Proceedings of the Edinburgh Mathematical Society (2007) **50**, 711–723 ©
DOI:10.1017/S0013091505000908 Printed in the United Kingdom

INCOMPLETE BESSEL FUNCTIONS. II. ASYMPTOTIC EXPANSIONS FOR LARGE ARGUMENT

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(Received 17 June 2005)

Abstract Asymptotic expansions for an incomplete Bessel function of large argument are derived when the parametric point (a) is well away from any saddle point, (b) coincides with a saddle point and (c) is in the neighbourhood of a saddle point.

Keywords: asymptotics; Bessel functions; modified Bessel functions

2000 *Mathematics subject classification:* Primary 33E20

1. Introduction

In an earlier paper [**1**] definitions were given of various incomplete Bessel functions and some of their properties were derived. A typical definition is

$$K_\nu(z, w) = K_\nu(z) - J(z, \nu, w), \tag{1.1}$$

where $K_\nu(z)$ is the modified Bessel function and

$$J(z, \nu, w) = \int_0^w \mathrm{e}^{-z \cosh t} \cosh \nu t \, \mathrm{d}t. \tag{1.2}$$

Among the properties established was the asymptotic behaviour as $|z| \to \infty$ when w is real. The purpose of the following is to extend the analysis to complex values of w. This involves consideration of the integral representation

$$K_\nu(z, w) = \int_w^{\infty + \mathrm{i}\sigma} \mathrm{e}^{-z \cosh t} \cosh \nu t \, \mathrm{d}t \tag{1.3}$$

for $|\mathrm{ph}\, z + \sigma| < \pi/2$.

It will be convenient to suppose that the phase of the symbol Z satisfies $|\mathrm{ph}\, Z| < \pi/2$, while z may have any phase unless it is specifically limited.

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2. Preliminary results

The integral in (1.3) has saddle points where $\sinh t = 0$, i.e. for $t = n\pi\mathrm{i}$ with n an integer. The asymptotic behaviour changes as w approaches one of the saddle points. To avoid such a complication in this section it will be assumed that w is kept away from the saddle points. The first restriction that will be imposed on w is that $\alpha_0 > 0$ when $w = \alpha_0 + \mathrm{i}\beta_0$.

Three types of integral are relevant to (1.3) and they will be discussed separately.

2.1. Case 1

The first integral that will be discussed is

$$L_1(Z) = \int_w^\infty \mathrm{e}^{-Z\cosh t - \nu t}\,\mathrm{d}t \tag{2.1}$$

subject to $\alpha_0 > 0$. If the path of integration from w can be chosen so that $\mathcal{I}(\cosh t - \cosh w) = 0$ and $\mathcal{R}(\cosh t - \cosh w)$ increases, the main contribution as $|Z| \to \infty$ will come from the neighbourhood of w. The equation of the desired path when $t = \alpha + \mathrm{i}\beta$ is $\sinh\alpha \sin\beta = \sinh\alpha_0 \sin\beta_0$.

When $0 \leqslant \beta_0 < \pi$, the desired path is a horizontal U-bend ($\subset$) with endpoints at infinity where $\beta = 0$ and $\beta = \pi$. The desired direction of integration goes towards the endpoint with $\beta = 0$. For $-\pi < \beta \leqslant 0$ the desired path is also a horizontal U-bend and the desired direction goes from the endpoint at $\beta = -\pi$ to that at $\beta = 0$. Hence, when $|\beta_0| < \pi$, the path of integration can be deformed as desired and

$$L_1(Z) = \mathrm{e}^{-Z\cosh w} \int_0^\infty \frac{\mathrm{e}^{-Zu - \nu t}}{\sinh t}\,\mathrm{d}u$$

after the substitution $\cosh t = \cosh w + u$.

For small u the expansion

$$\frac{\mathrm{e}^{-\nu t}}{\sinh t} = \mathrm{e}^{-\nu w} \sum_{s=0} a_s(\nu, w) u^s$$

is available. Here

$$a_0(\nu, w) = 1/\sinh w, \qquad a_1(\nu, w) = -(\nu + \coth w)/\sinh^2 w, \tag{2.2}$$

$$a_2(\nu, w) = (\nu^2 + 3\nu\coth w + 2 + 3/\sinh^2 w)/2\sinh^3 w. \tag{2.3}$$

The expansion leads to

$$L_1(z) \sim f(\nu, Z),$$

where

$$f(\nu, Z) = \mathrm{e}^{-Z\cosh w - \nu w} \sum_{s=0} s!\frac{a_s(\nu, w)}{Z^{s+1}}. \tag{2.4}$$

When $(2n-1)\pi < \beta_0 < (2n+1)\pi$ the paths just described are replicated but go off to infinity near $\beta = 2n\pi$. Therefore, deformation of the path in L_1 to the desired direction

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cannot be carried out. Instead

$$\int_w^{\infty+2n\pi\mathrm{i}} \mathrm{e}^{-Z\cosh t-\nu t}\,\mathrm{d}t \sim f(\nu, Z). \tag{2.5}$$

