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A SUFFICIENT CONDITION FOR NEVANLINNA PARAMETRIZATION AND AN EXTENSION OF HEINS THEOREM

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Abstract. An extended interpolation problem on a Riemann surface is formulated in terms of local rings and ideals. A sufficient condition for Nevanlinna parametrization is obtained. By means of this, Heins theorem on Pick-Nevanlinna interpolation in doubly connected domains is generalized to extended interpolation.

§1. Introduction

In order to consider extended interpolation problems on Riemann surfaces and to make simple the expressions concerning transformations, we wish at first to give a formulation of extended interpolation problems in terms of local rings and ideals.

Let X be a Riemann surface, i.e. a connected 1-dimensional complex manifold. For each $x \in X$, $\mathcal{O}_x$ denotes the ring of germs of holomorphic functions at x. Consider for each $x \in X$ a nonzero ideal $\mathcal{I}_x$ of $\mathcal{O}_x$ and an element $\mathfrak{c}_x$ of the quotient ring $\mathcal{O}_x/\mathcal{I}_x$. The collection $(\mathcal{I}, \mathfrak{c})$, where $\mathcal{I} = (\mathcal{I}_x)_{x\in X}$ and $\mathfrak{c} = (\mathfrak{c}_x)_{x\in X}$, will be called extended interpolation problem on X. Let $\mathcal{B}$ denote the set of all holomorphic functions f on X such that $|f| \leq 1$ on X. Our problem is to find a function $f \in \mathcal{B}$ which satisfies the condition

$$f_x + \mathcal{I}_x = \mathfrak{c}_x \qquad (\forall x \in X)\,, \tag{1.1}$$

where f_x is the germ at x represented by f and $f_x + \mathcal{I}_x$ is the coset of f_x modulo $\mathcal{I}_x$. Such a function will be called solution in $\mathcal{B}$ of the problem $(\mathcal{I}, \mathfrak{c})$. Let

$$\mathcal{E} = \{\, f \in \mathcal{B} \;:\; f \;\text{ satisfies (1.1)}\,\}$$

denote the set of solutions in $\mathcal{B}$ of the problem $(\mathcal{I}, \mathfrak{c})$. Some remarks will be added in §2.

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In §4, we suppose that X is biholomorphically equivalent to the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and that $\mathcal{E}$ has at least two elements. We showed in [14] that, under these hypotheses, there exists a Nevanlinna parametrization π of $\mathcal{E}$, which is by definition a bijection $\pi : \mathcal{B} \longrightarrow \mathcal{E}$ such that we have for any $g \in \mathcal{B}$

$$\pi(g) = \frac{Pg+Q}{Rg+S} \quad \text{with} \quad Rg+S \not\equiv 0 \text{ (not equal to the constant zero)},$$

where P, Q, R, and S are holomorphic functions on X, independent of g. The quadruple (P, Q, R, S) is said to represent π. In §4, under those hypotheses, we give a sufficient condition for a quadruple of the form $(P, Q, R, 1)$, where P, Q, and R are holomorphic on X, to represent a Nevanlinna parametrization of $\mathcal{E}$.

In §5, supposing X is as above, one considers an analytic automorphism T of X and a Möbius transformation U of the closed unit disc $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. In the case of classical Pick-Nevanlinna interpolation, where $\mathcal{I}_x$ is equal to the unique maximal ideal $\mathfrak{m}_\mathfrak{x}$ of $\mathcal{O}_x$ or to the whole ring $\mathcal{O}_x$, Heins showed in [5] that, under the consistency conditions, imposed on $\mathcal{I}$ and $\mathfrak{c}$, and the assumption $\mathcal{E} \neq \emptyset$, there exists a solution $f \in \mathcal{E}$ such that $f \circ T = U \circ f$. We generalize Heins theorem to our extended interpolation, based essentially on Heins' method. But, we use the result obtained in §4, which would make the proof transparent. The result of Heins on the classical interpolation in annuli in [5] is also generalized.

§2. Preliminaries

As in §1, consider an extended interpolation problem $(\mathcal{I}, \mathfrak{c})$ on a Riemann surface X. Let $\mathcal{B}$ and $\mathcal{E}$ be as in §1 and consider the set $\sigma = \{x \in X : \mathcal{I}_x \neq \mathcal{O}_x\}$.

