac{\pi}{2};$$

see [**13**, p. 242]. Note that the second term in (1.10) differs from the corresponding term in (1.9), and that the remainder $\sigma'$ is only of the order $n^{-3/2}$ for $\theta \in \left[0, \frac{\pi}{2}\right]$ and not of the order $n^{-2}$.

Another useful consequence of our main theorem is the following expansion for the zeros of the Jacobi polynomials.

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COROLLARY 2. *Let* $\alpha > -\frac{1}{2}$, $\alpha + \beta \geqq -1$, *and let* $0 < \theta_1 < \theta_2 < \ldots < \theta_n < \pi$ *be the zeros of* $P_n^{(\alpha,\beta)}(\cos\theta)$. *Then, as* $n \to \infty$,

$$\theta_l = \frac{j_{\alpha,l}}{N} + \frac{1}{N^2}\left\{\left(\alpha^2 - \frac{1}{4}\right)\frac{1 - t\cot t}{2t} - \frac{\alpha^2 - \beta^2}{4}\tan\frac{t}{2}\right\} + t^2 O\left(\frac{1}{N^3}\right), \tag{1.11}$$

*where* $j_{\alpha,l}$ *is the l-th positive zero of the Bessel function* $J_\alpha(x)$ *and* $t = j_{\alpha,l}/N$. *The* 0-*term is uniformly bounded for all values of* $l = 1, 2, \ldots [\gamma n]$, *where* $\gamma \in (0, 1)$.

The above results will all be used in a subsequent paper on the derivation of the asymptotic expansion of the Lebesgue constants for Jacobi series.

For completeness, we also mention two related papers on the same subject. The first one is by Hahn [**9**], who recently obtained an asymptotic expansion of Stieltjes type for Jacobi polynomials, that is, an expansion which is uniformly valid in any compact subinterval of $(0, \pi)$ and whose error is numerically less than twice the first neglected term. The second paper is by Elliot [**4**], who supplied a uniform asymptotic expansion of $P_n^{(\alpha,\beta)}(z)$ for $z \notin [-1, 1]$. A combination of the results in [**4**], [**9**] and the present paper gives a rather complete description of the behavior of the Jacobi polynomials.

**2. Some preliminary lemmas.** The starting point of our derivation of (1.3) is the following extension of Mehler's integral due to Gasper [**5**]; see also [**2**, p. 21].

LEMMA 1. *For* $\mathrm{Re}\,\alpha > -\frac{1}{2}$ *and* $0 < \theta < \pi$, *we have*

$$\begin{aligned}\frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)} &= \frac{2^{(\alpha+\beta+1)/2}\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}(1-\cos\theta)^{-\alpha}(1+\cos\theta)^{-(\alpha+\beta)/2} \\ &\quad \times \int_0^\theta \frac{\cos[n + (\alpha+\beta+1)/2]\phi}{(\cos\phi - \cos\theta)^{1/2-\alpha}} \\ &\quad \times {}_2F_1\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{1+\cos\theta}\right)d\phi.\end{aligned} \tag{2.1}$$

Later in our derivation, we shall replace the hypergeometric function ${}_2F_1$ in (2.1) by its Taylor series with remainder. An estimate for the remainder is given in the following lemma.

LEMMA 2. *Let $m$ be a positive integer,* $\operatorname{Re} c > \operatorname{Re} b > -m$, *and* $\operatorname{Re} a \geqq -m$. *Then, for real and negative $z$, we have*

$$(2.2) \quad {}_2F_1(a, b; c; z) = \sum_{k=0}^{m-1} \frac{(a)_k(b)_k}{(c)_k k!} z^k + z^m F_m(z),$$

*where*

$$(2.3) \quad |F_m^{(\eta)}(z)| \leqq \left| \frac{(a)_{m+\eta}(b)_m}{(c)_m(\eta+1)_m} \right|, \quad \eta = 0, 1, 2, \ldots .$$

*Proof.* The binomial expansion gives

$$(2.4) \quad (1 + x)^{-a} = \sum_{k=0}^{m-1} \frac{(-1)^k\Gamma(k + a)}{k!\Gamma(a)} x^k + x^m R_m(x)$$

for $x \geqq 0$, where

$$R_m(x) = \frac{(-1)^m(a)_m}{(m - 1)!} \int_0^1 (1 - u)^{m-1}(1 + xu)^{-a-m} du.$$

Since $\operatorname{Re} a \geqq -m$, the remainder satisfies

$$(2.5) \quad |R_m^{(\eta)}(x)| \leqq \frac{|(a)_{m+\eta}|}{(\eta + 1)_m}, \quad \eta = 0, 1, 2, \ldots .$$

Now insert (2.4) in the well-known identity

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \times \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} dt,$$

which is valid under the conditions

$$\operatorname{Re} c > \operatorname{Re} b > 0 \quad \text{and} \quad |\arg(1 - z)| < \pi.$$

Using the Pochhammer notation $(\alpha)_0 = 1$,

$$(\alpha)_n = \alpha(\alpha + 1) \ldots (\alpha + n - 1) = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)}.$$

we obtain (2.2) with

$$F_m(z) = \frac{(-1)^m\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b+m-1}(1 - t)^{c-b-1} R_m(-tz) dt.$$

Note that the last integral exists for Re $b > -m$. Thus the condition Re $b > 0$ can now be removed. The estimates in (2.3) follow immediately from those in (2.5).

Throughout the paper, the parameters $a$, $b$ and $c$ are fixed, and are taken to be

$$(2.6)\qquad a = \frac{\alpha + \beta}{2},\ b = \frac{\alpha - \beta}{2} \quad \text{and} \quad c = \alpha + \frac{1}{2}.$$

Therefore, there is no ambiguity in suppressing the dependence of $F_m$ on these parameters.

LEMMA 3. *Let $F_m(z)$ be defined as in Lemma 2, and put*

$$(2.7)\qquad f_{m,\theta}(\phi) = F_m\left(\frac{\cos\theta - \cos\phi}{1 + \cos\theta}\right).$$

(i) *For $p = 0, 1, 2, \ldots$, we have*

$$(2.8)\qquad f_{m,\theta}^{(2p+1)}(0) = 0.$$

(ii) *There exists a constant $C_{m,\nu}(\theta)$, depending on a, b and c, such that*

$$(2.9)\qquad \sup_{0 \leqq \phi \leqq \theta} |f_{m,}^{(\nu)}(\phi)| \leqq C_{m,\nu}(\theta), \quad \nu = 0, 1, \ldots, m.$$

*An explicit expression for $C_{m,\nu}(\theta)$ is given in* (2.14).

*Proof.* (i) The following identity is given in [**8**, p. 19, Eq. (1)]: If $f(x) = F(y)$ and $y = y(x)$, then

$$(2.10)\qquad \begin{aligned} \frac{d^n}{dx^n}f(x) &= \frac{U_1}{1!}F'(y) + \frac{U_2}{2!}F''(y) \\ &\quad + \frac{U_3}{3!}F'''(y) + \ldots + \frac{U_n}{n!}F^{(n)}(y), \end{aligned}$$

where

$$(2.11)\qquad \begin{aligned} U_k &= \frac{d^n}{dx^n}y^k - \frac{k}{1!}y\frac{d^n}{dx^n}y^{k-1} \\ &\quad + \frac{k(k-1)}{2!}y^2\frac{d^n}{dx^n}y^{k-2} - \ldots + (-1)^{k-1}ky^{k-1}\frac{d^n y}{dx^n}. \end{aligned}$$

In the present case, $x = \phi$, $y = (\cos\theta - \cos\phi)/(1 + \cos\theta)$, $F = F_m$ and $f = f_{m,\theta}$. By expanding $\cos\phi$ into a Maclaurin series, it is easily seen that for $n \geqq 1$ and $n$ odd,

$$(2.12)\qquad \left(\frac{d}{d\phi}\right)^n \left(\frac{\cos\theta - \cos\phi}{1 + \cos\theta}\right)^l \Bigg|_{\phi=0} = 0, \quad l = 1, \ldots, k.$$

Therefore all the $U_k$'s in (2.11), $k = 1, \dots, n$, vanish when we set $x = \phi = 0$. This proves the statement in (i).