Now, for integer m,

$$\int_{\infty+2(m-1)\pi\mathrm{i}}^{\infty+2m\pi\mathrm{i}} \mathrm{e}^{-Z\cosh t-\nu t}\,\mathrm{d}t = 2\pi\mathrm{i}\mathrm{e}^{(1-2m)\nu\pi\mathrm{i}} I_\nu(Z), \tag{2.6}$$

where $I_\nu(Z)$ is the usual modified Bessel function. Consequently,

$$\int_{\infty}^{\infty+2n\pi\mathrm{i}} \mathrm{e}^{-Z\cosh t-\nu t}\,\mathrm{d}t = 2\pi\mathrm{i}\mathrm{e}^{-n\nu\pi\mathrm{i}} \frac{\sin n\nu\pi}{\sin\nu\pi} I_\nu(Z). \tag{2.7}$$

From (2.5) and (2.7),

$$L_1(Z) \sim f(\nu, Z) - 2\pi\mathrm{i}\mathrm{e}^{-n\nu\pi\mathrm{i}} \frac{\sin n\nu\pi}{\sin\nu\pi} I_\nu(Z) \tag{2.8}$$

when $(2n-1)\pi < \beta_0 < (2n+1)\pi$.

The general asymptotic formula (see [**2**])

$$I_\nu(z) \sim \frac{\mathrm{e}^z}{(2\pi z)^{1/2}} \sum_{s=0} (-)^s \frac{A_s(\nu)}{z^s} - \mathrm{i}\frac{\mathrm{e}^{-\nu\pi\mathrm{i}-z}}{(2\pi z)^{1/2}} \sum_{s=0} \frac{A_s(\nu)}{z^s}, \tag{2.9}$$

valid for $|\mathrm{ph}\, z| < 3\pi/2$ with $A_0(\nu) = 1$ and

$$A_s(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots\{4\nu^2-(2s-1)^2\}}{s!8^s}, \tag{2.10}$$

can be employed in (2.8) but it is more economical to retain the form (2.8) as long as possible.

Notice that, at boundaries where the coefficient of $I_\nu(Z)$ in (2.8) alters, $\cosh w = -\cosh\alpha_0$, so $I_\nu(Z)$ is exponentially smaller than $f(\nu, Z)$. In other words, the changes in (2.8) should take place relatively smoothly. On the other hand, $I_\nu(Z)$ is likely to dominate near $\beta_0 = 2n\pi$ $(n \neq 0)$.

2.2. Case 2

The integral to be considered here is

$$L_2(Z) = \int_w^{\infty+\pi\mathrm{i}} \mathrm{e}^{Z\cosh t-\nu t}\,\mathrm{d}t. \tag{2.11}$$

In this case the preferred paths of integration are the same as in Case 1 but traversed in the opposite direction. Accordingly, if $2n\pi < \beta_0 < (2n+2)\pi$,

$$\int_w^{\infty+(2n+1)\pi\mathrm{i}} \mathrm{e}^{Z\cosh t-\nu t}\,\mathrm{d}t \sim f(\nu, -Z). \tag{2.12}$$

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From (2.6),

$$\int_{\infty+(2m-1)\pi\mathrm{i}}^{\infty+(2m+1)\pi\mathrm{i}} \mathrm{e}^{Z\cosh t-\nu t}\,\mathrm{d}t = 2\pi\mathrm{i}\mathrm{e}^{-2m\nu\pi\mathrm{i}} I_\nu(Z), \tag{2.13}$$

and so

$$L_2(Z) \sim f(\nu,-Z) - 2\pi\mathrm{i}\mathrm{e}^{-(n+1)\nu\pi\mathrm{i}} \frac{\sin n\nu\pi}{\sin\nu\pi} I_\nu(Z) \tag{2.14}$$

when $2n\pi < \beta_0 < (2n+2)\pi$.

A deduction from (2.11) and (2.14) is

$$\int_w^{\infty-\pi\mathrm{i}} \mathrm{e}^{Z\cosh t-\nu t}\,\mathrm{d}t \sim f(\nu,-Z) - 2\pi\mathrm{i}\mathrm{e}^{-n\nu\pi\mathrm{i}} \frac{\sin(n+1)\nu\pi}{\sin\nu\pi} I_\nu(Z) \tag{2.15}$$

when $2n\pi < \beta_0 < (2n+2)\pi$.

2.3. Case 3

Another integral for which the asymptotic behaviour is required is

$$L_3(Z) = \int_w^{\infty+\frac{1}{2}\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t. \tag{2.16}$$

The equation of the desired path of integration is $\cosh\alpha\cos\beta = \cosh\alpha_0\cos\beta_0$ now and, as a result, the pattern of paths is more complicated than in the preceding cases.

If $|\cosh\alpha_0\cos\beta_0| > 1$ the path is a horizontal U-bend. Typically, it will start at $\infty + (2n-\frac{1}{2})\pi\mathrm{i}$ or $\infty + (2n+\frac{3}{2})\pi\mathrm{i}$ and end at $\infty + (2n+\frac{1}{2})\pi\mathrm{i}$ when traversed in the desired direction.