We would like to give some remarks on the above formulation. If $\mathcal{I}_x = \mathcal{O}_x$ then $\mathcal{O}_x/\mathcal{I}_x$ and hence $\mathfrak{c}_x$ reduce to zeros, so that there is no requirements at x. Suppose $x \in \sigma$. Then, as $\{0\} \neq \mathcal{I}_x \neq \mathcal{O}_x$, there is a unique positive integer n_x such that $\mathcal{I}_x = \mathfrak{m}_\mathfrak{x}^{\mathfrak{n}_\mathfrak{x}}$, where $\mathfrak{m}_\mathfrak{x}$ is the maximal ideal of the local ring $\mathcal{O}_x$. Associating to x a local coordinate z such that $z(x) = 0$, we obtain

$$\mathfrak{c}_x = (c_0 + c_1 z + \cdots + c_{n_x-1} z^{n_x-1})_x + \mathcal{I}_x ,$$

where the n_x and the constants $c_0, c_1, \cdots, c_{n_x-1}$ are uniquely determined by $\mathfrak{c}_x$ and z. To give an extended interpolation problem $(\mathcal{I}, \mathfrak{c})$ is thus to give for each $x \in \sigma$ a positive integer n_x, a local coordinate z at x, and first n_x Taylor coefficients $c_0, c_1, \cdots, c_{n_x - 1}$ with respect to z.

For an arbitrary $\mathcal{I} = (\mathcal{I}_x)_{x \in X}$ and for any $f \in \mathcal{B}$, setting $\mathfrak{c}_x = f_x + \mathcal{I}_x$ for each $x \in X$, we have an extended interpolation problem $(\mathcal{I}, \mathfrak{c})$, which admits at least f as a solution in $\mathcal{B}$. In this case, σ may be arbitrary.

However, the case where σ is nonempty and has no limit points in X, i.e. σ is a discrete closed set of X, is of prime importance. For example, when $\mathcal{E}$ has at least two elements, σ has no limit points in X. In such a case, $\mathcal{I} = \bigcup_{x \in X} \mathcal{I}_x$ may be regarded as a coherent analytic subsheaf of the structure sheaf $\mathcal{O} = \bigcup_{x \in X} \mathcal{O}_x$ of X, $\mathfrak{c}$ as an element of the cohomology group $H^0(X, \mathcal{O}/\mathcal{I})$, and $f \in \mathcal{E}$ as an element of $H^0(X, \mathcal{O}) \cap \mathcal{B}$ which is mapped to $\mathfrak{c}$ by the canonical homomorphism $H^0(X, \mathcal{O}) \longrightarrow H^0(X, \mathcal{O}/\mathcal{I})$.

Let X, Y be Riemann surfaces and let $\varphi : X \longrightarrow Y$ be a holomorphic mapping. For each $x \in X$, setting $y = \varphi(x)$, we have the canonical ring homomorphism $\varphi_x^* : \mathcal{O}_y \longrightarrow \mathcal{O}_x$ defined by

$$\varphi_x^*(g_y) = (g \circ \varphi)_x ,$$

where g is a holomorphic function at y on Y. Moreover, when an ideal $\mathcal{I}_x$ of $\mathcal{O}_x$ and an ideal $\mathcal{I}_y$ of $\mathcal{O}_y$ are given in such a way that $\varphi_x^*(\mathcal{I}_y) \subset \mathcal{I}_x$, we have a canonical ring homomorphism $\mathcal{O}_y/\mathcal{I}_y \longrightarrow \mathcal{O}_x/\mathcal{I}_x$, which will be denoted by the same symbol φ_x^*.

In §5, we shall be concerned with a Möbius transformation of functions, so that we wish here to introduce a notation and to make some remarks. Consider a linear transformation U of the Riemann sphere:

$$U(w) = \frac{pw + q}{rw + s} \qquad (p, q, r, s \in \mathbb{C}\,;\, ps - qr \neq 0)\,.$$

Let $x \in X$ and let $\mathcal{I}_x$ be an ideal of $\mathcal{O}_x$ such that $\{0\} \neq \mathcal{I}_x \neq \mathcal{O}_x$. For an element $\mathfrak{c}_x \in \mathcal{O}_x/\mathcal{I}_x$, whenever $r\mathfrak{c}_x + s$ is a unit of the ring $\mathcal{O}_x/\mathcal{I}_x$, we can define $U_x(\mathfrak{c}_x) \in \mathcal{O}_x/\mathcal{I}_x$ by

$$U_x(\mathfrak{c}_x) = \frac{p\,\mathfrak{c}_x + q}{r\,\mathfrak{c}_x + s}\,,$$

where $p, q, r,$ and s are regarded as elements of $\mathcal{O}_x/\mathcal{I}_x$ represented by constant functions. If f is a holomorphic function in a neighborhood of x and if $\mathfrak{c}_x = f_x + \mathcal{I}_x$, then one sees immediately that $r\mathfrak{c}_x + s$ is a unit if and only if $rf(x) + s \neq 0$ and that, if so, we have

$$U_x(\mathfrak{c}_x) = \left(\frac{pf + q}{rf + s}\right)_x + \mathcal{I}_x\,.$$

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We would like to point out that, if U is a Möbius transformation, i.e. $U(\overline{D}) = \overline{D}$, and if we have $\mathfrak{c}_x = f_x + \mathcal{I}_x$ for some holomorphic function f in a neighborhood of x with $|f(x)| \le 1$, then we have $|r| < |s|$ and hence $r\mathfrak{c}_x + s$ is a unit, so that $U_x(\mathfrak{c}_x)$ can be defined.