(ii) Here we use the following alternative identity, also given in [**8**, p. 19, Eq. (2)]:

$$(2.13)\quad \frac{d^\nu}{dx^\nu}f(x) = \sum \frac{\nu!}{i!j!h!\dots k!}\,\frac{d^\eta F}{dy^\eta}\left(\frac{y'}{1!}\right)^i\left(\frac{y''}{2!}\right)^j\left(\frac{y'''}{3!}\right)^h\dots\left(\frac{y^{(l)}}{l!}\right)^k,$$

where $\sum$ sums over all solutions in non-negative integers of the two equations

$$i + 2j + 3h + \dots + lk = \nu \quad\text{and}\quad \eta = i + j + h + \dots + k.$$

The estimate in (2.9) now follows with

$$(2.14)\quad \begin{aligned} C_{m,\nu}(\theta) &= \sum \frac{\nu!}{i!j!h!\dots k!}\left|\frac{(a)_{m+\eta}(b)_m}{(c)_m(\eta + 1)_m}\right| \\ &\times \frac{(1 + \cos\theta)^{-\eta}}{(1!)^i(2!)^j(3!)^h\dots(l!)^k}. \end{aligned}$$

Note that $C_{m,\nu}(\theta)$ depends on $a$, $b$ and $c$.

For convenience of later calculation, we give the following values of $C_{m,\nu}(\theta)$:

$$(2.15)\quad \begin{aligned} &C_{1,0}\left(\frac{\pi}{2}\right) = \left|\frac{a\cdot b}{c}\right|,\ C_{1,1}\left(\frac{\pi}{2}\right) = \left|\frac{(a)_2\cdot b}{2c}\right|, \\ &C_{2,0}\left(\frac{\pi}{2}\right) = \left|\frac{(a)_2(b)_2}{(c)_2 2}\right|,\ C_{2,1}\left(\frac{\pi}{2}\right) = \left|\frac{(a)_3(b)_2}{(c)_2(2)_2}\right|, \\ &C_{2,2}\left(\frac{\pi}{2}\right) = \left|\frac{(a)_4(b)_2}{(c)_2(3)_2}\right| + \left|\frac{(a)_3(b)_2}{(c)_2(2)_2}\right|. \end{aligned}$$

LEMMA 4. *For* $\alpha > -1/2$, *put*

$$(2.16)\quad R_m(\theta; N) = \int_0^\infty \cos N\phi(\cos\phi - \cos\theta)^{m+\alpha-1/2} f_{m,\theta}(\phi)d\phi,$$

*where* $f_{m,\theta}(\phi)$ *is defined as in Lemma* 3. *Then there exists a constant* $C_m(\theta)$, *depending on a, b and c, such that*

$$(2.17)\quad |R_m(\theta; N)| \leqq \frac{C_m(\theta)}{N^m},\quad 0 \leqq \theta < \pi.$$

*An explicit formula for* $C_m(\theta)$ *is given in* (2.21).

*Proof.* Let

$$g_{m,\theta}(\phi) = (\cos\phi - \cos\theta)^{m+\alpha-1/2}$$

and recall the formula

$$\left(\frac{d}{dy}\right)^s y^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-s+1)} y^{\lambda-s}.$$

From (2.13), it follows that

$$|g_{m,\theta}^{(\mu)}(\phi)| \leqq \sum \frac{\mu!}{i!j!h!\dots k!} \frac{\Gamma\left(m+\alpha+\frac{1}{2}\right)}{\Gamma\left(m+\alpha+\frac{1}{2}-s\right)} \times \left(\frac{1}{1!}\right)^i \left(\frac{1}{2!}\right)^j \left(\frac{1}{3!}\right)^h \dots \left(\frac{1}{l!}\right)^k (\cos\phi - \cos\theta)^{m+\alpha-1/2-s},$$

where $\sum$ sums over all solutions in

$$i + 2j + 3h + \dots + lk = \mu \quad \text{and} \quad s = i + j + \dots + k.$$

In deriving this, we have taken $y = \cos\phi - \cos\theta$ and $F(y) = y^{m+\alpha-1/2}$ in (2.13). Since

$$0 \leqq \cos\phi - \cos\theta \leqq 1 - \cos\theta \quad \text{for } 0 \leqq \phi \leqq \theta < \pi,$$

we have

$$(\cos\phi - \cos\theta)^{m-s} \leqq (1 - \cos\theta)^{m-s} \quad \text{for } s \leqq \mu \leqq m.$$

Therefore,

$$(2.18) \quad |g_{m,\theta}^{(\mu)}(\phi)| \leqq D_{m,\mu}(\theta)(\cos\phi - \cos\theta)^{\alpha-1/2},$$

where

$$(2.19) \quad D_{m,\mu}(\theta) = \sum \frac{\mu!}{i!j!h!\dots k!} \frac{\Gamma\left(m+\alpha+\frac{1}{2}\right)}{\Gamma\left(m+\alpha+\frac{1}{2}-s\right)} \times \frac{(1-\cos\theta)^{m-s}}{(1!)^i(2!)^j(3!)^h\dots(l!)^k}.$$

Now we return to (2.16) and apply integration by parts $m$ times. On account of (2.8), all the integrated terms vanish. Thus

$$(2.20)\quad R_m(\theta; N) = \frac{\pm 1}{N^m}\int_0^\theta \begin{Bmatrix} \sin N\phi \\ \cos N\phi \end{Bmatrix} \frac{d^m}{d\phi^m}[g_{m,\theta}(\phi)f_{m,\theta}(\phi)]d\phi.$$

By (2.9) and (2.18), we have

$$|R_m(\theta; N)| \leqq \frac{1}{N^m}\left[\sum_{\nu+\mu=m}\binom{m}{\nu}C_{m,\nu}(\theta)D_{m,\mu}(\theta)\right] \times \int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}d\phi.$$

Taking

$$(2.21)\quad C_m(\theta) = \left[\sum_{\nu+\mu=m}\binom{m}{\nu}C_{m,\nu}(\theta)D_{m,\mu}(\theta)\right] \times \int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}d\phi,$$

we complete the proof of the lemma.

For $\theta$ in the interval $\left[0, \frac{\pi}{2}\right]$, the integral in (2.21) can be estimated by using the following lemma.

LEMMA 5. *For* $0 \leqq \phi \leqq \theta \leqq \frac{\pi}{2}$,

$$(2.22)\quad \frac{2}{\pi^2} \leqq \frac{\cos\phi - \cos\theta}{\theta^2 - \phi^2} \leqq \frac{1}{2}.$$

*Proof.* This follows immediately from the identity

$$\frac{\cos\phi - \cos\theta}{\theta^2 - \phi^2} = \frac{1}{2}\,\frac{\sin\left(\frac{\phi+\theta}{2}\right)}{\frac{\phi+\theta}{2}}\,\frac{\sin\left(\frac{\phi-\theta}{2}\right)}{\frac{\phi-\theta}{2}}$$

and the fact that

$$(2.23)\quad \frac{2}{\pi} \leqq \frac{\sin\psi}{\psi} \leqq 1$$

for $0 \leqq \psi \leqq \frac{\pi}{2}$.