If $|\cosh\alpha_0\cos\beta_0| \leqslant 1$ the path can cross the imaginary axis. There are two typical paths when the desired direction is taken into account. One starts at $-\infty + (2n+\frac{1}{2})\pi\mathrm{i}$ and goes to $\infty + (2n+\frac{1}{2})\pi\mathrm{i}$. The other is traversed in the opposite direction going from $\infty + (2n-\frac{1}{2})\pi\mathrm{i}$ to $-\infty + (2n-\frac{1}{2})\pi\mathrm{i}$. Thus the paths on which $|\cosh\alpha_0\cos\beta_0| < 1$ and $(2n-1)\pi < \beta_0 < 2n\pi$ are quite different from the other curves because they end up on opposite sides of the imaginary axis. These exceptional values of w will require a separate treatment.

When the exceptional paths are ignored and $(2n-\frac{1}{2})\pi < \beta_0 < (2n+\frac{3}{2})\pi$,

$$\int_w^{\infty+(2n+\frac{1}{2})\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t \sim f(\nu,-\mathrm{i}Z). \tag{2.17}$$

Since

$$\int_{\infty+(2m+\frac{1}{2})\pi\mathrm{i}}^{\infty+(2m+\frac{5}{2})\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t = 2\pi\mathrm{i}\mathrm{e}^{-(2m+\frac{3}{2})\nu\pi\mathrm{i}} J_\nu(Z), \tag{2.18}$$

with $J_\nu(Z)$ the customary Bessel function, it follows that

$$L_3(Z) \sim f(\nu,-\mathrm{i}Z) - 2\pi\mathrm{i}\mathrm{e}^{-(n+\frac{1}{2})\nu\pi\mathrm{i}} \frac{\sin n\nu\pi}{\sin\nu\pi} J_\nu(Z) \tag{2.19}$$

for $(2n-\frac{1}{2})\pi < \beta_0 < (2n+\frac{3}{2})\pi$ and w not on an exceptional path.

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For the exceptional paths on which $(2n-1)\pi < \beta_0 < 2n\pi$,

$$\int_w^{-\infty+(2n-\frac{1}{2})\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t \sim f(\nu,-\mathrm{i}Z).$$

Also,

$$\int_{-\infty+(2n-\frac{1}{2})\pi\mathrm{i}}^{\infty+(2n+\frac{1}{2})\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t = \pi\mathrm{i}\mathrm{e}^{-(2n-\frac{1}{2})\nu\pi\mathrm{i}} H_\nu^{(1)}(Z), \tag{2.20}$$

$H_\nu^{(1)}(Z)$ being the standard Hankel function, so that

$$\int_w^{\infty+(2n+\frac{1}{2})\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t \sim f(\nu,-\mathrm{i}Z) + \pi\mathrm{i}\mathrm{e}^{-(2n-\frac{1}{2})\nu\pi\mathrm{i}} H_\nu^{(1)}(Z).$$

Consequently, when $|\cosh\alpha_0\cos\beta_0| < 1$ and $(2n-1)\pi < \beta_0 < 2n\pi$,

$$L_3(Z) \sim f(\nu,-\mathrm{i}Z) + \pi\mathrm{i}\mathrm{e}^{-(2n-\frac{1}{2})\nu\pi\mathrm{i}} H_\nu^{(1)}(Z) - 2\pi\mathrm{i}\mathrm{e}^{-(n+\frac{1}{2})\nu\pi\mathrm{i}} \frac{\sin n\nu\pi}{\sin\nu\pi} J_\nu(Z). \tag{2.21}$$

It may be inferred from (2.19) and (2.21) that, when $(2n-\frac{3}{2})\pi < \beta_0 < (2n+\frac{1}{2})\pi$ in the normal case,

$$\int_w^{\infty-\frac{1}{2}\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t \sim f(\nu,\mathrm{i}Z) - 2\pi\mathrm{i}\mathrm{e}^{-(n-\frac{1}{2})\nu\pi\mathrm{i}} \frac{\sin n\nu\pi}{\sin\nu\pi} J_\nu(Z) \tag{2.22}$$

and, when $|\cosh\alpha_0\cos\beta_0| < 1$ with $(2n-2)\pi < \beta_0 < (2n-1)\pi$,

$$\int_w^{\infty-\frac{1}{2}\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}Z\cosh t-\nu t}\,\mathrm{d}t$$
$$\sim f(\nu,\mathrm{i}Z) + \pi\mathrm{i}\mathrm{e}^{-(2n-\frac{3}{2})\nu\pi\mathrm{i}} H_\nu^{(1)}(Z) - 2\pi\mathrm{i}\mathrm{e}^{-(n-\frac{1}{2})\nu\pi\mathrm{i}} \frac{\sin n\nu\pi}{\sin\nu\pi} J_\nu(Z). \tag{2.23}$$