For the holomorphic mapping $\varphi : X \longrightarrow Y$, let $y = \varphi(x)$ and suppose $\{0\} \ne \mathcal{I}_y \ne \mathcal{O}_y$ and $\varphi_x^*(\mathcal{I}_y) \subset \mathcal{I}_x$. Let $\mathfrak{c}_y \in \mathcal{O}_y/\mathcal{I}_y$. If $r\mathfrak{c}_y + s$ is a unit of $\mathcal{O}_y/\mathcal{I}_y$, then one sees at once that $r\varphi_x^*(\mathfrak{c}_y) + s$ is also a unit of $\mathcal{O}_x/\mathcal{I}_x$ and that we have

$$U_x(\varphi_x^*(\mathfrak{c}_y)) = \frac{p\,\varphi_x^*(\mathfrak{c}_y) + q}{r\,\varphi_x^*(\mathfrak{c}_y) + s} = \varphi_x^*\left(\frac{p\,\mathfrak{c}_y + q}{r\,\mathfrak{c}_y + s}\right) = \varphi_x^*(U_y(\mathfrak{c}_y))\,.$$

§3. Nevanlinna parametrizations

Based on Nevanlinna's method([9]), we defined and studied Nevanlinna parametrizations in [14]. In this section, we wish to restate, in terms of the formulation introduced in the previous sections, the definition and some results, obtained in [14] and needed below.

Let X be a simply connected Riemann surface of hyperbolic type; that is, X is biholomorphically equivalent to the open unit disc D. Let $(\mathcal{I}, \mathfrak{c})$, where $\mathcal{I} = (\mathcal{I}_x)_{x\in X}$ and $\mathfrak{c} = (\mathfrak{c}_x)_{x\in X}$, be an extended interpolation problem on X. Let $\mathcal{B}$ denote the set of all holomorphic functions f on X with $|f| \le 1$ on X and let $\mathcal{E}$ denote the set of all solutions in $\mathcal{B}$ of the problem $(\mathcal{I}, \mathfrak{c})$.

A Nevanlinna parametrization π is by definition a bijection $\pi : \mathcal{B} \longrightarrow \mathcal{E}$ represented by a quadruple (P, Q, R, S) of four holomorphic functions on X in such a way that

$$\pi(g) = \frac{Pg + Q}{Rg + S}\,, \qquad Rg + S \not\equiv 0 \qquad (\forall g \in \mathcal{B})\,.$$

It is well known ([9] and [14]) that $\mathcal{E}$ admits a Nevanlinna parametrization if and only if $\mathcal{E}$ has at least two elements. From now on in §3 as well as in §4, we assume that $\mathcal{E}$ has at least two elements and that the set $\sigma = \{x \in X : \mathcal{I}_x \ne \mathcal{O}_x\}$ is nonempty. Then σ has no limit points in X and there exists a Blaschke product B on X associated to $\mathcal{I} = (\mathcal{I}_x)_{x\in X}$. In fact, for two distinct elements f_0 and f_1 of $\mathcal{E}$, we see that the function $h = f_0 - f_1 (\not\equiv 0)$ is bounded on X, and that each $x \in \sigma$ is a zero of h of order at least n_x, where n_x is the positive integer such that $\mathcal{I}_x = \mathfrak{m}_{\mathfrak{x}}^{\mathfrak{n}_{\mathfrak{x}}}$. Take a biholomorphic mapping $\theta : X \longrightarrow D$ and write

$$B(x) = \prod_{x_i \in \sigma} \left(\lambda_i \frac{\theta(x) - z_i}{1 - \overline{z_i}\, \theta(x)} \right)^{n_i} ,$$

$$\lambda_i = \begin{cases} 1 & (\text{if } z_i = 0) \\ -|z_i|/z_i & (\text{if } z_i \neq 0) \end{cases} ,$$

where $z_i = \theta(x_i)$, $n_i = n_{x_i}$. It is not hard to see that B is a required Blaschke product on X.