The first inequality in (2.22) gives

$$\int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}d\phi \leqq \frac{2^{\alpha-1/2}}{\pi^{2\alpha-1}}\int_0^\theta (\theta^2 - \phi^2)^{\alpha-1/2}d\phi$$

$$= \frac{2^{\alpha-3/2}}{\pi^{2\alpha-1}}\theta^{2\alpha}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)}$$

if $\alpha < 1/2$, and the second inequality gives

$$\int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}d\phi \leqq 2^{-\alpha-1/2}\theta^{2\alpha}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)}$$

if $\alpha > 1/2$. Here we have used the definition of the Beta function [**10**, p. 37]. If $\alpha = 1/2$, then trivially the above integral is equal to $\theta$. Put

$$(2.24)\quad \Gamma_1(\alpha) = \begin{cases} 1 & \text{if } \alpha = \frac{1}{2} \\ 2^{-\alpha-1/2}\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)} & \text{if } \alpha > \frac{1}{2} \\ \dfrac{2^{\alpha-3/2}}{\pi^{2\alpha-1}}\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)} & \text{if } \alpha < \frac{1}{2}. \end{cases}$$

Then we have

$$(2.25)\quad \int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}d\phi \leqq \Gamma_1(\alpha)\theta^{2\alpha}$$

for $0 \leqq \theta \leqq \frac{\pi}{2}$. Define

$$(2.26)\quad \Gamma_2(\alpha) = \max_{(\pi/2)\leqq\theta\leqq\pi} \theta^{-2\alpha}\int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}d\phi.$$

Then we also have

$$(2.27)\quad \int_0^\theta (\cos\phi - \cos\theta)^{\alpha-1/2}d\phi \leqq \Gamma_2(\alpha)\theta^{2\alpha}$$

for $\frac{\pi}{2} \leqq \theta \leqq \pi$. A combination of (2.21), (2.25) and (2.27) gives

$$(2.28)\quad C_m(\theta) \leqq \Gamma^*(\alpha)\theta^{2\alpha}\left[\sum_{\nu+\mu=m}\binom{m}{\nu}C_{m,\nu}(\theta)D_{m,\mu}(\theta)\right]$$

for $0 \leqq \theta \leqq \pi$, where $\Gamma^*(\theta) = \max\{\Gamma_1(\alpha), \Gamma_2(\alpha)\}$.

LEMMA 6. *For $\alpha > -1/2$, $m = 1, 2, \ldots,$ and $k = 0, 1, \ldots, m - 1$, we have*

$$(2.29)\quad [2(\cos\phi - \cos\theta)]^{k+\alpha-1/2} = \left(\frac{\sin\theta}{\theta}\right)^{k+\alpha-1/2}\left[\sum_{\nu=0}^{m-k-1}\psi_{k,\nu}(\theta)(\theta^2-\phi^2)^{k+\nu+\alpha-1/2} + (\theta^2-\phi^2)^{m+\alpha-1/2}\Delta_{k,m}(\theta,\phi)\right],$$

*where the coefficients $\psi_{k,\nu}(\theta)$ are analytic functions for $0 \leqq \theta < \pi$ and the remainder satisfies*

$$(2.30)\quad \left|\frac{d^m}{d\phi^m}[(\theta^2-\phi^2)^{m+\alpha-1/2}\Delta_{k,m}(\theta,\phi)]\right| \leqq M_{k,m}(\theta)\theta^m(\theta^2-\phi^2)^{\alpha-1/2}$$

*for some constant $M_{k,m}(\theta)$. An explicit expression for $M_{k,m}(\theta)$ is given in* (2.39).

Here are some of the coefficients:

$$(2.31)\quad \psi_{k,0}(\theta) = 1$$

$$(2.32)\quad \psi_{k,1}(\theta) = \frac{1}{4}\left(k+\alpha-\frac{1}{2}\right)\frac{1-\theta\cot\theta}{\theta^2}$$

$$(2.33)\quad \psi_{k,2}(\theta) = \frac{k+\alpha-\frac{1}{2}}{2^2\cdot 3!}\left[\frac{1}{\theta}\left(\frac{3}{\theta^3}-\frac{1}{\theta}\right)-\frac{3}{\theta^3}\cot\theta\right] + \frac{\left(k+\alpha-\frac{1}{2}\right)\left(k+\alpha-\frac{3}{2}\right)}{2!\cdot 4^2}\left(\frac{1-\theta\cot\theta}{\theta^2}\right)^2.$$

*Proof.* From [**14**, p. 140, Eq. (3)], we have

$$(2.34)\quad 2(\cos\phi - \cos\theta) = (\theta^2-\phi^2)\frac{\sin\theta}{\theta}\left[1+\sum_{\nu=2}^{\infty}(\theta^2-\phi^2)^{\nu-1}\phi_\nu(\theta)\right],$$

where

$$(2.35)\quad \phi_\nu(\theta) = \frac{1}{\nu!}\frac{1}{2^{\nu-1}}\frac{J_{\nu-1/2}(\theta)}{\theta^{\nu-1}J_{1/2}(\theta)},\quad 0 \leqq \theta < \pi.$$

(Equation (2.35) was also used by Szegö in the derivation of (1.2).) Thus, for $\phi$ near $\theta$, we get

(2.36) $$[2(\cos\phi - \cos\theta)]^{k+\alpha-1/2}$$
$$= (\theta^2 - \phi^2)^{k+\alpha-1/2}\left(\frac{\sin\theta}{\theta}\right)^{k+\alpha-1/2}$$
$$\times \left[1 + \sum_{\nu=2}^{\infty} (\theta^2 - \phi^2)^{\nu-1}\phi_\nu(\theta)\right]^{k+\alpha-1/2}$$
$$= \left(\frac{\sin\theta}{\theta}\right)^{k+\alpha-1/2} \sum_{\nu=0}^{\infty} \psi_{k,\nu}(\theta)(\theta^2 - \phi^2)^{k+\nu+\alpha-1/2},$$

where $\psi_{k,\nu}(\theta)$ can be expressed as a linear combination of products of $\phi_\nu(\theta)$. The first few are given in (2.31)-(2.33).

Define $\Delta_{k,m}(\theta, \phi)$ as in (2.29), i.e.,

(2.37) $$(\theta^2 - \phi^2)^{m-k}\Delta_{k,m}(\theta, \phi)$$
$$= \left[\frac{2(\cos\phi - \cos\theta)}{\theta^2 - \phi^2}\right]^{k+\alpha-1/2}\left(\frac{\theta}{\sin\theta}\right)^{k+\alpha-1/2}$$
$$- \sum_{\nu=0}^{m-k-1} \psi_{k,\nu}(\theta)(\theta^2 - \phi^2)^{\nu}.$$

Since the sum in (2.37) is finite, there is no problem with convergence. Furthermore, it is easily seen that $\Delta_{k,m}(\theta, \phi)$ is a $C^\infty$-function of $\phi$ in $[0, \theta]$. (For $\phi$ near $\theta$, we can argue this by using the infinite convergent series in (2.36).)

To derive the estimate in (2.30), we note that

$$\left(\frac{d}{d\phi}\right)^l (\theta^2 - \phi^2)^{m+\alpha-1/2}$$
$$= \sum_{\lambda=0}^{l} \binom{l}{\lambda}\left(\frac{d}{d\phi}\right)^\lambda (\theta - \phi)^{m+\alpha-1/2}\left(\frac{d}{d\phi}\right)^{l-\lambda}(\theta + \phi)^{m+\alpha-1/2}$$
$$= \sum_{\lambda=0}^{l} (-1)^\lambda \binom{l}{\lambda}\left(m + \alpha - \frac{1}{2}\right)$$
$$\times \left(m + \alpha - \frac{3}{2}\right)\dots\left(m + \alpha - \frac{1}{2} - \lambda + 1\right)$$
$$\times (\theta - \phi)^{m+\alpha-1/2-\lambda}$$
$$\times \left(m + \alpha - \frac{1}{2}\right)\dots\left(m + \alpha - \frac{1}{2} - l + \lambda + 1\right)$$
$$\times (\theta + \phi)^{m+\alpha-1/2-l+\lambda}.$$

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Thus, if $m \geqq 1$ and $0 \leqq l \leqq m$,

$$(2.38)\quad \left|\left(\frac{d}{d\phi}\right)^l (\theta^2 - \phi^2)^{m+\alpha-1/2}\right|$$

$$\leqq (m + \alpha - 1/2)^l (\theta^2 - \phi^2)^{m+\alpha-1/2}\left[\frac{1}{\theta - \phi} + \frac{1}{\theta + \phi}\right]^l$$

$$= (m + \alpha - 1/2)^l (\theta^2 - \phi^2)^{m+\alpha-1/2-l}(2\theta)^l.$$

This implies

$$\left|\left(\frac{d}{d\phi}\right)^m [(\theta^2 - \phi^2)^{m+\alpha-1/2}\Delta_{k,m}(\theta, \phi)]\right|$$

$$= \left|\sum_{l=0}^{m}\binom{m}{l}\left(\frac{d}{d\phi}\right)^l (\theta^2 - \phi^2)^{m+\alpha-1/2}\left(\frac{d}{d\phi}\right)^{m-l}\Delta_{k,m}(\theta, \phi)\right|$$

$$\leqq (\theta^2 - \phi^2)^{\alpha-1/2}\sum_{l=0}^{m}\binom{m}{l}(m + \alpha - 1/2)^l(\theta^2 - \phi^2)^{m-l}(2\theta)^l$$

$$\times \left|\left(\frac{d}{d\phi}\right)^{m-l}\Delta_{k,m}(\theta, \phi)\right|.$$

The constant $M_{k,m}(\theta)$ should therefore be taken to be

$$(2.39)\quad M_{k,m}(\theta) = \sum_{l=0}^{m}\binom{m}{l}(2m + 2\alpha - 1)^l\theta^{m-l}$$

$$\times \max_{0 \leqq \phi \leqq \theta}\left|\left(\frac{d}{d\phi}\right)^{m-l}\Delta_{k,m}(\theta, \phi)\right|.$$

This completes the proof of Lemma 6.

