T. Hosoh
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# AMPLE VECTOR BUNDLES ON A RATIONAL SURFACE (HIGHER RANK)

TOSHIO HOSOH

## Introduction

In the previous paper [1], we showed that the set of simple vector bundles of rank 2 on a rational surface with fixed Chern classes is bounded and we gave a sufficient condition for an $H$-stable vector bundle of rank 2 on a rational surface to be ample. In this paper, we shall extend the results of [1] to the case of higher rank.

Let $k$ be an algebraically closed field of arbitrary characteristic. Throughout this paper, the ground field $k$ will be fixed.

In § 1, we shall prove the following;

THEOREM 1. *Let $X$ be the projective plane $\boldsymbol{P}^2$ or the rational ruled surface $\Sigma_n$. For a divisor $C_1$ on $X$ and integers $C_2, r\,(\geqq 2)$, put $\mathscr{F} = \{E;$ simple vector bundle of rank $r$ on $X$ with $C_i(E) = C_i$ for $i = 1, 2\}$, then $\mathscr{F}$ is bounded.*

For a vector bundle $E$ of rank $r$ on a non-singular projective surface, define an integer $\varDelta(E)$ to be $(r-1)C_1(E)^2 - 2rC_2(E)$. It is easy to see that $-\varDelta(E)$ is the second Chern class of $\mathrm{End}\,(E)$. Hence if $L$ is a line bundle, then $\varDelta(E \otimes L) = \varDelta(E)$. Let $H$ be a hyperplane of $\boldsymbol{P}^2$. For a vector bundle $E$ of rank $r$ on $\boldsymbol{P}^2$, there exists uniquely a line bundle $L$ on $\boldsymbol{P}^2$ such that $C_1(E \otimes L) = aH$ with $-r+1 \leqq a \leqq 0$. Put $a(E) = a$. In § 2, we shall prove the following;

THEOREM 2. *Let $E$ be an $H$-stable vector bundle of rank $r$ on $\boldsymbol{P}^2$. If $(C_1(E), H) \geqq -\frac{1}{2}\varDelta(E) + (a+2r)(2-a-r)/2$ then $E$ is ample where $a = a(E)$.*

Let $\Sigma_n = \boldsymbol{P}(O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1})$ be a rational ruled surface and let $M$ be a minimal section of $\Sigma_n$ and $N$ be a fibre of $\Sigma_n$. The divisor class group

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of $\Sigma_n$ is generated by the classes of $M$ and $N$. For a couple of integers $(\alpha, \beta)$, we denote $\alpha(M + nN) + \beta N$ by $H_{\alpha,\beta}$. $H_{\alpha,\beta}$ is ample if and only if $\alpha > 0, \beta > 0$. For a vector bundle $E$ of rank $r$ on $\Sigma_n$, there exists uniquely a line bundle $L$ on $\Sigma_n$ such that $C_1(E \otimes L) = aM + bN$ with $-r + 1 \leqq a$, $b \leqq 0$. Put $a(E) = a$ and $b(E) = b$. In § 3, we shall prove the following;

THEOREM 3. *Let $E$ be an $H_{\alpha,\beta}$-stable vector bundle of rank $r$ on $\Sigma_n$ $(\alpha > 0, \beta > 0)$. If $(C_1(E), N) \geqq -\frac{1}{2}\Delta(E) + c(a, b, r, n) + a$ and $(C_1(E), M) \geqq -\frac{1}{2}\Delta(E) + c(a, b, r, n) - an + b$ then $E$ is ample where $a = a(E), b = b(E)$ and $c(a, b, r, n) = \frac{1}{2}an(a + r) - r(a + b + ab + r - 2)$.*

In § 4, we shall show that Theorem 2 is best possible in some cases. If $E$ is an $H$-stable vector bundle of rank $r$ on $\boldsymbol{P}^2$ with $C_1(E) = \pm H$, then $C_2(E) \geqq r - 1$ (Lemma 4.1). Conversely for any couple of integers $(r, n)$ such that $n \geqq r - 1 \geqq 1$, there is an $H$-stable vector bundle $E$ of rank $r$ on $\boldsymbol{P}^2$ with $C_1(E) = H$ and $C_2(E) = n$ such that $E(t)$ is ample if and only if $E(t)$ satisfies the condition of Theorem 2 and $E^*(t)$ is ample if and only if $E^*(t)$ satisfies the condition of Theorem 2 (Theorem 4).

## § 1. Simple vector bundles

Let $S$ be a non-singular projective variety defined over $k$ and $E$ be a vector bundle (i.e. a locally free sheaf of finite rank) on $S$.

DEFINITION. $E$ is called simple if any global endomorphism of $E$ is constant i.e. $H^0(S, \mathrm{End}\,(E)) = k$.

DEFINITION. A set $\mathscr{F}$ of vector bundles on $S$ is bounded if there are an algebraic $k$-scheme $T$ and a vector bundle $V$ on $T \times S$ such that each $E$ in $\mathscr{F}$ is isomorphic to $V_t = V|_{t \times S}$ for some closed point $t$ in $T$.