If (P, Q, R, S) represents a Nevanlinna parametrization π of $\mathcal{E}$, then these functions have the following properties (Proposition 3 and Corollary 2 of [14]. See also Notes in §5 of [14]):

(a) $S \not\equiv 0$.
(b) $|P/S| < 1$, $|Q/S| < 1$, $|R/S| < 1$ on X.
(c) $Q/S \in \mathcal{E}$.
(d) We may write

$$\frac{PS - QR}{S^2} = U \cdot B ,$$

where U is a holomorphic function on X such that $0 < |U| \leq 1$ on X and B is a Blaschke product on X associated to $\mathcal{I}$.

By virtue of these properties, we may assume $S \equiv 1$. If two quadruples (P, Q, R, S) and $(\widehat{P}, \widehat{Q}, \widehat{R}, \widehat{S})$ represent the same π, then there exists a meromorphic function M on X such that $(\widehat{P}, \widehat{Q}, \widehat{R}, \widehat{S}) = M(P, Q, R, S)$. Hence, each Nevanlinna parametrization of $\mathcal{E}$ is represented by one and only one quadruple of the form $(P, Q, R, 1)$.

Let $\mathcal{P}$ denote the set of all Nevanlinna parametrizations of $\mathcal{E}$ and let G denote the group of all Möbius transformations

$$\tau(w) = \lambda \frac{w + a}{1 + \overline{a} w} \qquad (\lambda, a \in \mathbb{C}\,; |\lambda| = 1,\ |a| < 1) ,$$

regarded as analytic automorphisms of the closed unit disc $\overline{D} = D \bigcup \partial D$. The group G operates on $\mathcal{P}$ in the following way:

$$(\tau^*(\pi))(g) = \pi(\tau \circ g) \qquad\qquad (\tau \in G,\ \pi \in \mathcal{P},\ g \in \mathcal{B}) .$$

We obtained (Theorem 1 of [14])

THEOREM 1. *Let π, $\pi_0 \in \mathcal{P}$. Then there exists one and only one $\tau \in G$ such that $\pi = \tau^*(\pi_0)$.*

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For each $x \in X$, let

$$W(x) = \{f(x) : f \in \mathcal{E}\}$$

denote the set of values at x taken by all solutions $f \in \mathcal{E}$. We have $W(x) \subset D$. To each $\pi \in \mathcal{P}$ and to each $x \in X$, we associate the mapping $\pi_x : \overline{D} \longrightarrow W(x)$ defined by

$$\pi_x(\zeta) = \pi(\zeta)(x) \qquad (\zeta \in \overline{D}),$$

where ζ is regarded as a constant function. If π is represented by (P, Q, R, S), we have

$$\pi_x(\zeta) = \frac{P(x)\zeta + Q(x)}{R(x)\zeta + S(x)} \qquad (\zeta \in \overline{D}).$$

If $x \in X \setminus \sigma$, then π_x is bijective and $W(x)$ is a nondegenerate closed disc in D.

We showed (Lemma 1 of [14])

THEOREM 2. *Let $x \in X \setminus \sigma$, $w \in \partial W(x)$. Then there is a unique $f \in \mathcal{E}$ such that $f(x) = w$.*

About the extremal solutions, we established (Corollary 3 of [14])

THEOREM 3. *Assume that $\mathcal{E}$ has at least two elements. Let $\pi \in \mathcal{P}$ and $f \in \mathcal{E}$. The following three conditions are equivalent:*

(a) *$f(x) \in \partial W(x)$ for some $x \in X \setminus \sigma$.*

(b) *$f(x) \in \partial W(x)$ for all $x \in X \setminus \sigma$.*

(c) *There exists a $\zeta \in \partial D$ such that $f = \pi(\zeta)$.*

If one of these conditions is satisfied, ζ in (c) *is uniquely determined by π and f.*

§4. A sufficient condition for parametrization

Let $(\mathcal{I}, \mathfrak{c})$ be an extended interpolation problem on a simply connected Riemann surface X of hyperbolic type. Let $\mathcal{B}$, $\mathcal{E}$, and σ be as in §3 and assume $\mathcal{E}$ has at least two elements. Consider three holomorphic functions P, Q, and R on X. In this section, we want to give a sufficient condition for the quadruple $(P, Q, R, 1)$ to represent a Nevanlinna parametrization of $\mathcal{E}$. To this end, consider for each $x \in X$ two sets

$$\delta(x) = \left\{ \frac{P(x)\zeta + Q(x)}{R(x)\zeta + 1} : \zeta \in \overline{D}\,,\ R(x)\zeta + 1 \neq 0 \right\}$$

and

$$W(x) = \{f(x) : f \in \mathcal{E}\}\,.$$

We recall here well-known properties concerning linear transformations:

Proposition. *Let*

$$\tau(w) = \frac{pw + q}{rw + s} \qquad (p, q, r, s \in \mathbb{C}\,;\, ps - qr \neq 0)$$

be a linear transformation of the Riemann sphere. If $\tau(\overline{D}) \subset D$ *then we have* $|p| < |s|$, $|q| < |s|$, *and* $|r| < |s|$

We begin with the following

Lemma. *Suppose* $\delta(x) \subset W(x)$ *for all* $x \in X$.
(a) *Then we have*

$$\frac{Pg + Q}{Rg + 1} \in \mathcal{E} \qquad (\forall g \in \mathcal{B})\,.$$

(b) *Moreover, if there is a point* $x_0 \in X \setminus \sigma$ *such that* $\delta(x_0) = W(x_0)$, *then we have* $\delta(x) = W(x)$ *for all* $x \in X$.