**3. Proof of the theorem.** We are now ready to prove the main result of this paper. First, we replace ${}_2F_1$ in (2.1) by it series expansion (2.2). Thus

$$(3.1)\quad \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)}$$

$$= \frac{2^{(\alpha+\beta+1)/2}\Gamma(\alpha + 1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)}(1 - \cos\theta)^{-\alpha}(1 + \cos\theta)^{-(\alpha+\beta)/2}$$

$$\times \left[\sum_{k=0}^{m-1} b_k(\theta)I_k(\theta; N) + R_m(\theta; N)\right],$$

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where

$$(3.2) \quad b_k(\theta) = \frac{(-1)^k\left(\frac{\alpha+\beta}{2}\right)_k\left(\frac{\alpha-\beta}{2}\right)_k}{k!\left(\alpha+\frac{1}{2}\right)_k(1+\cos\theta)^k},$$

$$(3.3) \quad I_k(\theta; N) = \int_0^\theta (\cos\phi - \cos\theta)^{k+\alpha-1/2}\cos N\phi d\phi,$$

and $R_m(\theta; N)$ is given in (2.16). Next, we insert (2.29) in (3.3) and obtain

$$(3.4) \quad I_k(\theta; N) = \left(\frac{\sin\theta}{2\theta}\right)^{k+\alpha-1/2}\left[\sum_{\nu=0}^{m-k-1}\theta^{k+\nu+\alpha}\psi^*_{k,\nu}(\theta)\frac{J_{k+\nu+\alpha}(N\theta)}{N^{k+\nu+\alpha}} + S_{k,m}(\theta; N)\right],$$

where

$$(3.5) \quad \psi^*_{k,\nu}(\theta) = 2^{k+\alpha+\nu-1}\Gamma\left(\frac{1}{2}\right)\Gamma\left(k+\nu+\alpha+\frac{1}{2}\right)\psi_{k,\nu}(\theta)$$

and

$$(3.6) \quad S_{k,m}(\theta; N) = \int_0^\theta [(\theta^2-\phi^2)^{m+\alpha-1/2}\Delta_{k,m}(\theta; \phi)]\cos N\phi d\phi,$$

$\psi_{k,\nu}(\theta)$ being the coefficient given in (2.29). To the last integral, we shall apply repeated integration by parts, and show that for $k = 0, 1, \ldots, m-1$,

$$(3.7) \quad |S_{k,m}(\theta; N)| \leqq \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha+1)}\theta^{2\alpha+m}\frac{M_{k,m}(\theta)}{N^m},$$

where $M_{k,m}(\theta)$ has the same meaning as given in (2.30). To do this, we need the following two results:

$$(3.8) \quad \left(\frac{d}{d\phi}\right)^j[(\theta^2-\phi^2)^{m+\alpha-1/2}\Delta_{k,m}(\theta; \phi)]|_{\phi=0} = 0, \quad j \text{ odd}$$

and

$$(3.9) \quad \left(\frac{d}{d\phi}\right)^j[(\theta^2-\phi^2)^{m+\alpha-1/2}\Delta_{k,m}(\theta; \phi)]|_{\phi=\theta} = 0, \quad j = 0, 1, \ldots, m-1.$$

The first result follows from (2.29), (2.12) and the identity [**8**, Eq. 1, Section 0.432, p. 20]

$$(3.10)\quad \frac{d^n}{dx^n} F(x^2) = (2x)^n F^{(n)}(x^2)$$
$$+ \frac{n(n-1)}{1}(2x)^{n-2}F^{(n-1)}(x^2)$$
$$+ \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}F^{(n-2)}(x^2)$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!}$$
$$\times (2x)^{n-6}F^{(n-3)}(x^2) + \ldots .$$

with $x = \phi$ and $F(\phi^2) = (\theta^2 - \phi^2)^\lambda$, $\lambda$ being an arbitrary real number. The second result is a consequence of (2.38). Upon integration by parts $m$ times, (3.6) becomes

$$(3.11)\quad S_{k,m}(\theta; N)$$
$$= \frac{\pm 1}{N^m} \int_0^\theta \left\{ \begin{matrix} \sin \\ \cos \end{matrix} N\phi \right\} \left(\frac{d}{d\phi}\right)^m [\,(\theta^2 - \phi^2)^{m+\alpha-1/2}\Delta_{k,m}(\theta; \phi)\,]d\phi,$$

the integrated terms all vanishing due to (3.8) and (3.9). Coupling (2.30) and (3.11), we have

$$(3.12)\quad |S_{k,m}(\theta; N)| \leqq \frac{M_{k,m}(\theta)}{N^m}\theta^m \int_0^\theta (\theta^2 - \phi^2)^{\alpha-1/2}d\phi.$$

The estimate in (3.7) now follows from the definition of the Beta function [**10**, p. 37].

Finally, we substitute (3.4) in (3.1), and put

$$(3.13)\quad B_l(\theta) \equiv \sum_{k+\nu=l} b_k(\theta)\left(\frac{\sin\theta}{2\theta}\right)^k \psi^*_{k,\nu}(\theta).$$

The result is

$$(3.14)\quad P_n^{(\alpha,\beta)}(\cos\theta)$$
$$= \frac{\Gamma(n+\alpha+1)}{n!}\left(\sin\frac{\theta}{2}\right)^{-\alpha}\left(\cos\frac{\theta}{2}\right)^{-\beta}\left(\frac{\theta}{\sin\theta}\right)^{1/2}$$
$$\times \left[\sum_{l=0}^{m-1} A_l(\theta)\frac{J_{\alpha+l}(N\theta)}{N^{\alpha+l}} + E_m(\theta; N)\right],$$

where

$$(3.15)\quad A_l(\theta) = \frac{2^{1-\alpha}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}\theta^l B_l(\theta)$$

and

(3.16) $E_m(\theta; N)$

$$= \frac{2^{1-\alpha}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)} \theta^{-\alpha}\left[\sum_{k=0}^{m-1} b_k(\theta)\left(\frac{\sin\theta}{2\theta}\right)^k S_{k,m}(\theta; N) + \left(\frac{2\theta}{\sin\theta}\right)^{\alpha-1/2} R_m(\theta; N)\right].$$

Clearly, the functions $A_l(\theta)$ are analytic for $0 \leqq \theta < \pi$, and from (2.17), (2.28) and (3.7) it readily follows that

(3.17) $$E_m(\theta; N) = O(\theta^\alpha/N^m),$$

the $O$-symbol being independent of $\theta$ for all $\theta$ in $[0, \pi - \epsilon]$, $\epsilon > 0$. This completes the proof of the theorem.