3. Formulae for $K_\nu(z,w)$

An asymptotic expression for $K_\nu(Z,w)$ can be obtained from (1.3) by means of (2.8). The linear combination $f(\nu,Z) + f(-\nu,Z)$ occurs; it may be rewritten by defining

$$b_s(\nu,w) = \tfrac{1}{2}\{\mathrm{e}^{-\nu w}a_s(\nu,w) + \mathrm{e}^{\nu w}a_s(-\nu,w)\}.$$

Then

$$b_0(\nu,w) = \frac{\cosh\nu w}{\sinh w}, \qquad b_1(\nu,w) = \frac{\nu\sinh\nu w - \cosh\nu w\coth w}{\sinh^2 w}, \tag{3.1}$$

$$b_2(\nu,w) = \frac{\nu^2\cosh\nu w - 3\nu\sinh\nu w\coth w + (2+3/\sinh^2 w)\cosh\nu w}{2\sinh^3 w} \tag{3.2}$$

from (2.2) and (2.3).

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One concludes that, when $|\mathrm{ph}\, z| < \pi/2$ and $(2n-1)\pi < \beta_0 < (2n+1)\pi$,

$$K_\nu(z,w) \sim g(z,\nu,w) - \pi\mathrm{i}\frac{\sin n\nu\pi}{\sin\nu\pi}\{\mathrm{e}^{-n\nu\pi\mathrm{i}}I_\nu(z) + \mathrm{e}^{n\nu\pi\mathrm{i}}I_{-\nu}(z)\}, \tag{3.3}$$

where

$$g(z,\nu,w) = \mathrm{e}^{-z\cosh w}\sum_{s=0} s!\frac{b(\nu,w)}{z^{s+1}}. \tag{3.4}$$

An alternative version of (3.3) is supplied by the substitution

$$I_{-\nu}(z) = I_\nu(z) + (2/\pi)\sin\nu\pi K_\nu(z). \tag{3.5}$$

It is

$$K_\nu(z,w) \sim g(z,\nu,w) + (1-\mathrm{e}^{2n\nu\pi\mathrm{i}})K_\nu(z) - \pi\mathrm{i}\frac{\sin 2n\nu\pi}{\sin\nu\pi}I_\nu(z). \tag{3.6}$$

Note that (2.9) can be inserted into (3.6) and also

$$K_\nu(z) \sim \mathrm{e}^{-z}\left(\frac{\pi}{2z}\right)^{1/2}\sum_{s=0}\frac{A_s(\nu)}{z^s} \tag{3.7}$$

for $|\mathrm{ph}\, z| < 3\pi/2$.

The range of $\mathrm{ph}\, z$ can be extended by taking advantage of (2.14) and (1.3). Put $Z = z\mathrm{e}^{\pi\mathrm{i}}$ so that $-3\pi/2 < \mathrm{ph}\, z < -\pi/2$. By means of the general relations

$$I_\nu(z\mathrm{e}^{m\pi\mathrm{i}}) = \mathrm{e}^{m\nu\pi\mathrm{i}}I_\nu(z), \tag{3.8}$$

$$K_\nu(z\mathrm{e}^{m\pi\mathrm{i}}) = \mathrm{e}^{-m\nu\pi\mathrm{i}}K_\nu(z) - \pi\mathrm{i}\frac{\sin m\nu\pi}{\sin\nu\pi}I_\nu(z), \tag{3.9}$$

it is found that (2.14) leads to (3.6). Thus (3.6) holds for $-3\pi/2 < \mathrm{ph}\, z < -\pi/2$ and $2n\pi < \beta_0 < (2n+2)\pi$.

Similarly, it can be deduced from (2.15) with $Z = z\mathrm{e}^{-\pi\mathrm{i}}$ that (3.6) is valid for $\pi/2 < \mathrm{ph}\, z < 3\pi/2$ and $(2n-2)\pi < \beta_0 < 2n\pi$.

Another way of obtaining the result of the preceding paragraph is to start with the formula for $-3\pi/2 < \mathrm{ph}\, z < -\pi/2$ and invoke the general relation

$$K_\nu(z\mathrm{e}^{2\pi\mathrm{i}},w) = K_\nu(z,w) + (\mathrm{e}^{-2\nu\pi\mathrm{i}}-1)K_\nu(z) - 2\pi\mathrm{i}\cos\nu\pi I_\nu(z). \tag{3.10}$$

The expansion (3.6) is also valid for $-\pi < \mathrm{ph}\, z < 0$ and $(2n-\frac{1}{2})\pi < \beta_0 < (2n+\frac{3}{2})\pi$ by (2.19) as well as for $0 < \mathrm{ph}\, z < \pi$ and $(2n-\frac{3}{2})\pi < \beta_0 < (2n+\frac{1}{2})\pi$ by (2.22) provided that w is not exceptional.

