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A NOTE ON STRONG MARKUŠEVIČ DECOMPOSITIONS OF BANACH SPACES

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The space ℓ^∞ is known to have no Schauder decomposition. It is proved here that ℓ^∞ does not even possess any strong Markuševič decomposition.

1. INTRODUCTION

A sequence (G_n) of non-zero linear subspaces of a Banach space E is said to be a decomposition of E if for each $x \in E$ there exists a unique sequence (x_n) with $x_n \in G_n$ $(n = 1, 2, \ldots)$ such that the series $\sum_{n=1}^{\infty} x_n$ converges to x. This gives a unique sequence (v_n) of linear projections on E satisfying $v_i.v_j = 0$, whenever $i \neq j$. If each v_n is continuous then (G_n) is said to be a Schauder decomposition. It is known that every infinite dimensional Banach space has a decomposition and that every decomposition may not be Schauder ([9], Theorem 1 and Example 2) so much so, that every Banach space need not possess a Schauder decomposition. In fact, the space ℓ^∞ does not possess any Schauder decomposition (Dean [2]). A detailed account of the theory of Schauder decompositions can be found in [10]. A more general concept than that of Schauder decompositions, namely Markuševič decompositions was introduced under the name "complete biorthogonal decompositions" in [1]. These decompositions together with a particular class of them, called the strong Markuševič decompositions, have been discussed in detail by the authors in [5, 6]. It is natural to ask whether ℓ^∞ possesses a Markuševič decomposition. In the present note, we establish that ℓ^∞ does not have any strong Markuševič decomposition, countable or uncountable. The proof of our result is a little cumbersome and makes use of certain ideas developed by Lindenstrauss [7].

2. MAIN RESULT

DEFINITION 1: An indexed collection $(G_\lambda)_{\lambda\in\Lambda}$ of non-zero closed linear subspaces of a Banach space E is said to be a Markuševič decomposition (M-decomposition)

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of E if there exist bounded linear projections $(v_\lambda)_{\lambda\in\Lambda}$ with $v_\lambda(E) = G_\lambda$ such that $\overline{\text{span}} \bigcup_{\lambda\in\Lambda} G_\lambda$ is dense in E and $v_\lambda(x) = 0(\lambda \in \Lambda)$, imply $x = 0$.

It has been shown in [5] that the collection $(v_\lambda)_{\lambda\in\Lambda}$, called the associated family of coordinate projections, is uniquely determined by the M-decomposition and that every weakly compactly generated Banach space admits of a countable M-decomposition. Note that ℓ^∞ being the non-separable dual of a separable Banach space cannot be contained in any weakly compactly generated Banach space.

Definition 2: An M-decomposition $(G_\lambda)_{\lambda\in\Lambda}$ of a Banach space E with the associated family of coordinate projections $(v_\lambda)_{\lambda\in\Lambda}$ is said to be a strong M-decomposition if each $x \in \overline{\text{span}}_{\lambda\in\Lambda(x)} G_\lambda$, where $\Lambda(x) = \{\lambda \in \Lambda : v_\lambda(x) \neq 0\}$.

Clearly every Schauder decomposition is a countable strong M-decomposition. Every M-decomposition may not be a strong M-decomposition and a countable strong M-decomposition may not be a Schauder decomposition [6]. In the present discussion we shall write

$$\sigma(x) = \{n : \alpha_n \neq 0\}, \quad (x = (\alpha_n) \in \ell^\infty).$$

Theorem. *The space ℓ^∞ has no countable or uncountable strong M-decomposition.*

Proof: Write $E = \ell^\infty$. Let $(G_\lambda)_{\lambda\in\Lambda}$ be a strong M-decomposition of E with the associated family of projections $(v_\lambda)_{\lambda\in\Lambda}$. Define seminorms on E by

$$t_\lambda(x) = \|v_\lambda\|^{-1} \|v_\lambda(x)\|, \quad (\lambda \in \Lambda).$$

Since $(v_\lambda)_{\lambda\in\Lambda}$ is total on E, the set

$$(x, \varepsilon) = \{\lambda \in \Lambda : t_\lambda(x) > \varepsilon\}$$

is finite for each $x \in E$ and $\varepsilon > 0$. Without any loss of generality we may assume that $\mathbb{N}$ and Λ are disjoint sets. Let $\Delta = \{0\} \cup \mathbb{N} \cup \Lambda$. Define a mapping $Q_0 : E \to c_0(\Delta)$ by

$$(Q_0 x)(\delta) = \begin{cases} 2^{-n}(1+n)^2 \|x\|, & (n = \delta/2 \text{ and } \delta = 0, 2, 4 \ldots), \\ \alpha_n/n, & (n = (\delta+1)/2, \quad \delta = 1, 3, 5, \ldots \text{ and } x = (\alpha_i)), \\ t_\delta(x), & (\delta \in \Lambda). \end{cases}$$