Let $X$ be the projective plane $\boldsymbol{P}^2$ or a rational ruled surface $\Sigma_n = \boldsymbol{P}(O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1})$ $(n \geqq 0)$. Let $M$ be a minimal section of $\Sigma_n$ and $N$ be a fibre of $\Sigma_n$. By the same symbol $H$, we denote a hyperplane of $\boldsymbol{P}^2$ when $X = \boldsymbol{P}^2$, $H_{1,1} = (M + nN) + N$ when $X = \Sigma_n$. $H$ is a very ample divisor on $X$ and a general member of the complete linear system $|H|$ is isomorphic to the projective line $\boldsymbol{P}^1$. If $K_X$ is the canonical divisor on $X$, then $K_X \sim -3H$ when $X = \boldsymbol{P}^2$, $K_X \sim -2M - (n + 2)N$ when $X = \Sigma_n$. For a divisor $D$ on $X$ and a coherent sheaf $E$ on $X$, we denote $E \otimes O_X(D)$ by $E(D)$, $E \otimes O_X(mH)$ by $E(m)$ and the dual sheaf $\mathrm{Hom}_{O_X}(E, O_X)$ of $E$

by $E^*$. The aim of this section is;

THEOREM 1. *Let $X$ be $\boldsymbol{P}^2$ or $\Sigma_n$. For a divisor $C_1$ on $X$ and integers $C_2, r\ (\geqq 2)$, put $\mathscr{F} = \{E$; simple vector bundle of rank $r$ on $X$ with $C_i(E) = C_i$ for $i = 1, 2\}$ then $\mathscr{F}$ is bounded.*

*Proof.* For an integer $d$, let $\mathscr{F}_d$ be the subset of $\mathscr{F}$ which consists of $E$ in $\mathscr{F}$ such that $H^0(X, E(d)) = (0)$ and $H^0(X, E(d+1)) \neq (0)$, then $\mathscr{F} = \cup \mathscr{F}_d$. We separate the proof into two steps;

(a) For almost all $d$, $\mathscr{F}_d$ is empty,

(b) $\mathscr{F}_d$ is bounded for all $d$.

If (a) and (b) are proved then $\mathscr{F}$ is considered as a finite union of bounded families and so $\mathscr{F}$ is bounded. Before proving (a) and (b), we introduce one more notation. For $E$ in $\mathscr{F}$, let $P$ be the numerical polynomial defined by $P(m) = \chi(X, E(m)) = \Sigma(-1)^i h^i(X, E(m))$ where $h^i(X, E(m)) = \dim_k H^i(X, E(m))$. Since $H$ is ample and $X$ is a surface, $P$ is of degree two and $P(m) \to \infty$ if $m \to \pm\infty$. $P$ is independent from a choice of $E$ in $\mathscr{F}$.

(a) We shall prove that if $\mathscr{F}_d$ is not empty then $P(d) \leqq 0$. Hence such $d$'s are finite. Assume that $\mathscr{F}_d$ is not empty. Let $E$ be an element of $\mathscr{F}_d$, then $H^0(X, E(d)) = (0)$ and $H^0(X, E(d+1)) \neq (0)$. We want to prove that $H^2(X, E(d)) = (0)$. If this is proved, then $P(d) = -h^1(X, E(d)) \leqq 0$. The dual of $H^2(X, E(d))$ is isomorphic to $H^0(X, E(d)^* \otimes O_X(K_X))$ (Serre duality) and $E(d)^* \otimes O_X(K_X) \cong E(d+1)^* \otimes O_X(K_X + H)$. Since $-K_X - H$ is linearly equivalent to an effective divisor, $H^0(X, E(d)^* \otimes O_X(K_X)) \subset H^0(X, E(d+1)^*)$. Therefore it suffices to prove that $H^0(X, E(d+1)^*) = (0)$. This follows from;

LEMMA 1.1. ((4) Proposition 1.) *Let $E'$ be a vector bundle on a non-singular variety $S$ defined over $k$. If $H^0(S, E') \neq (0)$, $H^0(S, E'^*) \neq (0)$ and $E'$ is not a line bundle than $E'$ is not simple.*

Since $E$ is simple and $H^0(X, E(d+1)) \neq (0)$, $H^0(X, (X, E(d+1)^*) = (0)$ by Lemma 1.1.

(b) By a theorem of Kleiman ((2) Theorem 1.13), it is sufficient to show that there are integers $m_1, m_2$ such that for any $E$ in $\mathscr{F}_d(d)$, i) $h^0(X, E) \leqq m_1$ ii) $h^0(\ell, E|_\ell) \leqq m_2$ for a general member $\ell$ in $|H|$ where $\mathscr{F}_d(d) = \{E(d); E$ in $\mathscr{F}_d\}$. By the definition of $\mathscr{F}_d$, $m_1 = 0$ satisfies i). We now show ii). For a general member $\ell$ in $|H|$ and $E$ in $\mathscr{F}_d(d)$,