In the exceptional case ($\mathcal{R}(\cosh w) < 1$), (2.21) is relevant when $-\pi < \mathrm{ph}\, z < 0$. Then

$$K_\nu(z,w) \sim g(z,\nu,w) + (1+\mathrm{e}^{-2n\nu\pi\mathrm{i}})K_\nu(z) - \pi\mathrm{i}\frac{\sin 2n\nu\pi}{\sin\nu\pi}I_\nu(z) \tag{3.11}$$

for $(2n-1)\pi < \beta_0 < 2n\pi$. Furthermore, when $0 < \mathrm{ph}\, z < \pi$, (2.23) gives (3.11) subject to $2n\pi < \beta_0 < (2n+1)\pi$.

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The formulae (3.6) and (3.11) cover the range $-3\pi/2 < \operatorname{ph} z < 3\pi/2$. Other ranges of the phase can be handled by calling on (3.10).

The expansions are governed by the restriction $\mathcal{R}(w) > 0$. However, it is straightforward to check that $J(z, \nu, -w) = -J(z, \nu, w)$ and so

$$K_\nu(z, -w) = 2K_\nu(z) - K_\nu(z, w). \tag{3.12}$$

Thus the asymptotic behaviour for $\mathcal{R}(w) < 0$ can be written down from the foregoing and (3.12).

When $\mathcal{R}(w) = 0$, the formulae can be expected to be applicable so long as w is sufficiently distant from any saddle point. The behaviour at a saddle point is considered in the next section.

4. Saddle points

The integral representation of $K_\nu(z, w)$ has saddle point at $t = n\pi\mathrm{i}$. So far these have been excluded from the asymptotic expansions. This section is concerned with their contributions, which, as will be seen, are somewhat different from those already derived.

A change in the variable of integration gives

$$\int_{2n\pi\mathrm{i}}^{\infty+2n\pi\mathrm{i}} \mathrm{e}^{-Z\cosh t} \cosh\nu t \, \mathrm{d}t = K_\nu(Z)\cos 2n\nu\pi + \mathrm{i}\sin 2n\nu\pi \int_0^\infty \mathrm{e}^{-Z\cosh t} \sinh\nu t \, \mathrm{d}t. \tag{4.1}$$

The asymptotic performance of the integral on the right-hand side is obtained by putting $\cosh t = 1 + \frac{1}{2}u^2$ or $u = 2\sinh(t/2)$ and expanding the factor of the exponential in powers of u. This requires the expansion of $\sinh\nu t/\cosh(t/2)$ but it is more useful to consider $\mathrm{e}^{\nu t}/\cosh(t/2)$. Let

$$F(u) = \frac{\mathrm{e}^{\nu t}}{\cosh(t/2)} = \sum_{m=0} c_m u^m$$

for small u. Then

$$\cosh^2(t/2)F''(u) + \tfrac{3}{2}\sinh(t/2)F'(u) = (\nu^2 - \tfrac{1}{4})F(u)$$

or

$$(u^2 + 4)F''(u) + 3uF'(u) = (4\nu^2 - 1)F(u).$$

Take a derivative m times by Leibnitz's theorem and then set $u = 0$. There results

$$F^{(m+2)}(0) = \{4\nu^2 - (m+1)^2\}F^{(m)}(0)/4.$$

Since $F(0) = 1$ it follows that, for $m > 0$,

$$\begin{aligned} F^{(2m)}(0) &= (4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots\{4\nu^2 - (2m-1)^2\}/4^m \\ &= m!2^m A_m(\nu) \end{aligned}$$

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with $A_m(\nu)$ defined as in (2.10). Hence

$$c_{2m} = \pi^{1/2} A_m(\nu)/(m - \tfrac{1}{2})!2^m. \tag{4.2}$$

On the other hand, $F'(0) = \nu$ so that

$$F^{(2m+1)}(0) = \nu(\nu^2 - 1^2)(\nu^2 - 2^2) \cdots (\nu^2 - m^2).$$

Therefore,

$$c_{2m+1} = \nu B_m(\nu)/m!2^m, \tag{4.3}$$

where $B_0(\nu) = 1$ and, for $m > 0$,

$$B_m(\nu) = \pi^{1/2}(\nu^2 - 1^2)(\nu^2 - 2^2) \cdots (\nu^2 - m^2)/(m + \tfrac{1}{2})!2^{m+1}. \tag{4.4}$$

The coefficients $A_m(\nu)$ and $B_m(\nu)$ are even functions of ν. It may therefore be deduced that

$$\frac{\cosh \nu t}{\cosh(t/2)} = \sum_{m=0} c_{2m} u^{2m} \tag{4.5}$$

with c_{2m} as in (4.2) and

$$\frac{\sinh \nu t}{\cosh(t/2)} = \sum_{m=0} c_{2m+1} u^{2m+1} \tag{4.6}$$

with c_{2m+1} given by (4.3).