First of all, we remark something about $P - QR$. If $P(x) - Q(x)R(x) \neq 0$ for a point $x \in X \setminus \sigma$, then, by the assumption $\delta(x) \subset W(x) \subset D$ and by the above proposition, we see easily $|R(x)| < 1$. Thus, if $P - QR \not\equiv 0$, then we have $|R| < 1$ on X and the condition $R(x)\zeta + 1 \neq 0$ in the definition of $\delta(x)$ can be omitted. If $P - QR \equiv 0$, then $\delta(x)$ degenerates to a sole point $Q(x)$ for any $x \in X$ and, in (a) of the lemma, we may set $(Pg + Q)/(Rg + 1) = Q$ for any $g \in \mathcal{B}$, even in the case $Rg + 1 \equiv 0$.

Proof. To see (a), take a Nevanlinna parametrization $\widehat{\pi}$ of $\mathcal{E}$ represented by a quadruple $(\widehat{P}, \widehat{Q}, \widehat{R}, 1)$ and let $g \in \mathcal{B}$. In any case, $f = (Pg + Q)/(Rg + 1)$ is holomorphic on X and we have $f(x) \in \delta(x) \subset W(x)$ for any $x \in X$.

Since, for each $x \in X \setminus \sigma$, the mapping $\widehat{\pi}_x : \overline{D} \longrightarrow W(x)$, defined in §3, is bijective, the function

$$\widehat{g}(x) = \widehat{\pi}_x^{-1}(f(x)) = \frac{f(x) - \widehat{Q}(x)}{-\widehat{R}(x)f(x) + \widehat{P}(x)}$$

is well-defined and holomorphic on $X \setminus \sigma$ and satisfies $|\widehat{g}| \leq 1$ there. Hence $\widehat{g}$ may be regarded as a function in $\mathcal{B}$. Clearly $f = \widehat{\pi}(\widehat{g}) \in \mathcal{E}$.

To see (b), let $x \in X \setminus \sigma$ and $w \in \partial W(x)$. Note that $P - QR \not\equiv 0$ since $\delta(x_0) = W(x_0)$ is nondegenerate. By Theorem 2, there is a unique $f_0 \in \mathcal{E}$ such that $f_0(x) = w$ and, by Theorem 3, we have $f_0(x_0) \in \partial W(x_0) = \partial\delta(x_0)$. Therefore, there exists a unique $\zeta_0 \in \partial D$ such that

$$f_0(x_0) = \frac{P(x_0)\zeta_0 + Q(x_0)}{R(x_0)\zeta_0 + 1}.$$

The function $f = (P\zeta_0 + Q)/(R\zeta_0 + 1)$ belongs to $\mathcal{E}$ by (a). As $f(x_0) = f_0(x_0)$, Theorem 2 yields $f = f_0$ and $w = f_0(x) = f(x) \in \delta(x)$. This implies $\delta(x) = W(x)$. For $x \in \sigma$, $W(x)$ and hence $\delta(x)$ consist of the same single point. □

We proceed to present a sufficient condition for parametrization.

THEOREM 4. *If we have $\delta(x) \subset W(x)$ for any $x \in X \setminus \sigma$ and $\delta(x_0) = W(x_0)$ for at least one $x_0 \in X \setminus \sigma$, then the quadruple $(P, Q, R, 1)$ represents a Nevanlinna parametrization of $\mathcal{E}$.*

Proof. Note that we see $P - QR \not\equiv 0$ as in (b) of Lemma. For each $g \in \mathcal{B}$, we have $(Pg + Q)/(Rg + 1) \in \mathcal{E}$ by (a) of Lemma. Conversely, let $f \in \mathcal{E}$. By (b) of Lemma, $\delta(x) = W(x)$ for all $x \in X$. Hence, for any $x \in X \setminus \sigma$, the linear mapping $\pi_x : \overline{D} \longrightarrow W(x)$ defined by

$$\pi_x(\zeta) = \frac{P(x)\zeta + Q(x)}{R(x)\zeta + 1} \qquad (\zeta \in \overline{D})$$