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there is a long exact sequence of cohomologies;

$$\cdots \to H^0(X,E) \to H^0(\ell, E|_\ell) \to H^1(X, E(-1)) \to \cdots .$$

Since $H^0(X,E) = (0)$,

$$h^0(\ell, E|_\ell) \leqq h^1(X, E(-1)) . \tag{1}$$

If $X = \boldsymbol{P}^2$ then $h^2(\boldsymbol{P}^2, E(-1)) = h^0(\boldsymbol{P}^2, E(-1)^* \otimes O_{\boldsymbol{P}^2}(K_{\boldsymbol{P}^2})) = h^0(\boldsymbol{P}^2, E(1)^* \otimes O_{\boldsymbol{P}^2}(-1))$. Since $h^0(\boldsymbol{P}^2, E(1)) \neq 0$ and $E$ is simple, $h^0(\boldsymbol{P}^2, E(1)^*) = 0$ by Lemma 1.1. Hence $h_2(\boldsymbol{P}^2, E(-1)) = 0$ and also $h^0(\boldsymbol{P}^2, E(-1)) = 0$. Therefore $h^1(\boldsymbol{P}^2, E(-1)) = -P(d-1)$. This and (1) show that $m_2 = -P(d-1)$ satisfies ii) when $X = \boldsymbol{P}^2$. Now assume $X = \Sigma_n$. Put $F = E(1)^*$ and consider the following long exact sequence of cohomologies;

$$\cdots \to H^0(\Sigma_n, F) \to H^0(N, F|_N) \to H^1(\Sigma_n, F(-N)) \to \cdots .$$

Since $H^0(\Sigma_n, F) = (0)$, we have $h^0(N, F|_N) \leqq h^1(\Sigma_n, F(-N))$. On the other hand, $h^2(\Sigma_n, F(-N)) = h^0(\Sigma_n, F(-N)^* \otimes O_{\Sigma_n}(K_{\Sigma_n})) = h^0(\Sigma_n, E \otimes O_{\Sigma_n}(-M)) = 0$ and $h^0(\Sigma_n, F(-N)) = 0$, therefore $h^0(N, F|_N) \leqq -\chi(\Sigma_n, F(-N))$. Note that $\chi(\Sigma_n, F(-N))$ is dependent only on $\mathscr{F}$ and $d$. Since $N$ is a fibre of $\Sigma_n$, $F(mN)|_N \cong F|_N$ for any integer $m$. Now consider the following long exact sequences of cohomologies;

$$0 \to H^0(\Sigma_n, F((m-1)N)) \to H^0(\Sigma_n, F(mN)) \to H^0(N, F(mN)|_N) \to \cdots$$

for $m = 0, \cdots, n$, then we have;

$$\begin{aligned} h^0(\Sigma_n, F(nN)) &\leqq h^0(\Sigma_n, F((n-1)N)) + h^0(N, F(nN)|_N) \\ &= h^0(\Sigma_n, F((n-1)N)) + h^0(N, F|_N) \\ &\cdots \\ &\leqq nh^0(N, F|_N) \leqq -n\chi(\Sigma_n, F(-N)) . \end{aligned} \tag{2}$$

Since $h^0(\Sigma_n, E(-1)) = 0$ and $h^2(\Sigma_n, E(-1)) = h^0(\Sigma_n, E(-1)^* \otimes O_{\Sigma_n}(K_{\Sigma_n})) = h^0(\Sigma_n, E(1)^* \otimes O_{\Sigma_n}(nN)) = h^0(\Sigma_n, F(nN))$, $h^1(\Sigma_n, E(-1)) = -P(d-1) + h^0(\Sigma_n, F(nN))$. Therefore, (1) and (2) show that $m_2 = -P(d-1) - n\chi(\Sigma_n, F(-N))$ satisfies ii) when $X = \Sigma_n$.

## §2. *H*-stable vector bundles on $\boldsymbol{P}^2$

Let $E$ be a vector bundle on a non-singular projective surface $S$ defined over $k$ and $H$ be an ample divisor on $S$.

DEFINITION. $E$ is $H$-stable if for every non-zero coherent subsheaf

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$F$ of $E$ of rank $< r(E), (C_1(F), H)/r(F) < (C_1(E), H)/r(E)$ where $r(F)$ is the rank of $F$.

We refer to [5] for basic properties of $H$-stable vector bundles. For a vector bundle $E$ on $S$, put $\Delta(E) = (r-1)C_1(E)^2 - 2rC_2(E)$. This integer is equal to $-C_2(\mathrm{End}\, E)$. If $E$ is a vector bundle of rank $r$ on $\boldsymbol{P}^2$ then there exists uniquely a line bundle $L$ on $\boldsymbol{P}^2$ such that $C_1(E \otimes L) = aH$ with $-r+1 \leqq a \leqq 0$, where $H$ is a hyperplane of $\boldsymbol{P}^2$. Put $a(E) = a$. The aim of this section is;

THEOREM 2. *Let $E$ be an $H$-stable vector bundle of rank $r$ on $\boldsymbol{P}^2$. If $(C_1(E), H) \geqq -\frac{1}{2}\Delta(E) + (a+2r)(2-a-r)/2$ then $E$ is ample where $a = a(E)$.*

In order to prove Theorem 2, we need the following lemma.