By virtue of (4.6),

$$\int_0^\infty \mathrm{e}^{-Z \cosh t} \sinh \nu t \, \mathrm{d}t \sim h(\nu, Z), \tag{4.7}$$

where

$$h(\nu, z) = \frac{\nu}{z} \mathrm{e}^{-z} \sum_{m=0} \frac{B_m(\nu)}{z^m}. \tag{4.8}$$

Hence, via (2.7), (3.5) and (4.1),

$$K_\nu(Z, 2n\pi \mathrm{i}) \sim (1 - \mathrm{i} \sin 2n\nu\pi) K_\nu(Z) - \pi \mathrm{i} \frac{\sin 2n\nu\pi}{\sin \nu\pi} I_\nu(Z) + \mathrm{i} h(\nu, Z) \sin 2n\nu\pi. \tag{4.9}$$

As regards the remaining saddle points, commence with

$$\begin{aligned}
&\int_{(2n+1)\pi\mathrm{i}}^{\infty+2n\pi\mathrm{i}} \mathrm{e}^{-Z\cosh t} \cosh \nu t \, \mathrm{d}t \\
&\qquad = \int_0^{\infty-\pi\mathrm{i}} \mathrm{e}^{Z \cosh t} \{\cosh \nu t \cos(2n+1)\nu\pi + \mathrm{i} \sinh \nu t \sin(2n+1)\nu\pi\} \, \mathrm{d}t.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^{\infty-\pi\mathrm{i}} \mathrm{e}^{Z\cosh t} \cosh \nu t \, \mathrm{d}t &= \frac{1}{2} \int_{-\infty+\pi\mathrm{i}}^{\infty-\pi\mathrm{i}} \mathrm{e}^{Z\cosh t - \nu t} \, \mathrm{d}t \\
&= \frac{1}{2} \int_{-\infty+\pi\mathrm{i}}^{\infty+\pi\mathrm{i}} \mathrm{e}^{Z\cosh t - \nu t} \, \mathrm{d}t - \frac{1}{2} \int_{\infty-\pi\mathrm{i}}^{\infty+\pi\mathrm{i}} \mathrm{e}^{Z\cosh t - \nu t} \, \mathrm{d}t \\
&= \mathrm{e}^{-\nu\pi\mathrm{i}} K_\nu(Z) - \pi \mathrm{i} I_\nu(Z).
\end{aligned}$$

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Also,

$$\int_0^{\infty-\pi\mathrm{i}} \mathrm{e}^{Z\cosh t}\sinh \nu t\,\mathrm{d}t \sim h(\nu,-Z).$$

Consequently,

$$K_\nu(Z,(2n+1)\pi\mathrm{i}) \sim \{1-\mathrm{i}\mathrm{e}^{-\nu\pi\mathrm{i}}\sin(2n+1)\nu\pi\}K_\nu(Z) \\ -\pi\mathrm{i}\sin(2n+1)\nu\pi\cot\nu\pi I_\nu(Z)+\mathrm{i}h(\nu,-Z)\sin(2n+1)\nu\pi. \quad (4.10)$$

5. Transition formulae

An expansion of $K_\nu(Z,w)$ when w is not near a saddle point is supplied by (3.6). In contrast, (4.9) is available when w coincides with a saddle point. How the expansion (3.6) transforms into (4.9) as w approaches a saddle point is the topic of this section.

Consider $K_\nu(Z,w)$ as w approaches the saddle point at the origin. Change the variable of integration to v, where $v=2\sinh(t/2)-2\sinh(w/2)$. Then

$$K_\nu(Z,w)=\mathrm{e}^{-Z}\int_0^\infty \mathrm{e}^{-Z(v-b)^2/2}f_0(v)\,\mathrm{d}v,$$

where $b=-2\sinh(w/2)$ and

$$f_0(v)=\cosh\nu t/\cosh(t/2).$$

Let

$$f_0(v)=C_0+D_0(v-b)+v(v-b)g_0(v)$$

so that $C_0=f_0(b)$ and $C_0-bD_0=f_0(0)$. Then

$$K_\nu(Z,w)=C_0\mathrm{e}^{-Z}\left(\frac{\pi}{2Z}\right)^{1/2}\operatorname{erfc}\{-b(Z/2)^{1/2}\} \\ +\frac{D_0}{Z}\mathrm{e}^{-Z(1+\frac{1}{2}b^2)}+\mathrm{e}^{-Z}\int_0^\infty v(v-b)g_0(v)\mathrm{e}^{-Z(v-b)^2/2}\,\mathrm{d}v, \quad (5.1)$$

where

$$\operatorname{erfc}(z)=\frac{2}{\pi^{1/2}}\int_z^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y.$$

Integration by parts provides

$$\int_0^\infty v(v-b)g_0(v)\mathrm{e}^{-Z(v-b)^2/2}\,\mathrm{d}v=\frac{1}{Z}\int_0^\infty f_1(v)\mathrm{e}^{-Z(v-b)^2/2}\,\mathrm{d}v,$$

where

$$f_1(v)=\frac{\mathrm{d}}{\mathrm{d}v}\{vg_0(v)\}. \quad (5.2)$$

Repeat the process carried out with f_0 by putting

$$f_1(v)=C_1+D_1(v-b)+v(v-b)g_1(v).$$

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This changes (5.1) by replacing C_0, D_0 and $g_0(v)$ by $C_0 + C_1/Z$, $D_0 + D_1/Z$ and $g_1(v)$, respectively. Further repetition leads to

$$K_\nu(Z,w) \sim \mathrm{e}^{-Z}\left(\frac{\pi}{2Z}\right)^{1/2} \operatorname{erfc}\{-b(Z/2)^{1/2}\} \sum_{n=0} \frac{C_n}{Z^n} + \frac{\mathrm{e}^{-Z\cosh w}}{Z} \sum_{n=0} \frac{D_n}{Z^n}. \tag{5.3}$$