Let J denote the Day's locally uniformly convex norm as shown in [8] and $\|.\|$, the usual sup norm on $c_0(\Delta)$. Then

$$\|x\|/2 \leqslant J(x) \leqslant \|x\|/\sqrt{3}, \quad (x \in c_0(\Delta)).$$

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The function on E given by

$$|||x|||_0 = 2J(Q_0 x)$$

defines a norm on E ([11], Lemma) and there is a $K > 1$ such that

$$\|x\| \leqslant |||x|||_0 \leqslant K\,\|x\|\,, \quad (x \in E).$$

Write $M = \sup\{|||x|||_0 : \|x\| = 1\}$. Since $(3M+1)/4 < M$, there is an $x_1 = \left(\alpha_i^{(1)}\right) \in E$ with $\|x_1\| = 1$ such that

$$(3M+1)/4 \leqslant |||x_1|||_0\,.$$

We may assume that $\mathbb{N} \setminus \sigma(x_1)$ is infinite. Let S_1 be an infinite subset of $\mathbb{N} \setminus \sigma(x_1)$ such that $\mathbb{N} \setminus (\sigma(x) \cup S_1)$ is infinite and let i_1 be an integer in $\mathbb{N} \setminus (\sigma(x_1) \cup S_1)$. Writing

$$F_1 = \Big\{ y = (\beta_i) \in E : \|y\| = 1, \quad |\beta_{i_1}| = 1, \quad \beta_i = \alpha_i^{(1)}, \\ \text{for all } i \in \sigma(x) \cup S_1 \text{ and } \mathbb{N} \setminus (\sigma(y) \cup \sigma(x_1) \cup S_1) \text{ is infinite}\}$$

and $K_1 = \sup\{|||y|||_0 : y \in F_1\}$ and since for each $y \in F_1$, $2x_1 - y \in F_1$, we have $\|2x_1\| - 1 \leqslant K_1$. This gives

$$K_1 - 1 \leqslant ((4K-3)M - 1)/2K \leqslant (M-1)/2.$$

Again, since $(3K_1 + 1)/4 < K_1$, there is an $x_2 = \left(\alpha_i^{(2)}\right) \in E$ with $\|x_2\| = 1$ such that

$$(3K_1 - 1)/4 \leqslant |||x|||_0\,.$$

Again, we may assume without any loss of generality that $\mathbb{N} \setminus (\sigma(x_1) \cup \sigma(x_2) \cup S_1)$ is infinite. Let S_2 be an infinite subset of $\mathbb{N} \setminus (\sigma(x_1) \cup \sigma(x_2) \cup S_1)$ such that the set $\mathbb{N} \setminus (\sigma(x_1) \cup \sigma(x_2) \cup S_1 \cup S_2)$ is infinite and let $i_2 \in \mathbb{N} \setminus (\sigma(x_1) \cup \sigma(x_2) \cup S_1 \cup S_2)$. Writing

$$F_2 = \Big\{ y = (\beta_i) \in E : |\beta_{i_2}| = 1, \quad \beta_i = \alpha_i^{(2)}, \quad \text{for all } i \in \sigma(x_1) \cup \sigma(x_2) \cup S_1 \cup S_2 \\ \text{and the set} \quad \mathbb{N} \setminus (\sigma(y) \cup \sigma(x_1) \cup \sigma(x_2) \cup S_1 \cup S_2) \text{ is infinite}\}$$

and $K_2 = \sup\{|||y|||_0 : y \in F_2\}$, we have $\|2x_2\| - 1 \leqslant K_2$. This gives

$$K_2 - 1 \leqslant ((4M-3)K_1 - 1)/2M \leqslant (K_1 - 1)/2 \leqslant (M-1)/4.$$

Continuing in this way, we get for each n, $x_n = \left(\alpha_i^{(n)}\right) \in E$, $S_n \subset \mathbb{N}$, $F_n \subset E$, a real number $K_n > 1$ and a positive integer i_n such that

(a) $x_n \in F_{n-1}$, $1 \leqslant |||x_n|||_0 \leqslant K_{n-1}$ and $K_{n-1} \leqslant (M-1)/2^n$, where $K_0 = M$,

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(b) if $M_n = \left(\bigcup_{j=1}^{n} \sigma(x_j)\right) \cup \left(\bigcup_{j=1}^{n-1} S_j\right)$, then S_n is an infinite subset of $\mathbb{N} \setminus M_n$ such that $\mathbb{N} \setminus (M_n \cup S_n)$ is infinite,

(c) $i_n \in \mathbb{N} \setminus (M_n \cup S_n)$ and $\left|\alpha_{i_n}^{(n-1)}\right| = 1$,

(d) $\alpha_k^{(n)} = \alpha_k^{(n-1)}$, for $k \in M_{n-1} \cup S_{n-1}$.