LEMMA 2.1. *Let $E$ be an $H$-stable vector bundle of rank $r$ on $\boldsymbol{P}^2$ such that $C_1(E) = aH$ with $a = a(E)$ then;*

(1) $h^0(\boldsymbol{P}^2, E) = 0$,

(2) $h^2(\boldsymbol{P}^2, E(m)) = 0$ *for any* $m \geqq 0$,

(3) $h^1(\boldsymbol{P}^2, E(m)) \leqq h^1(\boldsymbol{P}^2, E(m-1))$ *for any* $m \geqq 1$,

(4) *If* $h^1(\boldsymbol{P}^2, E(m)) = h^1(\boldsymbol{P}^2, E(m-1))$ *for some* $m \geqq 1$,

*then $E(m)$ is generated by its global sections.*

*Proof.* (1) If $h^0(\boldsymbol{P}^2, E) \neq 0$ then $E$ contains $O_{\boldsymbol{P}^2}$ as a subsheaf but $(C_1(E), H) = a \leqq 0$. Since $E$ is $H$-stable, this cannot occur. (2) Since $E^*$ is also $H$-stable and $(C_1(E(m)^* \otimes O_{\boldsymbol{P}^2}(-3)), H) = -a - r(m+3) \leqq 0$ for any $m \geqq 0$, $h^2(\boldsymbol{P}^2, E(m)) = 0$ for any $m \geqq 0$ by the Serre duality. (3) Let $F_m$ be the smallest subsheaf of $E(m)$ such that $H^0(\boldsymbol{P}^2, F_m) = H^0(\boldsymbol{P}^2, E(m))$ and $E(m)/F_m$ is torsion free. Note that $H^0(\boldsymbol{P}^2, F_m(-1)) = H^0(\boldsymbol{P}^2, E(m-1))$. Let $\ell$ be a general member of $|H|$ such that $F_m|_\ell$ is locally free on $\ell$ and $0 \to F_m(-1) \to F_m \to F_m|_\ell \to 0$ is exact. Since $F_m$ is generically generated by its global sections and $\ell \cong \boldsymbol{P}^1$, $F_m|_\ell$ is generated by its global sections and $h^1(\ell, F_m|_\ell) = 0$ for a suitable choice of $\ell$. Considering the following long exact sequence of cohomologies;

$$\cdots \to H^1(\boldsymbol{P}^2, F_m(-1)) \to H^1(\boldsymbol{P}^2, F_m) \to H^1(\ell, F_m|_\ell)$$
$$\to H^2(\boldsymbol{P}^2, F_m(-1)) \to H^2(\boldsymbol{P}^2, F_m) \to 0$$

we have $h^1(\boldsymbol{P}^2, F_m) \leqq h^1(\boldsymbol{P}^2, F_m(-1))$ and $h^2(\boldsymbol{P}^2, F_m) = h^2(\boldsymbol{P}^2, F_m(-1))$. Hence we have;

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$$\begin{aligned} h^1(\boldsymbol{P}^2, E(m)) - h^1(\boldsymbol{P}^2, E(m-1)) \\ &= h^0(\boldsymbol{P}^2, E(m)) - h^0(\boldsymbol{P}^2, E(m-1)) - (\chi(\boldsymbol{P}^2, E(m)) - \chi(\boldsymbol{P}^2, E(m-1))) \\ &= h^0(\boldsymbol{P}^2, F_m) - h^0(\boldsymbol{P}^2, F_m(-1)) - (r + (C_1(E(m)), H)) \\ &= h^1(\boldsymbol{P}^2, F_m) - h^1(\boldsymbol{P}^2, F_m(-1)) + (\chi(\boldsymbol{P}^2, F_m) \\ &\quad - \chi(\boldsymbol{P}^2, F_m(-1))) - (r + (C_1(E(m)), H)) \\ &\leqq (r' + (C_1(F_m), H)) - (r + (C_1(E(m)), H)) \end{aligned}$$

where $r' =$ rank of $F_m$. Since $E$ is $H$-stable and $(C_1(E(m)), H) = a + rm > 0$, $(C_1(F_m), H) \leqq (C_1(E(m)), H)$ therefore $h^1(\boldsymbol{P}^2, E(m)) \leqq h^1(\boldsymbol{P}^2, E(m-1))$.

(4) If $h^1(\boldsymbol{P}^2, E(m)) = h^1(\boldsymbol{P}^2, E(m-1))$ then $F_m = E(m)$ by the above inequality. Hence for a general member $\ell$ in $|H|$, $E(m)|_\ell$ is generated by its global sections and $h^1(\ell, E(m)|_\ell) = 0$. Consider the following long exact sequence of cohomologies;

$$\begin{aligned} \cdots \to H^0(\boldsymbol{P}^2, E(m)) \to H^0(\ell, E(m)|_\ell) \\ \to H^1(\boldsymbol{P}^2, E(m-1)) \to H^1(\boldsymbol{P}^2, E(m)) \to H^1(\ell, E(m)|_\ell) \to \cdots . \end{aligned}$$

Since $h^1(\ell, E(m)|_\ell) = (0)$ and $h^1(\boldsymbol{P}^2, E(m)) = h^1(\boldsymbol{P}^2, E(m-1))$, $H^0(\boldsymbol{P}^2, E(m)) \to H^0(\ell, E(m)|_\ell)$ is surjective. Hence for any closed point $x$ in $\ell$, $H^0(\boldsymbol{P}^2, E(m)) \to E(m) \otimes k(x)$ is surjective. On the other hand for any closed point $y$ in $X - \ell$, take a member $\ell'$ in $|H|$ such that $\ell'$ contains $y$ and take $x$ in $\ell \cap \ell'$. Now consider the following commutative diagram;

$$\begin{array}{ccc} H^0(\boldsymbol{P}^2, E(m)) & \longrightarrow & E(m) \otimes k(x) \\ & \searrow \quad \nearrow & \\ & H^0(\ell', E(m)|_{\ell'}) & \end{array}$$

Since $H^0(\boldsymbol{P}^2, E(m)) \to E(m) \otimes k(x)$ is surjective, $H^0(\ell', E(m)|_{\ell'}) \to E(m) \otimes k(x)$ is surjective therefore $E(m)|_{\ell'}$ is generated by its global sections and $h^1(\ell', E(m)|_{\ell'}) = 0$. As the above argument for $E(m)|_\ell$, we have that $H^0(\boldsymbol{P}^2, E(m)) \to E(m) \otimes k(y)$ is surjective. Hence $E(m)$ is generated by its global sections by Nakayama's lemma.