is bijective and we have $P(x) - Q(x)R(x) \neq 0$. The function

$$g(x) = \pi_x^{-1}(f(x)) = \frac{f(x) - Q(x)}{-R(x)f(x) + P(x)}$$

is then holomorphic in $X \setminus \sigma$ and satisfies $|g| \leq 1$ there. The function g may be regarded as a function in $\mathcal{B}$. Clearly $f = (Pg + Q)/(Rg + 1)$. This shows that $(P, Q, R, 1)$ represents a Nevanlinna parametrization $\pi : \mathcal{B} \longrightarrow \mathcal{E}$ with its inverse $\pi^{-1} : \mathcal{E} \longrightarrow \mathcal{B}$ given by

$$\pi^{-1}(f) = \frac{f - Q}{-Rf + P} \qquad (f \in \mathcal{E}).$$

$\square$

Here, we would like to observe that there are many quadruples $(P, Q, R, 1)$ satisfying the condition $\delta(x) \subset W(x)$ for any $x \in X$ but representing no Nevanlinna parametrization of $\mathcal{E}$. Let $(P_0, Q_0, R_0, 1)$ be a quadruple representing a Nevanlinna parametrization of $\mathcal{E}$ and let $h \in \mathcal{B}$. Set $(P, Q, R, 1) = (P_0 h, Q_0, R_0 h, 1)$. For these functions P, Q, and R holomorphic on X and for $x \in X$, consider the set

$$\begin{aligned}\delta(x) &= \left\{ \frac{P(x)\zeta + Q(x)}{R(x)\zeta + 1} : \zeta \in \overline{D} \right\} \\ &= \left\{ \frac{P_0(x)h(x)\zeta + Q_0(x)}{R_0(x)h(x)\zeta + 1} : \zeta \in \overline{D} \right\}.\end{aligned}$$

Taking account of $|R(x)\zeta| < 1$ for $\zeta \in \overline{D}$, we see at once that $\delta(x) \subset W(x)$ for any $x \in X$ and that, for any $x \in X \setminus \sigma$, $\delta(x) = W(x)$ if and only if $|h(x)| = 1$. By $h \in \mathcal{B}$ and by Theorem 4, $(P, Q, R, 1)$ represents a Nevanlinna parametrization of $\mathcal{E}$ if and only if h is a constant with $|h| = 1$. This affirms the required assertion, since there are many $h \in \mathcal{B}$ such that $|h| < 1$ on X. Finally, note that these functions P, Q, and R with $S \equiv 1$ satisfy evidently the properties (a), (b), (c) and (d) indicated in the previous §3 if h has no zeros on X. Thus, these properties give no sufficient condition for Nevanlinna parametrization.

§5. An extension of Heins theorem

In this section, let us extend Heins theorem to our extended interpolation problem.

Let $(\mathcal{I}, \mathfrak{c})$, where $\mathcal{I} = (\mathcal{I}_x)_{x \in X}$ and $\mathfrak{c} = (\mathfrak{c}_x)_{x \in X}$, be an extended interpolation problem on a simply connected Riemann surface X of hyperbolic type. Let $\mathcal{B}, \mathcal{E}$, and σ be as in the preceding section. We assume always $\sigma \neq \emptyset$. On the other hand, consider an analytic automorphism T of X and a Möbius transformation U:

$$U(w) = \lambda \frac{w + a}{1 + \overline{a}w} \qquad (\lambda, a \in \mathbb{C}\,; |\lambda| = 1,\ |a| < 1).$$

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Our present problem is to find a solution $f \in \mathcal{B}$ satisfying the interpolation condition

$$f_x + \mathcal{I}_x = \mathfrak{c}_x \qquad (\forall x \in X)$$

and moreover the condition

$$f \circ T = U \circ f .$$

For this purpose, it would be normal to impose on $(\mathcal{I}, \mathfrak{c})$ the following consistency conditions (5.1) and (5.2) and moreover the hypothesis of existence $\mathcal{E} \neq \emptyset$:

$$T_x^*(\mathcal{I}_{T(x)}) = \mathcal{I}_x \qquad (\forall x \in X)\ ; \tag{5.1}$$

$$T_x^*(\mathfrak{c}_{T(x)}) = U_x(\mathfrak{c}_x) \qquad (\forall x \in \sigma)\ . \tag{5.2}$$

Note that, under the hypothesis $\mathcal{E} \neq \emptyset$, $U_x(\mathfrak{c}_x)$ is defined for each $x \in \sigma$, as was pointed out in §2, because we have $\{0\} \neq \mathcal{I}_x \neq \mathcal{O}_x$ and $\mathfrak{c}_x = f_x + \mathcal{I}_x$ for some $f \in \mathcal{B}$, which implies that we have $1 + \overline{a} f(x) \neq 0$ and hence $1 + \overline{a}\mathfrak{c}_x$ is a unit of $\mathcal{O}_x/\mathcal{I}_x$.