The coefficients C_n and D_n are derived from the function $f_n(v)$. As v tends to b, $t \to 0$, and so, by (4.5),

$$f_0(v) = \sum_{m=0} c_{2m}(v-b)^{2m}. \tag{5.4}$$

Now assume that

$$f_n(v) = \sum_{m=0} f_{nm}(v-b)^{2m},$$

which implies that $C_n = f_{n0}$ and $f_{0m} = c_{2m}$. The analogue of (5.2) is

$$\begin{aligned} f_{n+1}(v) &= \frac{\mathrm{d}}{\mathrm{d}v}\frac{f_n(v) - C_n}{v-b} \\ &= \sum_{m=0}(2m+1)f_{n,m+1}(v-b)^{2m}. \end{aligned} \tag{5.5}$$

Hence,

$$\begin{aligned} f_{n+1,m} = (2m+1)f_{n,m+1} &= (2m+1)(2m+3)f_{n-1,m+2} \\ &= \frac{(m+n+\frac{1}{2})!}{(m-\frac{1}{2})!}2^{n+1}f_{0,m+n+1}. \end{aligned}$$

It follows that

$$C_n = \frac{(n-\frac{1}{2})!}{\pi^{1/2}}2^n c_{2n} = A_n(\nu) \tag{5.6}$$

from (4.2).

The coefficient D_n satisfies

$$C_n - bD_n = \sum_{m=0} f_{nm}b^{2m}, \tag{5.7}$$

which shows that $D_n \to 0$ as $b \to 0$.

An alternative expansion for $f_n(v)$ stems from Case 1 and

$$\frac{\cosh \nu t}{\sinh t} = \sum_{m=0} b_m(\nu,w)u^m,$$

where $u = \cosh t - \cosh w$. Thus

$$f_0(v) = (v-b)\sum_{m=0} b_m(\nu,w)u^m.$$

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Now make the assumption that

$$f_n(v) = (v-b)\sum_{m=0} b_{nm}u^m + \sum_{m=1}^{n} \frac{B_{nm}}{(v-b)^{2m}},$$

the last term being absent for $n = 0$. From (5.5),

$$f_{n+1}(v) = (v-b)\sum_{m=0}(m+1)b_{n,m+1}u^m - \sum_{m=2}^{n+1}\frac{(2m-1)B_{n,m-1}}{(v-b)^{2m}} + \frac{C_n}{(v-b)^2}.$$

Consequently,

$$\begin{aligned} b_{n+1,m} &= (m+1)b_{n,m+1} = (m+1)(m+2)b_{n-1,m+2} \\ &= (m+n+1)!b_{m+n+1}(\nu,w)/m!. \end{aligned} \tag{5.8}$$

Also, $B_{n+1,1} = C_n$ and, for $m = 2,\dots,n+1$,

$$\begin{aligned} B_{n+1,m} &= (-)(2m-1)B_{n,m-1} = (-)^2(2m-1)(2m-3)B_{n-1,m-2} \\ &= (m-\tfrac{1}{2})!(-2)^{m-1}C_{n+1-m}/\tfrac{1}{2}!. \end{aligned} \tag{5.9}$$

Since u and v both vanish when $t = w$,

$$C_n - bD_n = -bb_{n0} + \sum_{m=1}^{n}\frac{B_{nm}}{b^{2m}},$$

whence

$$D_n - \sum_{m=0}^{n}\frac{(m-\frac{1}{2})!(-2)^m}{\pi^{1/2}b^{2m+1}}C_{n-m} = n!b_n(\nu,w) \tag{5.10}$$

via (5.8) and (5.9).

On account of (3.7) and (5.6), (5.3) can be rewritten as

$$K_\nu(Z,w) \sim K_\nu(Z)\,\mathrm{erfc}\{-b(Z/2)^{1/2}\} + \frac{\mathrm{e}^{-Z\cosh w}}{Z}\sum_{n=0}\frac{D_n}{Z^n}. \tag{5.11}$$

As $w \to 0$, $b \to 0$ and $D_n \to 0$. Also, $\mathrm{erfc}(0) = 1$ so that, in the limit, (5.11) agrees with (4.9). On the other hand, when w is moved sufficiently far from the origin for $|bZ^{1/2}|$ to be large, the formulae

$$\mathrm{erfc}(z) \sim \frac{\mathrm{e}^{-z^2}}{\pi^{1/2}z}\left\{1 + \sum_{p=1}\frac{(p-\frac{1}{2})!(-)^p}{\pi^{1/2}z^{2p}}\right\}, \quad |\mathrm{ph}\,z| < 3\pi/4$$

and

$$\mathrm{erfc}(-z) = 2 - \mathrm{erfc}(z)$$

can be employed. Then, if $|\mathcal{I}(w)| < \pi$ but w is not near $\pm\pi\mathrm{i}$, (3.7), (5.10), (5.6) and (3.4) reproduce (3.6) and (3.12).