COROLLARY 2.2. *Let $E$ be as in Lemma 2.1 then $E(-\chi(P^2, E) + 2)$ is ample.*

*Proof.* $h^1(\boldsymbol{P}^2, E) = -\chi(P^2, E)$ by Lemma 2.1 (1) and (2). Put $c = -\chi(\boldsymbol{P}^2, E)$, then by Lemma 2.1 (3) we have;

$$c = h^1(\boldsymbol{P}^2, E) \geqq h^1(\boldsymbol{P}^2, E(1)) \geqq \cdots \geqq h^1(\boldsymbol{P}^2, E(c)) \geqq h^1(\boldsymbol{P}^2, E(c+1)) \geqq 0 .$$

Hence there must be an integer $m$ $(1 \leqq m \leqq c+1)$ such that $h^1(\boldsymbol{P}^2, E(m)) = h^1(\boldsymbol{P}^2, E(m-1))$. Hence $E(m)$ is generated by its global sections by Lemma 2.1 (4) therefore $E(-\chi(\boldsymbol{P}^2, E) + 2)$ is ample.

*Proof of Theorem* 2. Let $E$ be as in Theorem 2, then there is a line bnndle $L$ on $\boldsymbol{P}^2$ such that for $E' = E \otimes L$, $C_1(E') = aH$. It is easily calculated that $(C_1(E'(-\chi(\boldsymbol{P}^2, E') + 2)), H) = -\frac{1}{2}\Delta(E) + (a+2r)(2-a-r)/2$. For $E'' = E'(-\chi(\boldsymbol{P}^2, E') + 2)$, there is a line bundle $L'$ on $\boldsymbol{P}^2$ such that $E = E'' \otimes L'$. By the condition of Theorem 2, we have $(C_1(E'' \otimes L'), H) \geqq (C_1(E''), H)$. Hence $(C_1(L'), H) \geqq 0$. This is equivalent to that $L'$ is generated by its global sections. Since $E''$ is ample by Corollary 2.2, $E = E'' \otimes L'$ is ample.

## § 3. $H_{\alpha,\beta}$-stable vector bundles on $\Sigma_n$

Let $\Sigma_n = \boldsymbol{P}(O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1})$ $(n \geqq 1)$ be a rational ruled surface and let $M$ be a minimal section of $\Sigma_n$ and $N$ be a fibre of $\Sigma_n$. The divisor class group of $\Sigma_n$ is generated by the classes of $M$ and $N$. For a couple of integers $(\alpha, \beta)$, we denote $\alpha(M + nN) + \beta N$ by $H_{\alpha,\beta}$. The intersection numbers $(H_{\alpha,\beta}, N)$ and $(H_{\alpha,\beta}, M)$ are $\alpha$ and $\beta$ respectively. $H_{\alpha,\beta}$ is ample if and only if $\alpha > 0$, $\beta > 0$ and the complete linear system $|H_{\alpha,\beta}|$ is base point free if and only if $\alpha \geqq 0$, $\beta \geqq 0$ ((1) Lemma (3.1)). For a vector bundle $E$ of rank $r$ on $\Sigma_n$, there exists uniquely a line bundle $L$ on $\Sigma_n$ such that $C_1(E \otimes L) = aM + bN$ with $-r+1 \leqq a, b \leqq 0$. Put $a(E) = a$ and $b(E) = b$. The aim of this section is;

THEOREM 3. *Let $E$ be an $H_{\alpha,\beta}$-stable vector bundle of rank $r$ on $\Sigma_n$ $(\alpha > 0, \beta > 0)$. If $(C_1(E), N) \geqq -\frac{1}{2}\Delta(E) + c(a,b,r,n) + a$ and $(C_1(E), M) \geqq -\frac{1}{2}\Delta(E) + c(a,b,r,n) - an + b$ then $E$ is ample where $a = a(E)$, $b = b(E)$ and $c(a,b,r,n) = \frac{1}{2}an(a+r) - r(a+b+ab+r-2)$.*

In order to prove Theorem 3, we need some lemmas.

LEMMA 3.1. *Let $E$ be an $H_{\alpha,\beta}$-stable vector bundle of rank $r$ on $\Sigma_n$ with $C_1(E) = aM + bN$ such that $a = a(E)$, $b = b(E)$, then;*

(1) $h^0(\Sigma_n, E) = 0$

(2) $h^2(\Sigma_n, E(D)) = 0$ *for any effective divisor $D$ on $\Sigma_n$.*

*Proof.* The proof is similar to that of Lemma 2.1 (1), (2).