The first three coefficients in expansion (1.3) are given by $A_0(\theta) = 1$,

(3.18) $$A_1(\theta) = \left(\alpha^2 - \frac{1}{4}\right)\frac{1 - \theta\cot\theta}{2\theta} - \frac{\alpha^2 - \beta^2}{4}\tan\frac{\theta}{2}$$

and

(3.19) $A_2(\theta)$

$$= \left(\alpha^2 - \frac{1}{4}\right)\left[\frac{1}{6}\left(\alpha + \frac{3}{2}\right)\left(\frac{3}{\theta 2} - 1 - \frac{3}{\theta}\cot\theta\right) + \frac{1}{8}\left(\alpha^2 - \frac{9}{4}\right)\left(\frac{1 - \theta\cot\theta}{\theta}\right)^2\right] - (\alpha^2 - \beta^2)$$
$$\times \left[\left(\alpha + \frac{3}{2}\right)\left(\alpha + \frac{1}{2}\right)\tan\frac{\theta}{2}\,\frac{1 - \theta\cot\theta}{8\theta} - \frac{1}{32}\left((\alpha - 2)^2 - \beta^2\right)\tan^2\frac{\theta}{2}\right].$$

**4. Proof of corollary 1.** Before proceeding, one further result is required.

LEMMA 7. *Put*

(4.1) $$\frac{2(\cos\phi - \cos\theta)}{\theta^2 - \phi^2}\,\frac{\theta}{\sin\theta} = 1 + \sigma(\theta, \phi).$$

*For* $0 \leqq \phi \leqq \theta \leqq \frac{\pi}{2}$, *we have*

(4.2) $$0 < \sigma(\theta, \phi) \leqq \frac{\pi}{24}(\theta^2 - \phi^2)$$

(4.3) $$-\frac{\pi}{12}\theta < \sigma_\phi(\theta, \phi) < 0$$

(4.4) $$-\frac{\pi}{12} < \sigma_{\phi\phi}(\theta, \phi) < 0$$

(4.5) $$-\frac{\pi}{360}\theta < \frac{d}{d\phi}\frac{\sigma(\theta, \phi)}{\theta^2 - \phi^2} < 0$$

(4.6) $$-\frac{\pi}{360} < \frac{d^2}{d\phi^2}\frac{\sigma(\theta, \phi)}{\theta^2 - \phi^2} < 0.$$

*Proof.* From

$$\cos\phi - \cos\theta = \frac{1}{2!}(\theta^2 - \phi^2) - \frac{1}{4!}(\theta^4 - \phi^4) + \frac{1}{6!}(\theta^6 - \phi^6) - + \dots,$$

we have

$$\frac{\cos\phi - \cos\theta}{\theta^2 - \phi^2} = \sum_{n=1}^{\infty} (-1)^{n+1}\rho_n,$$

where

$$\rho_n = \frac{1}{(2n)!}(\theta^{2(n-1)} + \theta^{2(n-2)}\phi^2 + \dots + \theta^2\phi^{2(n-2)} + \phi^{2(n-1)}).$$

Thus

$$\frac{2(\cos\phi - \cos\theta)}{\theta^2 - \phi^2} - \frac{\sin\theta}{\theta} = \sum_{n=1}^{\infty} (-1)^{n+1}\alpha_n,$$

where

$$\alpha_n = \frac{\theta^{2n}}{(2n+1)!} - \frac{\theta^{2n} + \theta^{2(n-1)}\phi^2 + \dots + \theta^2\phi^{2(n-1)} + \phi^{2n}}{(n+1)(2n+1)!}$$

This together with (4.1) gives

(4.7) $$\sigma(\theta, \phi) = \frac{\theta}{\sin\theta}\sum_{n=1}^{\infty} (-1)^{n+1}\alpha_n.$$

We wish to show that $\{\alpha_n\}$ is a monotonically decreasing sequence of non-negative numbers. To demonstrate the non-negativeness, we rewrite $\alpha_n$ in the form

$$\alpha_n = \frac{(\theta^2 - \phi^2)}{(2n + 1)!(n + 1)}\{\theta^{2(n-1)} + \theta^{2(n-2)}(\theta^2 + \phi^2) + \theta^{2(n-3)}(\theta^4 + \phi^2\theta^2 + \theta^4) + \ldots + (\theta^{2(n-1)} + \theta^{2(n-2)}\phi^2 + \ldots + \phi^{2(n-1)})\}.$$

It is clear that $\alpha_n \geqq 0$. To prove that $\alpha_n \geqq \alpha_{n+1}$, we write the last equation as

$$(4.8) \quad \alpha_n = \frac{(\theta^2 - \phi^2)}{(2n + 1)!(n + 1)}\{n\theta^{2(n-1)} + (n - 1)\theta^{2(n-2)}\phi^2 + (n - 2)\theta^{2(n-3)}\phi^4 + \ldots + 2\theta^2\phi^{2(n-2)} + \phi^{2(n-1)}\}.$$

Multiplying the right-hand side by $\theta^2 + \phi^2$ and dividing it by the same quantity gives

$$(4.9) \quad \alpha_n = \frac{\theta^2 - \phi^2}{(2n + 1)!(n + 1)(\theta^2 + \phi^2)} \times \{n\theta^{2n} + (2n - 1)\theta^{2(n-1)}\phi^2 + \ldots + 3\theta^2\phi^{2(n-1)} + \phi^{2n}\}.$$

We now replace $n$ by $n + 1$ in (4.8) and subtract the resulting expression from (4.9). Observe that

$$\theta^2 + \phi^2 \leqq 2\theta^2 < \frac{\pi^2}{2} \quad \text{and}$$

$$\pi^2 < 4n(n + 2)(2n + 3)/(n + 1) \quad \text{for } n = 1, 2, \ldots.$$

The non-negativeness of the differences $\alpha_n - \alpha_{n+1}$, $n = 2, 3, \ldots$, follows immediately upon comparing the coefficients of $\theta^{2n}$, $\theta^{2(n-1)}\phi^2, \ldots$, $\theta^2\phi^{2(n-1)}$ and $\phi^{2n}$. Therefore, we have from (4.7)

$$0 < \sigma(\theta, \phi) < \frac{\theta}{\sin\theta}\alpha_1.$$

Since

$$\alpha_1 = \frac{1}{12}(\theta^2 - \phi^2) \quad \text{and} \quad \frac{2}{\pi} \leqq \frac{\sin\theta}{\theta} \leqq 1,$$

the desired result (4.2) now follows.

To demonstrate (4.3), we first note that (4.7) gives

$$\sigma_\phi(\theta, \phi) = \frac{\theta}{\sin\theta}\sum_{n=1}^{\infty}(-1)^n\beta_n,$$

where $\beta_n = -d\alpha_n/d\phi$. From the equation preceding (4.7), we also have

$$\beta_n = \frac{1}{(n+1)(2n+1)!}[2\theta^{2(n-1)}\phi + \dots + (2l)\theta^{2(n-l)}\phi^{2l-1} + \dots + (2n)\phi^{2n-1}].$$

Hence, $\beta_n \geqq 0$ for all $n \geqq 1$. The fact that $\beta_n \geqq \beta_{n+1}$, $n \geqq 1$, is proved in exactly the same manner as before. Since $\beta_1 = \phi/6$, the results in (4.3) are now trivial.

The inequalities in (4.4) are verified in a similar fashion. That is, we first write

$$\sigma_{\phi\phi}(\theta, \phi) = \frac{\theta}{\sin\theta}\sum_{n=1}^{\infty}(-1)^n\gamma_n,$$

where

$$\gamma_n = \frac{1}{(n+1)(2n+1)!}[2\theta^{2(n-1)} + \dots + (2l)(2l-1)\theta^{2(n-l)}\phi^{2l-2} + \dots + (2n)(2n-1)\phi^{2n-2}],$$

and then show that $\gamma_n \geqq 0$ and $\gamma_n \geqq \gamma_{n+1}$.