We are now ready to establish

Theorem 5. *Suppose $\mathcal{E} \neq \emptyset$ and that the consistency conditions* (5.1) *and* (5.2) *are fulfilled. Then, there exists a function $f \in \mathcal{E}$ satisfying*

$$f \circ T = U \circ f . \tag{5.3}$$

Proof. Let $f \in \mathcal{E}$. We claim that $U^{-1} \circ f \circ T \in \mathcal{E}$ and $U \circ f \circ T^{-1} \in \mathcal{E}$. Clearly, $U^{-1} \circ f \circ T \in \mathcal{B}$ and $U \circ f \circ T^{-1} \in \mathcal{B}$. For any $x \in \sigma$, we have $(U^{-1} \circ f \circ T)_x + \mathcal{I}_x = (U^{-1})_x((f \circ T)_x + \mathcal{I}_x) = U_x^{-1}(T_x^*(f_{T(x)} + \mathcal{I}_{T(x)})) = U_x^{-1}(T_x^*(\mathfrak{c}_{T(x)})) = \mathfrak{c}_x$ and $(U \circ f \circ T^{-1})_{T(x)} + \mathcal{I}_{T(x)} = (T^{-1})^*_{T(x)}((U \circ f)_x + \mathcal{I}_x) = (T_x^*)^{-1}(U_x(\mathfrak{c}_x)) = \mathfrak{c}_{T(x)}$, by definition and by (5.1) and (5.2). For any $x \in X \setminus \sigma$, we have to say nothing.

In the case where $\mathcal{E}$ has only one element f, we have $f = U^{-1} \circ f \circ T$, which proves the theorem.

Now, assume $\mathcal{E}$ has at least two elements. Then, as was mentioned in §3, we may take a Nevanlinna parametrization π of $\mathcal{E}$ represented by a quadruple (P, Q, R, S). Consider for each $x \in X$ the set $W(x) = \{f(x) : f \in \mathcal{E}\}$. We claim that

$$W(x) = U^{-1}(W(T(x))) \qquad (\forall x \in X)\,. \tag{5.4}$$

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In fact, if $f \in \mathcal{E}$ and $x \in X$, then $f(T(x)) = U((U^{-1} \circ f \circ T)(x)) \in U(W(x))$ and $U(f(x)) = (U \circ f \circ T^{-1})(T(x)) \in W(T(x))$. This shows $W(T(x)) = U(W(x))$ and hence (5.4) holds.

Keeping in mind the expression

$$U^{-1}(w) = \frac{w - \lambda a}{\lambda - \overline{a}w}$$

and defining four holomorphic functions P_0, Q_0, R_0, and S_0 on X by

$$\begin{bmatrix} P_0 & Q_0 \\ R_0 & S_0 \end{bmatrix} = \begin{bmatrix} 1 & -\lambda a \\ -\overline{a} & \lambda \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

we can write by (5.4)

$$\begin{aligned} W(x) &= U^{-1}\left(\left\{\frac{P(T(x))\zeta + Q(T(x))}{R(T(x))\zeta + S(T(x))} : \zeta \in \overline{D}\right\}\right) \\ &= \left\{\frac{P_0(T(x))\zeta + Q_0(T(x))}{R_0(T(x))\zeta + S_0(T(x))} : \zeta \in \overline{D}\right\}, \end{aligned}$$

which is in D.

By Proposition in §4, the functions

$$\widehat{P}(x) = \frac{P_0(T(x))}{S_0(T(x))}, \qquad \widehat{Q}(x) = \frac{Q_0(T(x))}{S_0(T(x))}, \qquad \widehat{R}(x) = \frac{R_0(T(x))}{S_0(T(x))}$$

are holomorphic and verify $|\widehat{R}| < 1$, first in $X \setminus \sigma$ and hence on X entirely. From

$$W(x) = \left\{\frac{\widehat{P}(x)\zeta + \widehat{Q}(x)}{\widehat{R}(x)\zeta + 1} : \zeta \in \overline{D}\right\} \qquad (\forall x \in X)$$

and from Theorem 4 in §4, it follows that $(\widehat{P}, \widehat{Q}, \widehat{R}, 1)$ represents a Nevanlinna parametrization $\widehat{\pi}$ of $\mathcal{E}$. By Theorem 1, we find a unique Möbius transformation $\widehat{U}$ such that

$$\pi(g) = \widehat{\pi}(\widehat{U} \circ g) \qquad (\forall g \in \mathcal{B}).$$