LEMMA 3.2. *Let $E$ be an $H_{\alpha,\beta}$-stable vector bundle of rank $r$ on $\Sigma_n$ with $C_1(E) = aM + bN$ such that $a \geqq a(E)$, $b \geqq b(E)$ and let $F$ be the smallest subsheaf of $E(H_{1,1})$ such that $H^0(\Sigma_n, F) = H^0(\Sigma_n, E(H_{1,1}))$ and $E(H_{1,1})/F$ is torsion free, then;*

(1) *if $r' = \text{rank of } F < r$ then $h^1(\Sigma_n, E(H_{0,1})) < h^1(\Sigma_n, E)$ or $h^1(\Sigma_n, E(H_{1,0})) < h^1(\Sigma_n, E)$,*

(2) *if $r' = r$ (i.e. $E(H_{1,1})$ is generically generated by its global sections) then $h^1(\Sigma_n, E(H_{1,1})) \leqq h^1(\Sigma_n, E)$ and if $h^1(\Sigma_n, E(H_{1,1})) = h^1(\Sigma_n, E)$ then $E)H_{1,1})$ is generated by its global sections.*

*Proof.* (1) Put $C_1(E(H_{1,1})) = uM + vN$ and $C_1(F) = u'M + v'N$, then by the stability of $E$ we have;

$$\frac{\beta u' + \alpha v'}{r'} < \frac{\beta u + \alpha v}{r} .$$

Since $\alpha > 0$, $\beta > 0$, $u > 0$, $v > 0$ and $r' < r$, we have $u' < u$ or $v' < v$. We want to prove that (i) if $u' < u$ then $h^1(\Sigma_n, E(H_{0,1})) < h^1(\Sigma_n, E)$ (ii) if $v' < v$ then $h^1(\Sigma_n, E(H_{1,0})) < h^1(\Sigma_n, E)$.

(i) Assume $u' < u$. Let $\ell$ be a general member of $|H_{0,1}|$ such that $F|_\ell$ is locally free and $0 \to F(-H_{1,1}) \to F(-H_{1,0}) \to F(-H_{1,0})|_\ell \to 0$ is exact. Since $\ell$ is a fibre of $\Sigma_n$, $\ell$ is isomorphic to the projective line and since $F$ is generically generated by its global sections, $F|_\ell$ is generated by its global sections for a suitable choice of $\ell$. The intersection number $(-H_{1,0}, \ell)$ is $-1$ so we have $h^1(\ell, F(-H_{1,0})|_\ell) = 0$ for a suitable choice of $\ell$. Considering the following long exact sequence of cohomologies;

$$\begin{aligned}\cdots &\to H^1(\Sigma_n, F(-H_{1,1})) \to H^1(\Sigma_n, F(-H_{1,0})) \to H^1(\ell, F(-H_{1,0})|_\ell)\\ &\to H^2(\Sigma_n, F(-H_{1,1})) \to H^2(\Sigma_n, F(-H_{1,0})) \to 0\end{aligned}$$

we have $h^1(\Sigma_n, F(-H_{1,0})) \leqq h^1(\Sigma_n, F(-H_{1,1}))$ and $h^2(\Sigma_n, F(-H_{1,0})) = h^2(\Sigma_n, F(-H_{1,1}))$. Note that $h^0(\Sigma_n, E(H_{0,1})) = h^0(\Sigma_n, F(-H_{1,0}))$ and $h^0(\Sigma_n, E) = h^0(\Sigma_n, F(-H_{1,1}))$, hence we have;

$$\begin{aligned}&h^1(\Sigma_n, E(H_{0,1})) - h^1(\Sigma_n, E)\\ &\quad= h^0(\Sigma_n, E(H_{0,1})) - h^0(\Sigma_n, E) - (\chi(\Sigma_n, E(H_{0,1})) - \chi(\Sigma_n, E))\\ &\quad= h^0(\Sigma_n, F(-H_{1,0})) - h^0(\Sigma_n, F(-H_{1,1})) - (r + (C_1(E(H_{0,1})), H_{0,1}))\\ &\quad= h^1(\Sigma_n, F(-H_{1,0})) - h^1(\Sigma_n, F(-H_{1,1})) + (\chi(\Sigma_n, F(-H_{1,0}))\\ &\qquad - \chi(\Sigma_n, F(-H_{1,1})) - u\\ &\quad\leqq u' - u < 0 .\end{aligned}$$

(ii) Assume $v' < v$. A general member $\ell$ of $|H_{1,0}|$ is a section of $\Sigma_n$ so $\ell$ is isomorphic to the projective line and $(-H_{0,1}, \ell) = -1$. Hence $h^1(\Sigma_n, E(H_{1,0})) < h^1(\Sigma_n, E)$ is similarly obtained as above.

(2) The proof is similar to that of Lemma 2.1 (3), (4).