To prove (4.5) and (4.6), we observe that coupling (4.7) and (4.8) gives

$$(4.10)\quad \frac{\sigma(\theta, \phi)}{\theta^2 - \phi^2} = \frac{\theta}{\sin\theta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!(n+1)}\{n\theta^{2(n-1)} + (n-1)\theta^{2(n-2)}\phi^2 + \dots + 2\theta^2\phi^{2(n-2)} + \phi^{2(n-1)}\}.$$

Upon termwise differentiation, we obtain

$$\frac{d}{d\phi}\frac{\sigma(\theta, \phi)}{\theta^2 - \phi^2} = \frac{\theta}{\sin\theta}\sum_{n=2}^{\infty}(-1)^{n+1}\lambda_n$$

and

$$\frac{d^2}{d\phi^2}\frac{\sigma(\theta, \phi)}{\theta^2 - \phi^2} = \frac{\theta}{\sin\theta}\sum_{n=2}^{\infty}(-1)^{n+1}\epsilon_n,$$

where $\lambda_n$ and $\epsilon_n$ are, respectively, the first and second partial derivatives (with respect to $\phi$) of the quantity inside the curly brackets in (4.10). Using the argument for $\alpha_n$ again, it can be shown that $\lambda_n, \epsilon_n \geqq 0$, $\lambda_n \geqq \lambda_{n+1}$ and $\epsilon_n \geqq \epsilon_{n+1}$ for $n = 2, 3, \dots$. This completes the proof of Lemma 7.

To prove Corollary 1, we need the values of $M_{0,2}\left(\frac{\pi}{2}\right)$ and $M_{1,2}\left(\frac{\pi}{2}\right)$; see equation (2.39). From (2.37) with $k = 0$ and $m = 2$, and from (4.1), we have

$$\begin{aligned}(\theta^2 - \phi^2)^2\Delta_{0,2}(\theta, \phi) &= [1 + \sigma(\theta, \phi)]^{\alpha - 1/2} - 1 \\ &\quad - \frac{1}{4}\left(\alpha - \frac{1}{2}\right)\frac{1 - \theta \cot \theta}{\theta^2}(\theta^2 - \phi^2) \\ &= \left(\alpha - \frac{1}{2}\right)\sigma(\theta, \phi) + \sigma^2(\theta, \phi)R_2(\sigma(\theta, \phi)) \\ &\quad - \frac{1}{4}\left(\alpha - \frac{1}{2}\right)\frac{1 - \theta \cot \theta}{\theta^2}(\theta^2 - \phi^2).\end{aligned}$$

The second equality follows from (2.4). By using (2.34) and the fact that

$$\phi_2(\theta) = (1 - \theta \cot \theta)/4\theta^2,$$

it is also true that

$$\begin{aligned}&(\theta^2 - \phi^2)^2\Delta_{0,2}(\phi, \theta) \\ &= \left(\alpha - \frac{1}{2}\right)(\theta^2 - \phi^2)^2 \sum_{\mu=0}^{\infty} \phi_{\mu+3}(\theta)(\theta^2 - \phi^2)^{\mu} \\ &+ \sigma^2(\theta, \phi)R_2(\sigma(\theta, \phi)).\end{aligned}$$

Thus

$$\begin{aligned}(4.11)\quad &\Delta_{0,2}(\theta, \phi) \\ &= \left(\alpha - \frac{1}{2}\right) \sum_{\mu=0}^{\infty} \phi_{\mu+3}(\theta)(\theta^2 - \phi^2)^{\mu} \\ &+ \left[\frac{\sigma(\theta, \phi)}{\theta^2 - \phi^2}\right]^2 R_2(\sigma(\theta, \phi)).\end{aligned}$$

We now digress briefly to discuss the Bessel function

$$J_\nu(\theta) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{1}{2}\theta\right)^{2m+\nu}}{m!\Gamma(m + \nu + 1)}.$$

For $0 \leqq \theta \leqq \frac{\pi}{2}$ and $\nu \geqq 0$, the terms in the series alternate in sign and are monotonically decreasing. Hence

$$0 \leqq J_\nu(\theta) \leqq \frac{\left(\frac{1}{2}\theta\right)^\nu}{\Gamma(\nu + 1)}, \quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

Furthermore, it is well-known that

$$J_{1/2}(\theta) = \sqrt{\frac{2}{\pi\theta}} \sin \theta$$

and

$$\frac{2}{\pi} \leqq \frac{\sin \theta}{\theta} \leqq 1, \quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

A combination of these results gives

$$(4.12) \quad 0 \leqq \phi_\nu(\theta) \leqq \frac{\pi^{3/2}}{2^{2\nu} \cdot \nu!\Gamma\left(\nu + \frac{1}{2}\right)}, \quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

We now restrict $\theta$ to the interval $\left[0, \frac{\pi}{2}\right]$, and apply (4.12) and (4.2) to (4.11). This, together with (2.5) with $a = -\alpha + \frac{1}{2}$, gives

$$(4.13) \quad |\Delta_{0,2}(\theta, \phi)| \leqq K_1\left|\alpha - \frac{1}{2}\right| + \frac{\pi^2}{1152}\left|\alpha - \frac{1}{2}\right| \left|\alpha - \frac{3}{2}\right|,$$

where

$$(4.14) \quad K_1 = \frac{\pi^{3/2}}{2^6} \sum_{\mu=0}^{\infty} \left(\frac{\pi}{4}\right)^{2\mu} \frac{1}{\Gamma\left(\mu + 3 + \frac{1}{2}\right)(\mu + 3)!} = 0.00456094.$$

Upon differentiating (4.11) once, we have by Lemma 7

$$(4.15) \quad \left|\frac{d}{d\phi}\Delta_{0,2}(\theta, \phi)\right| \leqq K_2 \left|\alpha - \frac{1}{2}\right| + \frac{\pi^3}{17280} \left|\alpha - \frac{1}{2}\right| \left|\alpha - \frac{3}{2}\right| + \frac{\pi^4}{82944} \left|\alpha - \frac{1}{2}\right| \left|\alpha - \frac{3}{2}\right| \left|\alpha - \frac{5}{2}\right|,$$

where

$$(4.16) \quad K_2 = \frac{\sqrt{\pi}}{2^4} \sum_{\mu=1}^{\infty} \left(\frac{\pi}{4}\right)^{2\mu} \frac{\mu}{(\mu + 3)!\Gamma\left(\mu + 3 + \frac{1}{2}\right)} = 0.00025858,$$

and upon differentiating (4.11) twice, we get

$$(4.17)\quad \left|\frac{d^2}{d\phi^2}\Delta_{0,2}(\theta,\phi)\right|$$
$$\leqq \left(K_3 + \frac{2}{\pi}K_2\right)\left|\alpha - \frac{1}{2}\right|$$
$$+ \left(\frac{\pi^2}{24\cdot 360} + \frac{\pi^4}{4\cdot(360)^2}\right)\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|$$
$$+ \left(\frac{\pi^3}{6\cdot 12\cdot(24)^2} + \frac{\pi^5}{36\cdot 48\cdot 360}\right)\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|\left|\alpha-\frac{5}{2}\right|$$
$$+ \frac{\pi^6}{(24)^4\cdot 12}\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|\left|\alpha-\frac{5}{2}\right|\left|\alpha-\frac{7}{2}\right|,$$

where

$$(4.18)\quad K_3 = \frac{1}{4\sqrt{\pi}}\sum_{\mu=2}^{\infty}\left(\frac{\pi}{4}\right)^{2\mu}\frac{\mu(\mu-1)}{(\mu+3)!\Gamma\left(\mu+3+\frac{1}{2}\right)} = 0.00001807.$$

Substituting (4.13), (4.15) and (4.17) in (2.39) yields

$$M_{0,2}\left(\frac{\pi}{2}\right) = (3+2\alpha)^2\left[K_1\left|\alpha-\frac{1}{2}\right| + \frac{\pi^2}{1152}\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|\right]$$
$$+ \pi(3+2\alpha)\left[K_2\left|\alpha-\frac{1}{2}\right| + \frac{\pi^3}{17280}\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|\right.$$
$$\left.+ \frac{\pi^4}{82944}\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|\left|\alpha-\frac{5}{2}\right|\right]$$
$$(4.19)\quad + \left(\frac{\pi}{2}\right)^2\left[\left(K_3 + \frac{2}{\pi}K_2\right)\left|\alpha-\frac{1}{2}\right| + \left(\frac{\pi^2}{24\cdot 360} + \frac{\pi^4}{4\cdot(360)^2}\right)\right.$$
$$\times\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|$$
$$+ \left(\frac{\pi^3}{6\cdot 12\cdot(24)^2} + \frac{\pi^5}{36\cdot 48\cdot 360}\right)\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|\left|\alpha-\frac{5}{2}\right|$$
$$\left.+ \frac{\pi^6}{12\cdot(24)^4}\left|\alpha-\frac{1}{2}\right|\left|\alpha-\frac{3}{2}\right|\left|\alpha-\frac{5}{2}\right|\left|\alpha-\frac{7}{2}\right|\right].$$