Thus, there is an $x_0 = (\gamma_i) \in E$ such that

$$\gamma_k = \begin{cases} \alpha_k^{(n)}, & (k \in M_n \cup S_n, \quad n = 1, 2, \ldots), \\ 0, & \left(k \in \mathbb{N} \Big\backslash \left(\bigcup_{n=1}^{\infty} (\sigma(x_n)) \cup S_n\right)\right). \end{cases}$$

Note that $|\gamma_{i_n}| = 1$, $(n = 1, 2, \ldots)$ and consider the continuous linear functional on E given by the Banach limit ([4], p.73) defined by

$$f(y) = \operatorname{LIM}(\gamma_{i_n} . \alpha_{i_n}), \quad (y = (\alpha_i) \in E).$$

Then, $f(x_0) = 1$ and $f(x_n) = 0$, $(n = 1, 2, \ldots)$.

Let for each $\lambda \in \Lambda$, F_λ be a separable closed subspace of G_λ such that

$$\{x_0, x_1, \ldots\} \subset F = \overline{\operatorname{span}} \bigcup_{\lambda \in \Lambda} F_\lambda.$$

Also, let $w_\lambda = v_\lambda | F$. Then, $(F_\lambda)_{\lambda \in \Lambda}$ is a strong M-decomposition of F with the associated family of coordinate projections $(w_\lambda)_{\lambda \in \Lambda}$. For each $\lambda \in \Lambda$, let $\left(e_n^{(\lambda)}\right)_{n=1}^{\infty}$ be dense in F_λ. Write for each n, $U_n = \{A \subset \Lambda : \operatorname{card} A \leqslant n\}$ and $U = \bigcup_{n=1}^{\infty} U_n$. Let us define the following semi-norms on F

$$E_A^{(n)}(x) = \inf \left\{ \left\| x - \sum_{\lambda \in A} \beta_\lambda \sum_{i=1}^{n} \alpha_i^{(\lambda)} e_i^{(\lambda)} \right\| : \beta_\lambda, \alpha_i^{(\lambda)} \text{ are scalars} \right\}, (A \in U, n = 1, 2, \ldots)$$

$$F_A(x) = \sum_{\lambda \in A} t_\lambda(x), \quad (A \in U),$$

$$G_0(x) = \|x\|,$$

$$G_n(x) = \sup \left\{ E_A^{(n)}(x) + n F_A(x) : A \in U \right\}, \quad (n = 1, 2, \ldots).$$

Define a mapping $Q \colon F \to c_0(\Delta)$ by

$$(Qx)(\delta) = \begin{cases} 2^{-n} G_n(x), & (n = \delta/2, \quad \delta = 0, 2, 4, \ldots), \\ \alpha_n / n, & (n = (\delta+1)/2, \quad \delta = 1, 3, \ldots \text{ and } x = (\alpha_i)), \\ t_\delta(x), & (\delta \in \Lambda). \end{cases}$$

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Note that since $G_n(x) \leqslant (1+n^2)\,\|x\|$, $(n = 1, 2, \ldots)$, we have

$$(Qx)(\delta) \leqslant (Q_0 x)(\delta), \quad (x \in E,\, \delta \in \Lambda).$$

Thus, the function on F given by

$$|||x||| = 2J(Qx),$$

defines a norm on F ([11], Lemma) and, for each $x \in F$, satisfies

$$\|x\| \leqslant |||x||| \leqslant |||x|||_0 .$$

Finally, since $(\alpha_n) \to (\alpha_n/n)$ is an injective continuous linear operator of E into c_0, that $(F_\lambda)_{\lambda\in\Lambda}$ is a strong M-decomposition of F and that each F_λ is separable, it follows (for example see the proof of Theorem 2 in [3], p.101) that the norm $|||.|||$ is a locally uniformly convex on F. Now, note that for each n, the elements x and $(x_n + x)/2$ are in F_n. Therefore, $1 \leqslant |||x_n||| \leqslant K_{n-1}$, $1 \leqslant |||x||| \leqslant K_n$ and $2 \leqslant |||x_n + x||| \leqslant 2K_n$, where $K_0 = M$ and for all n. This gives that $|||x||| = \lim_{n\to\infty} |||x_n||| = \lim_{n\to\infty} K_n = 1$ and $\lim_{n\to\infty} |||x_n + x||| = 2$. Hence $\lim_{n\to\infty} x_n = x$. But this is a contradiction and hence the proof is complete. ▯

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