COROLLARY 3.3. *Let $E$ be as in Lemma* 3.1, *then* $E((-\chi(\Sigma_n, E) + 2)H_{1,1})$ *is ample.*

*Proof.* $h^1(\Sigma_n, E) = -\chi(\Sigma_n, E)$ by Lemma 3.1. Put $c = -\chi(\Sigma_n, E)$. By Lemma 3.2 (1), there are integers $p \geqq 0, q \geqq 0$ such that for $E' = E(H_{p,q})$, $h^1(\Sigma_n, E') \leqq c - (p + q)$ and $E'(H_{1,1})$ is generically generated by its global sections. Put $c' = h^1(\Sigma_n, E')$ then by Lemma 3.2 (2) we have;

$$\begin{aligned} c' = h^1(\Sigma_n, E') &\geqq h^1(\Sigma_n, E'(H_{1,1})) \geqq \cdots \\ &\geqq h^1(\Sigma_n, E'(c'H_{1,1})) \geqq h^1(\Sigma_n, E'((c' + 1)H_{1,1})) \geqq 0 . \end{aligned}$$

Hence there must be an integer $m$ $(1 \leqq m \leqq c' + 1)$ such that $h^1(\Sigma_n, E'((m - 1)H_{1,1})) = h^1(\Sigma_n, E'(mH_{1,1}))$. Hence by Lemma 3.2 (2), $E'(mH_{1,1})$ is generated by its global sections, therefore $E'((c' + 2)H_{1,1})$ is ample. On the other hand $E((c + 2)H_{1,1}) = E'((c' + 2)H_{1,1}) \otimes O_{\Sigma_n}(H_{q,p} + (c - (p + q) - c')H_{1,1})$ and $c - (p + q) - c' \geqq 0$, so $E((c + 2)H_{1,1})$ is ample.

*Proof of Theorem* 3. Let $E$ be as in Theorem 3, then there is a line bundle $L$ on $\Sigma_n$ such that for $E' = E \otimes L$, $C_1(E') = aM + bN$. It is easily calculated that for $E'' = E'((-\chi(\Sigma_n, E') + 2)H_{1,1})$, $(c_1(E''), N) = -\frac{1}{2}\Delta(E) + c(a, b, r, n) + a$ and $(C_1(E''), M) = -\frac{1}{2}\Delta(E) + c(a, b, r, n) - an + b$. There are integers $p, q$ such that $E = E''(H_{p,q})$. By the condition of Theorem 3, we have $(C_1(E''(H_{p,q})), N) \geqq (C_1(E''), N)$ and $(C_1(E''(H_{p,q})), M) \geqq (C_1(E''), M)$. Hence $(H_{p,q}, N) = p \geqq 0$ and $(H_{p,q}, M) = q \geqq 0$. This is equivalent to that $O_{\Sigma_n}(H_{p,q})$ is generated by its global sections. Since $E''$ is ample by Corollary 3.5, $E = E''(H_{p,q})$ is ample.

## § 4. Examples of $H$-stable vector bundles on $P^2$

In this section we shall show that Theorem 2 is best possible when $a = -r + 1$ or $-1$. Let $H$ be a hyperplane of $P^2$. We begin with a simple lemma.

LEMMA 4.1. *Let $E$ be an $H$-stable vector bundle of rank $r$ on $P^2$. If $C_1(E) = H$ or $-H$ then $C_2(E) \geqq r - 1$.*

*Proof.* Since $C_1(E^*) = -C_1(E)$ and $C_2(E^*) = C_2(E)$, we may assume

$C_1(E) = -H$. By Lemma 2.1 (1), (2), $h^0(\boldsymbol{P}^2, E) = h^2(\boldsymbol{P}^2, E) = 0$. Hence $-h^1(\boldsymbol{P}^2, E) = \chi(\boldsymbol{P}^2, E) = r + (C_1(E), 3H)/2 + (C_1(E)^2 - 2C_2(E))/2 = r - 1 - C_2(E)$ by the Riemann-Roch theorem. Therefore $C_2(E) \geqq r - 1$.

The following lemma is due to Maruyama ((3) Theorem 4.6).

LEMMA 4.2. *Let $\ell$ be a line on $\boldsymbol{P}^2$ and $n \geqq 1$ be an integer, then there is an $H$-stable vector bundle of rank 2 on $\boldsymbol{P}^2$ such that $C_1(E) = H$, $C_2(E) = n$ and $E|_\ell \cong O_\ell(-n+1) \oplus O_\ell(n)$ where $O_\ell(n)$ is the line bundle on $\ell$ with* $\deg(O_\ell(n)) = n$.

LEMMA 4.3. *Let $E$ be an $H$-stable vector bundle of rank $r$ on $\boldsymbol{P}^2$ with $C_1(E) = H$. If there is a short exact sequence of vector bundles;*

$$0 \to O_{p^2} \to E' \to E \to 0 \qquad (*)$$

*and this is not split then $E'$ is $H$-stable.*

*Proof.* Let $F$ be a non-trivial subsheaf of $E'$ such that the rank of $F < r+1$ and $E'/F$ is torsion free. Since $C_1(E') = H$, it is sufficient to show that $(C_1(F), H) \leqq 0$. Put $L = F \cap O_{\boldsymbol{P}^2}$ and $F'$ be the image of $F$ in $E$, then there is a short exact sequence $0 \to L \to F \to F' \to 0$. Since $O_{\boldsymbol{P}^2}$ and $E$ are $H$-stable, $(C_1(L), H) \leqq 0$ and $(C_1(F'), H) \leqq 1$ hence $(C_1(F), H) \leqq 1$. Therefore it is sufficient to show that $(C_1(F), H) \neq 1$. If it were happened then $(C_1(L), H) = 0$ and $(C_1(F'), H) = 1$. This is possible if and only if $L = (0)$ and $\dim \operatorname{supp}(E/F') \leqq 0$, by the $H$-stability of $E$. Since (*) is not split, $E/F' \neq (0)$. There is a short exact sequence $0 \to O_{\boldsymbol{P}^2} \to E'/F \to E/F' \to 0$. But $H^0(\boldsymbol{P}^2, (E/F')(m)) \neq (0)$ and $H^1(\boldsymbol{P}^2, O_{\boldsymbol{P}^2}(m)) = (0)$ for all $m$ and since $E'/F$ is torsion free, $H^0(\boldsymbol{P}^2(E'/F(m)) = (0)$ for $m \ll 0$. This is a contradiction.