Similarly, from (2.37) with $k = 1$ and $m = 2$, and from (4.1), we have

$$(\theta^2 - \phi^2)\Delta_{1,2}(\theta, \phi) = [1 + \sigma(\theta, \phi)]^{\alpha+1/2} - 1$$
$$= \sigma(\theta, \phi)R_1(\sigma(\theta, \phi)).$$

The last equality follows again from (2.4). Thus

$$(4.20) \quad \Delta_{1,2}(\theta, \phi) = \frac{\sigma(\theta, \phi)}{\theta^2 - \phi^2}R_1(\sigma(\theta, \phi)),$$

from which we can calculate

$$d\Delta_{1,2}(\theta, \phi)/d\phi \quad \text{and} \quad d^2\Delta_{1,2}(\theta, \phi)/d\phi^2.$$

By Lemma 7 and (2.5) with $a = -\alpha - \frac{1}{2}$, we get

$$|\Delta_{1,2}(\theta, \phi)| \leqq \frac{\pi}{24}\left(\alpha + \frac{1}{2}\right),$$

$$\left|\frac{d}{d\phi}\Delta_{1,2}(\theta, \phi)\right| \leqq \frac{\pi^2}{720}\left(\alpha + \frac{1}{2}\right) + \frac{\pi^3}{2 \cdot (24)^2}\left|\alpha^2 - \frac{1}{4}\right|,$$

and

$$\left|\frac{d^2}{d\phi^2}\Delta_{1,2}(\theta, \phi)\right| \leqq \frac{\pi}{360}\left(\alpha + \frac{1}{2}\right)$$
$$+ \left(\frac{\pi^2}{(24)^2} + \frac{\pi^4}{48 \cdot 360}\right)\left|\alpha^2 - \frac{1}{4}\right|$$
$$+ \frac{\pi^5}{3 \cdot (24)^3}\left|\alpha^2 - \frac{1}{4}\right|\left|\alpha - \frac{3}{2}\right|.$$

Therefore, according to (2.39),

$$(4.21) \quad M_{1,2}\left(\frac{\pi}{2}\right)$$
$$= \frac{\pi}{24}(3 + 2\alpha)^2\left(\alpha + \frac{1}{2}\right) + \pi(3 + 2\alpha)\left[\frac{\pi^2}{720}\left(\alpha + \frac{1}{2}\right)\right.$$
$$\left. + \frac{\pi^3}{2 \cdot (24)^2}\left|\alpha^2 - \frac{1}{4}\right|\right] + \left(\frac{\pi}{2}\right)^2\left[\frac{\pi}{360}\left(\alpha + \frac{1}{2}\right)\right.$$
$$+ \left(\frac{\pi^2}{(24)^2} + \frac{\pi^4}{48 \cdot 360}\right)\left|\alpha^2 - \frac{1}{4}\right|$$
$$\left. + \frac{\pi^5}{3 \cdot (24)^3}\left|\alpha^2 - \frac{1}{4}\right|\left|\alpha - \frac{3}{2}\right|\right].$$

*Proof of Corollary* 1. Take $m = 2$ in (3.14) and recall (3.18). This gives the approximation in (1.5) with

$$(4.22)\quad \sigma_2 = \frac{2^{1-\alpha}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)}\theta^{-\alpha}\left[b_0(\theta)S_{0,2}(\theta, N) + b_1(\theta)\frac{\sin\theta}{2\theta}S_{1,2}(\theta; N) + \left(\frac{2\theta}{\sin\theta}\right)^{\alpha-1/2}R_2(\theta; N)\right],$$

where

$$b_0(\theta) = 1 \quad \text{and} \quad b_1(\theta) = (\alpha^2 - \beta^2)/4\left(\alpha + \frac{1}{2}\right)(1 + \cos\theta).$$

With $\theta$ restricted to $\left[0, \frac{\pi}{2}\right]$, we have from (3.7)

$$(4.23)\quad |S_{i,2}(\theta, N)| \leqq \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(\alpha + 1)}\theta^{2\alpha+2}\frac{M_{i,2}\left(\frac{\pi}{2}\right)}{N^2},$$

$i = 0, 1$, the values of $M_{i,2}\left(\frac{\pi}{2}\right)$ being given in (4.19) and (4.21). Furthermore, equation (2.20) gives

$$(4.24)\quad |R_2(\theta; N)| \leqq \frac{1}{N^2}\int_0^\theta \left|\frac{d^2}{d\phi^2}\{g_{2,\theta}(\phi)f_{2,\theta}(\phi)\}\right| d\phi.$$

Using Lemma 5 and (2.23), it is easily shown from the equation following (2.17) that for $0 \leqq \phi \leqq \theta \leqq \frac{\pi}{2}$,

$$|g_{2,\theta}(\phi)| \leqq \frac{\theta^4}{4}(\cos\phi - \cos\theta)^{\alpha-1/2}$$

$$|g'_{2,\theta}(\phi)| \leqq \left(\alpha + \frac{3}{2}\right)\frac{\theta^2}{2}(\cos\phi - \cos\theta)^{\alpha-1/2}$$

and

$$|g''_{2,\theta}(\phi)| \leqq \left(\alpha + \frac{3}{2}\right)(\alpha + 1)\theta^2(\cos\phi - \cos\theta)^{\alpha-1/2}.$$

Thus, from (2.9), it follows that

$$|R_2(\theta; N)| \leqq \frac{\theta^2}{N^2}\left\{\frac{\pi^2}{16}C_{2,2}\left(\frac{\pi}{2}\right) + \left(\alpha + \frac{3}{2}\right)C_{2,1}\left(\frac{\pi}{2}\right) + \left(\alpha + \frac{3}{2}\right)(\alpha + 1)C_{2,0}\left(\frac{\pi}{2}\right)\right\} \times \int_0^\theta (\cos\phi - \cos\theta)^{\alpha - 1/2} d\phi.$$

The inequality in (2.25) then implies

$$(4.25) \quad |R_2(\theta; N)| \leqq \frac{C_2}{N^2}\theta^{2+2\alpha}, \quad 0 \leqq \theta \leqq \frac{\pi}{2},$$

where

$$(4.26) \quad C_2 = \Gamma_1(\alpha)\left\{\frac{\pi^2}{16}C_{2,2}\left(\frac{\pi}{2}\right) + \left(\alpha + \frac{3}{2}\right)C_{2,1}\left(\frac{\pi}{2}\right) + \left(\alpha + \frac{3}{2}\right)\left(\alpha + \frac{1}{2}\right)C_{2,0}\left(\frac{\pi}{2}\right)\right\},$$

$\Gamma_1(\alpha)$ being given in (2.24). Note that the estimate in (4.25) is sharper than that given in (2.17) when $\theta$ is small. Coupling (4.23) and (4.25), we obtain the desired estimate in (1.7) with

$$(4.27) \quad E_2 = \frac{1}{2^\alpha\Gamma(\alpha + 1)}\left[M_{0,2}\left(\frac{\pi}{2}\right) + \frac{|\beta^2 - \alpha^2|}{8\left(\alpha + \frac{1}{2}\right)}M_{1,2}\left(\frac{\pi}{2}\right)\right] + C_2\frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)} \max_{0 \leqq \theta \leqq (\pi/2)} \left(\frac{\theta}{\sin\theta}\right)^{\alpha - 1/2}$$

This completes the proof of the corollary.