Let us see that, for $g \in \mathcal{B}$, $f = \pi(g)$ satisfies $f \circ T = U \circ f$ if and only if g satisfies the condition

$$g \circ T = \widehat{U} \circ g. \tag{5.5}$$

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In fact, for $x \in X$, we have by definition

$$(U^{-1} \circ \pi(g) \circ T)(x) = U^{-1} \left(\frac{P(T(x))\, g(T(x)) + Q(T(x))}{R(T(x))\, g(T(x)) + S(T(x))} \right)$$

$$= \frac{\widehat{P}(x)\, g(T(x)) + \widehat{Q}(x)}{\widehat{R}(x)\, g(T(x)) + 1} = (\widehat{\pi}(g \circ T))(x) = \pi(\widehat{U}^{-1} \circ g \circ T)(x).$$

Therefore $f = \pi(g) = U^{-1} \circ f \circ T$ if and only if $g = \widehat{U}^{-1} \circ g \circ T$, which shows the assertion.

The theory of linear transformations permits us to take a fixed point ζ of $\widehat{U}$ in $\overline{D}$. When g is the constant ζ, g satisfies evidently (5.5) and hence $f = \pi(g) \in \mathcal{E}$ satisfies the condition (5.3). The theorem is established. □

We wish to conclude this paper with applying Theorem 5 to the extended interpolation problem on the annuli, just as Heins did in [5] for the Pick-Nevanlinna interpolation problem.

Let X be a doubly connected Riemann surface biholomorphically equivalent to an annulus $\{z \in \mathbb{C} : 1 < |z| < \rho\}$ $(1 < \rho < \infty)$ and let $(\mathcal{I}, \mathfrak{c})$, where $\mathcal{I} = (\mathcal{I}_x)_{x \in X}$ and $\mathfrak{c} = (\mathfrak{c}_x)_{x \in X}$, be an extended interpolation problem on X. Choose a holomorphic mapping $\varphi : D \longrightarrow X$ of the open unit disc D onto X such that (D, φ) is a universal covering of X. For each $z \in D$, setting

$$\widetilde{\mathcal{I}}_z = \varphi_z^*(\mathcal{I}_{\varphi(z)})\,, \quad \widetilde{\mathcal{I}} = (\widetilde{\mathcal{I}}_z)_{z \in D} \quad \text{and} \quad \widetilde{\mathfrak{c}}_z = \varphi_z^*(\mathfrak{c}_{\varphi(z)})\,, \quad \widetilde{\mathfrak{c}} = (\widetilde{\mathfrak{c}}_z)_{z \in D}\,,$$

we have an extended interpolation problem $(\widetilde{\mathcal{I}}, \widetilde{\mathfrak{c}})$ on D.

Corollary. *A necessary and sufficient condition that there exists a holomorphic function f on X satisfying*

$$|f| \le 1 \quad \textit{on } X \qquad \textit{and} \qquad f_x + \mathcal{I}_x = \mathfrak{c}_x \quad (\forall x \in X)$$

is that there exists a holomorphic function $\widetilde{f}$ on D satisfying

$$|\widetilde{f}| \le 1 \quad \textit{on } D \qquad \textit{and} \qquad \widetilde{f}_z + \widetilde{\mathcal{I}}_z = \widetilde{\mathfrak{c}}_z \quad (\forall z \in D)\,.$$

In fact, if f is such a function on X, then it is not hard to see that $\widetilde{f} = f \circ \varphi$ satisfies the second conditions. Conversely, let $\widetilde{f}$ be such a function on D. In Theorem 5, let $U : \overline{D} \longrightarrow \overline{D}$ be the identity, replace X by D, and let $T : D \longrightarrow D$ be a generator of the covering transformation group of (D, φ). We have $T_z^*(\widetilde{\mathcal{I}}_{T(z)}) = T_z^*(\varphi^*_{T(z)})(\mathcal{I}_{\varphi(T(z))}) = (\varphi \circ T)^*_z(\mathcal{I}_{(\varphi\circ T)(z)}) = \varphi^*_z(\mathcal{I}_{\varphi(z)}) = \widetilde{\mathcal{I}}_z$ and $T_z^*(\widetilde{\mathfrak{c}}_{T(z)}) = T_z^*(\varphi^*_{T(z)})(\mathfrak{c}_{\varphi(T(z))}) = \varphi^*_z(\mathfrak{c}_{\varphi(z)}) = \widetilde{\mathfrak{c}}_z$. As the consistency conditions are thus verified, we may assume $\widetilde{f} \circ T = \widetilde{f}$ by Theorem 5. Then there exists a holomorphic function f on X such that $f \circ \varphi = \widetilde{f}$. Clearly $|f| \le 1$ on X. As φ_z is isomorphic for any $z \in D$, we see at once $f_x + \mathcal{I}_x = \mathfrak{c}_x$ $(\forall x \in X)$. $\square$

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