The aim of this section is the following theorem which shows that the converse of Lemma 4.1 and that Theorem 2 is best possible when $a = -r+1$ or $-1$.

THEOREM 4. *Put $A = \{(r, n); n \geqq r - 1 \geqq 1\}$. Let $\ell$ be a line on $\boldsymbol{P}^2$. Then there is a set $S = \{E_{(r,n)}\}_{(r,n) \in A}$ of vector bundles on $\boldsymbol{P}^2$ which satisfies the following conditions;*

(1) *$S$ consists of $H$-stable vector bundles,*

(2) *the rank of $E_{(r,n)}$ is $r$, $C_1(E_{(r,n)}) = H$ and $C_2(E_{(r,n)}) = n$ for all $(r, n) \in A$,*

(3) *there is a short exact sequence $0 \to O_{\boldsymbol{P}^2} \to E_{(r,n)} \to E_{(r-1,n)} \to 0$ and*

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*this is not split,*

(4) $h^1(\boldsymbol{P}^2, E^*_{(r,n)}) = n - r + 1$,

(5) $E_{(r,n)}|_\ell \cong O_\ell(-n+1) \oplus O_\ell(n-r+2) \oplus \sum^{r-2} O_\ell(1)$ *where* $\sum^{r-2} O_\ell(1) = O_\ell(1) \oplus \cdots \oplus O_\ell(1)$ $(r-2$ *times*),

(6) $H^1(\boldsymbol{P}^2, E^*_{(r,n)}) \cong H^1(\ell, E^*_{(r,n)}|_\ell)$ *canonically,*

(7) $E_{(r,n)}(t)$ *is ample if and only if* $E_{(r,n)}(t)$ *satisfies the condition of Theorem* 2,

(8) $E^*_{(r,n)}(t)$ *is ample if and only if* $E^*_{(r,n)}(t)$ *satisfies the condition of Theorem* 2.

*Proof.* The above conditions are not independent each other. In fact;

( i ) (1), (2) and (3) for $E_{(r-1,n)} \Rightarrow$ (1) for $E_{(r,n)}$ by Lemma 4.3,

( ii ) (2) and (3) for $E_{(r-1,n)} \Rightarrow$ (2) for $E_{(r,n)}$,

(iii) (1) and (2) $\Rightarrow$ (4) by the Riemann-Roch theorem and Lemma 2.1 (1), (2),

(iv) (1), (2), (4) and (5) $\Rightarrow$ (6),

( v ) (1), (2) and (5) $\Rightarrow$ (7),

(vi) (1), (2) and (5) $\Rightarrow$ (8).

(v) and (vi) are easily checked by considering $E_{(r,n)}(t)|_\ell$ and $E^*_{(r,n)}(t)|_\ell$ respectively. We now show (iv). Consider the following long exact sequence of cohomologies;

$$\cdots \to H^1(\boldsymbol{P}^2, E^*_{(r,n)}) \to H^1(\ell, E^*_{(r,n)}|_\ell) \to H^2(\boldsymbol{P}^2, E^*_{(r,n)}(-1)) \to \cdots .$$

Since $(C_1(E_{(r,n)}(-2)), H) < 0$ by (2), $H^2(\boldsymbol{P}^2, E^*_{(r,n)}(-1)) = (0)$ by (1). Moreover $h^1(\boldsymbol{P}^2, E^*_{(r,n)}) = n - r + 1$ by (4) and $h^1(\ell, E^*_{(r,n)}|_\ell) = n - r + 1$ by (5) hence we have $H^1(\boldsymbol{P}^2, E^*_{(r,n)}) \cong H^1(\ell, E^*_{(r,n)}|_\ell)$ canonically.

By Lemma 4.2, for any $n \geqq 1$, there is a vector bundle $E_{(2,n)}$ such that $E_{(2,n)}$ satisfies (1), (2) and (5). Lastly we constant $E_{(r,n)}$ which satisfies (3) and (5) by (5) and (6) for $E_{(r-1,n)}$. There is a short exact sequence;

$$0 \to O_\ell \to O_\ell(-n+1) \oplus O_\ell(n-r+2) \oplus \sum^{r-1} \oplus O_\ell(1) \to E_{(r-1,n)}|_\ell \to 0 \quad (\divideontimes)$$

of vector bundles on $\ell$ by (5) for $E_{(r-1,n)}$. ($\divideontimes$) has an obstruction in $H^1(\ell, E^*_{(r-1,n)}|_\ell)$ hence there is a short exact sequence $0 \to O_{\boldsymbol{P}^2} \to E_{(r,n)} \to E_{(r-1,n)} \to 0$ such that its restriction to $\ell$ is isomorphic to ($\divideontimes$) by (6) for $E_{(r-1,n)}$. This short exact sequence is not split and $E_{(r,n)}$ satisfies (5) by

(⁑). All these together we have constructed $S = \{E_{(r,n)}\}_{(r,n)\in A}$ which satisfies (1)–(8).

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*Nagoya University*