In the important special case $\alpha = \beta = 0$, the second term inside the square bracket in (4.27) obviously vanishes. In view of its definition (4.26), the constant $C_2$ in the third term on the right-hand side of (4.27) is also zero. Thus we have from (4.19)

$$(4.28) \quad E_2 = M_{0,2}\left(\frac{\pi}{2}\right) = 0.1253\ldots.$$

This gives the result stated in (1.9).

For comparison, let us also compute a bound for the first error term

$$(4.29)\quad E_1(\theta; N) = \frac{2^{1-\alpha}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)}\theta^{-\alpha}\left[S_{0,1}(\theta; N) + \left(\frac{2\theta}{\sin\theta}\right)^{\alpha-1/2} R_1(\theta; N)\right];$$

see (3.16). By (3.7) and (2.20), we have

$$(4.30)\quad |S_{0,1}(\theta; N)| \leqq \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(\alpha + 1)}\theta^{2\alpha+1}\frac{M_{0,1}(\theta)}{N},$$

and

$$(4.31)\quad |R_1(\theta; N)| \leqq \frac{1}{N}\int_0^\theta \left|\frac{d}{d\phi}\{g_{1,\theta}(\phi)f_{1,\theta}(\phi)\}\right| d\phi.$$

Using the same argument as for (4.25), we obtain

$$(4.32)\quad |R_1(\theta; N)| \leqq \frac{C_1}{N}\theta^{1+2\alpha}, \quad 0 \leqq \theta \leqq \frac{\pi}{2},$$

where

$$(4.33)\quad C_1 = \Gamma_1(\alpha)\left\{\left(\alpha + \frac{1}{2}\right)C_{1,0}\left(\frac{\pi}{2}\right) + \frac{\pi}{4}C_{1,1}\left(\frac{\pi}{2}\right)\right\}.$$

Coupling (4.30) and (4.32) gives

$$(4.34)\quad |E_1(\theta; N)| \leqq E_1\frac{\theta^{1+\alpha}}{N}$$

with

$$(4.35)\quad E_1 = \frac{1}{2^\alpha\Gamma(\alpha + 1)}M_{0,1}\left(\frac{\pi}{2}\right) + C_1\frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)}\max_{0\leqq\theta\leqq(\pi/2)}\left(\frac{\theta}{\sin\theta}\right)^{\alpha-1/2}.$$

If $\alpha = 0$, then the constant $C_1$ in (4.33) is zero, in which case (4.34) becomes

$$(4.36)\quad |E_1(\theta; N)| \leqq M_{0,1}\left(\frac{\pi}{2}\right)\frac{\theta}{N}.$$

Simple calculation shows

$$M_{0,1}\left(\frac{\pi}{2}\right) = 0.0869\ldots.$$

Now set $\alpha = \beta = 0$ and $m = 1$ in (3.14). The resulting expression together with (4.36) yields

$$(4.37)\quad P_n(\cos\theta) = \left(\frac{\theta}{\sin\theta}\right)^{1/2} J_0\left\{\left(n + \frac{1}{2}\right)\theta\right\} + \sigma,$$

where

$$(4.38)\quad |\sigma| \leqq 0.1089\frac{\theta}{n + 1/2}, \quad 0 \leqq \theta \leqq \frac{\pi}{2};$$

compare with [**13**, p. 242].

**5. Proof of corollary 2.** We first observe that the error term $\theta^{\alpha}O(N^{-m})$ in (1.3) can be replaced by $\theta^{\alpha+m}O(N^{-m})$ for $\theta$ in the interval $[0, \pi - \epsilon]$, $\epsilon > 0$. This assertion can be proved in the same manner as given in the proof of Corollary 1. Next we choose $m > \frac{1}{2}(\beta - \alpha)$ and $m \geqq 2 + \alpha$. In view of (3.15), the $l$-th term under the summation sign in (1.3) is $\theta^{l}O(N^{-l-\alpha})$. This together with the above refined estimate for the error term gives the asymptotic approximation in Corollary 1 without the condition $\alpha - \beta > -4$; the remainder $\sigma_2$ in (1.5) now, however, satisfies only the order estimate

$$(5.1)\quad \sigma_2 = \theta^2 O(N^{-2-\alpha}),$$

instead of the numerical bound given in (1.7).

*Proof of Corollary* 2. Let $\theta_l$ be the $l$-th zero of $P_n^{(\alpha,\beta)}(\cos\theta)$. It is well-known [**1**, p. 787] that

$$(5.2)\quad \lim_{n\to\infty} N\theta_l = j_{\alpha,l},$$

where $j_{\alpha,l}$ is the $l$-th positive zero of the Bessel function $J_{\alpha}(x)$. Put

$$(5.3)\quad \theta_l = \frac{j_{\alpha,l}}{N} + \delta_{\alpha,l}$$

and

$$(5.4)\quad t = \frac{j_{\alpha,l}}{N}.$$

Equation (5.2) implies

$$(5.5)\quad N\delta_{\alpha,l} = o(1), \quad \text{as } n \to \infty.$$

Now we expand $A_1(\theta)$ at $\theta = t$. Thus

$$A_1(\theta) = A_1(t) + A_1'(\xi)(\theta - t),$$

where $\xi$ is between $\theta$ and $t$. From (5.3) and (5.4), we have

$$A_1(\theta_l) = A_1(t) + O(\delta_{\alpha,l}). \tag{5.6}$$

Replacing $\theta$ by $\theta_l$ in (1.5) yields

$$0 = J_\alpha(j_{\alpha,l} + N\delta_{\alpha,l}) + \frac{1}{N}[A_1(t) + O(\delta_{\alpha,l})]J_{\alpha+1}(j_{\alpha,l} + N\delta_{\alpha,l}) + N^\alpha \sigma_2.$$

Taylor's theorem then gives

$$\begin{aligned} 0 &= J_\alpha(j_{\alpha,l}) + J_\alpha'(j_{\alpha,l})N\delta_{\alpha,l} + \frac{1}{2!}J_\alpha''(\eta_1)(N\delta_{\alpha,l})^2 \\ &+ \frac{1}{N}[A_1(t) + O(\delta_{\alpha,l})]\{J_{\alpha+1}(j_{\alpha,l}) + J_{\alpha+1}'(\eta_2)(N\delta_{\alpha,l})\} \\ &+ N^\alpha \sigma_2, \end{aligned} \tag{5.7}$$

where $\eta_1$ and $\eta_2$ are between $j_{\alpha,l}$ and $j_{\alpha,l} + N\delta_{\alpha,l}$. Note that the first term on the right-hand side of (5.7) is zero, and that by a recurrence relation,

$$J_{\alpha+1}(j_{\alpha,l}) = -J_\alpha'(j_{\alpha,l}).$$

Therefore, upon dividing by $NJ_\alpha'(j_{\alpha,l})$ on both sides of (5.7), we obtain

$$\begin{aligned} 0 &= \delta_{\alpha,l} + O(N\delta_{\alpha,l}^2) - \frac{1}{N^2}\left[A_1(t) + O(\delta_{\alpha,l})\right] \\ &+ [A_1(t) + O(\delta_{\alpha,l})]O\left(\frac{\delta_{\alpha,l}}{N}\right) + O\left(\frac{\sigma_2}{N^{1-\alpha}}\right). \end{aligned} \tag{5.8}$$

Now observe that in view of (5.5), the second term on the right-hand side is $o(\delta_{\alpha,l})$, and that the fourth term is of the same order. Furthermore, by (5.1), (5.2) and (5.3), the last term is $O(t^2/N^3)$. Hence the main balance in (5.8) suggests that $\delta_{\alpha,l} \sim A_1(t)/N^2$, i.e.,

$$\delta_{\alpha,l} = O\left(\frac{t}{N^2}\right). \tag{5.9}$$

Inserting (5.9) in (5.8), we have

$$\delta_{\alpha,l} = \frac{1}{N^2}A_1(t) + tO\left(\frac{1}{N^4}\right) + t^2O\left(\frac{1}{N^3}\right). \tag{5.10}$$

The final result (1.11) now follows from (5.3) and (5.10).

For a similar result concerning the zeros of the Legendre function $P_n^{-m}(\cos\theta)$, see [**10**, p. 469, Ex. 12.5].

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