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The same technique may be applied to the integral in which $\cosh\nu t$ is replaced by $\sinh\nu t$. Here, (4.6) is pertinent. It shows that the analogue of C_n is zero. Moreover, the analogue of D_n does not tend to zero as $b\to 0$; instead, the limit is $\nu B_n(\nu)$. Since $b_n(\nu,w)$ is replaced by $d_n(\nu,w)$ where

$$d_n(\nu,w)=\tfrac{1}{2}\{\mathrm{e}^{\nu w}a_m(\nu,w)-\mathrm{e}^{-\nu w}a_m(\nu,w)\},$$

the result is

$$\int_w^\infty \mathrm{e}^{-Z\cosh t}\sinh\nu t\,\mathrm{d}t\sim\frac{\mathrm{e}^{-Z\cosh w}}{Z}\sum_{n=0}\frac{n!d_n(\nu,w)}{Z^n}\tag{5.12}$$

for w in the neighbourhood of the origin. As $w\to 0$, $n!d_n(\nu,w)\to\nu B_n(\nu)$ (cf. (5.10)) and the right-hand side of (5.12) goes to $h(\nu,Z)$, consistent with (4.7).

When w is near $2n\pi\mathrm{i}$, change the variable of integration so that

$$\int_w^{\infty+2n\pi\mathrm{i}}\mathrm{e}^{-Z\cosh t}\cosh\nu t\,\mathrm{d}t=\int_{w-2n\pi\mathrm{i}}^\infty \mathrm{e}^{-Z\cosh t}(\cosh\nu t\cos 2n\nu\pi+\mathrm{i}\sinh\nu t\sin 2n\nu\pi)\,\mathrm{d}t.$$

Since $w-2n\pi\mathrm{i}$ is in the neighbourhood of the origin, (5.11) and (5.12), with appropriate changes, can be used to supply a transition formula for the integral on the left-hand side. The upper limit of integration can be switched to ∞ through (2.7) and an expression for $K_\nu(Z,w)$ obtained when w is near $2n\pi\mathrm{i}$. It can be verified to give agreement with (4.9) and (3.6) as w moves away from $2n\pi\mathrm{i}$.

6. An error bound

Most asymptotic series fail to converge and computation is limited to a finite number of terms. Therefore, it is useful to have some idea of the error arising when an asymptotic expansion is truncated. Information for the modified Bessel functions is already available, so only $g(z,\nu,w)$ needs attention below. Since $g(z,\nu,w)$ is constructed from $f(\nu,z)$, it will suffice to discuss $f(\nu,z)$.

The expansion of $f(\nu,z)$ depends upon the representation of $\mathrm{e}^{-\nu t}/\sinh t$ (see Case 1). Let

$$\frac{\mathrm{e}^{-\nu t}}{\sinh t}=\mathrm{e}^{-\nu w}\left\{\sum_{s=0}^{n-1}a_s(\nu,w)u^s+u^n\phi_n(u)\right\}\tag{6.1}$$

with $\phi_n(0)=a_n(\nu,w)$. If

$$f(\nu,z)=\mathrm{e}^{-z\cosh w-\nu w}\left\{\sum_{s=0}^{n-1}s!\frac{a_s(\nu,w)}{z^{s+1}}+\epsilon_n(z)\right\},\tag{6.2}$$

then

$$\epsilon_n(Z)=\int_0^\infty \mathrm{e}^{-Zu}u^n\phi_n(u)\,\mathrm{d}u.$$

As $u\to 0$, $\phi_n(u)$ is bounded. Also, it is clear that $\phi_n(u)$ is bounded exponentially. Hence there is a $\mu_n(\nu,w)$ such that

$$|\phi_n(u)|\leqslant|a_n(\nu,w)|\mathrm{e}^{\mu_n(\nu,w)u}$$

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for $u \geqslant 0$. Consequently, if $\operatorname{ph} Z = \theta$,

$$|\epsilon_n(Z)| \leqslant n!|a_n(\nu, w)|/\{|Z| \cos\theta - \mu_n(\nu, w)\}^{n+1}$$

so long as $|Z| \cos\theta > \mu_n(\nu, w)$.

More generally, Cases 2 and 3 can be called on and a similar argument applied. Then

$$|\epsilon_n(z)| \leqslant n!|a_n(\nu, w)|/\{|z| \cos(\operatorname{ph} z + \delta) - \mu_n(\nu, w)\}^{n+1} \tag{6.3}$$

for $|z| \cos(\operatorname{ph} z + \delta) > \mu_n(\nu, w)$, δ being such that $|\operatorname{ph} z + \delta| < \pi/2$. Typical values for δ are 0, $\pm\pi/2$.

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