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THE SUBSEISMIC APPROXIMATION FOR LOW-FREQUENCY MODES OF THE EARTH APPLIED TO LOW-DEGREE, LOW-FREQUENCY G-MODES OF NON-ROTATING STARS

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ABSTRACT The subseismic approximation for low-frequency modes in the Earth is used in an asymptotic treatment of low-frequency g-modes belonging to low-degree spherical harmonics in a non-rotating star.

INTRODUCTION

In an investigation on normal modes of rotating, self-gravitating, compressible, stratified, fluid bodies, Rochester (1989) adopted the earlier suggestion of Smylie and Rochester (1981) that, for many low-frequency modes in the Earth, the contribution of the Eulerian perturbation of the pressure to the Lagrangian perturbation of that quantity is negligible. This approximation is referred to as the subseismic approximation since it is used for modes with frequencies that are low compared with the lower limit of the frequencies of acoustic modes usually studied in terrestrial seismology.

We examined the adequacy of the suggestion of Smylie and Rochester for low-frequency g-modes belonging to low-degree spherical harmonics in a *non-rotating* star by developing an asymptotic representation for these modes based on the subseismic approximation.

BASIC EQUATIONS

Let r be the radial distance from the center, ρ the mass density, P the pressure, Φ the gravitational potential, g the gravity, $\Gamma_1 \equiv (\partial \ln P/\partial \ln \rho)_S$ one of the generalized isentropic coefficients, and $c \equiv (\Gamma_1 P/\rho)^{1/2}$ the isentropic sound velocity. $\vec{\xi}$ is the Lagrangian displacement. We denote the Eulerian perturbation of a quantity by a prime on that quantity.

For normal modes depending on time t by $\exp(i\sigma t)$, the equations of motion, conservation of mass, and conservation of entropy can be combined to give

$$\nabla \cdot \vec{\xi} = -\frac{1}{\rho c^2}\left(P' + \frac{dP}{dr}\xi_r\right), \qquad (1)$$

$$\sigma^2 \vec{\xi} = \nabla\left(\Phi' + \frac{P'}{\rho}\right) - \frac{N^2}{\rho g}\left(P' + \frac{dP}{dr}\xi_r\right)\vec{1_r}, \qquad (2)$$

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where $\vec{1_r}$ is the local unit vector in the radial direction, and N^2 the square of the Brunt-Väisälä frequency defined as

$$N^2 = -g\left(\frac{g}{c^2} + \frac{d\ln\rho}{dr}\right). \tag{3}$$

Adopting the subseismic approximation, we neglect P' relative to $(dP/dr)\xi_r$ in the right-hand members of Eqs. (1) and (2). For modes with a Lagrangian displacement $\vec{\xi}$ related to a single spherical harmonic $Y_\ell^m(\theta,\phi)$ as

$$\left[\frac{u(r)}{r^2}, \frac{v(r)}{r}\frac{\partial}{\partial\theta}, \frac{v(r)}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]Y_\ell^m(\theta,\phi), \tag{4}$$

where θ and ϕ are the angular coordinates, we derive the second-order differential equation for the function u

$$\frac{d^2u}{dr^2} - \frac{g}{c^2}\frac{du}{dr} + \left[\frac{\ell(\ell+1)}{r^2}\left(\frac{N^2}{\sigma^2} - 1\right) - \frac{d}{dr}\frac{g}{c^2}\right]u = 0. \tag{5}$$

In the usual approximation in which the Eulerian perturbation of the gravitational potential is neglected, the equation for the function u takes the form

$$\frac{d^2u}{dr^2} + \frac{1}{\rho}\frac{d\rho}{dr}\frac{du}{dr} + \left[\frac{\ell(\ell+1)}{r^2}\left(\frac{N^2}{\sigma^2} - 1\right) - \frac{d}{dr}\frac{g}{c^2}\right]u$$
$$= -\frac{\sigma^2}{S_\ell^2}\left(1 - \frac{\sigma^2}{S_\ell^2}\right)^{-1}\left(\frac{c^2}{r^2}\frac{d}{dr}\frac{r^2}{c^2}\right)\left(\frac{du}{dr} - \frac{g}{c^2}u\right) - \frac{\sigma^2}{c^2}u, \tag{6}$$

where

$$S_\ell^2 = \frac{\ell(\ell+1)c^2}{r^2}. \tag{7}$$

The left-hand members of Eqs. (5) and (6) are identical apart from the coefficient of the first derivative du/dr. The right-hand member of Eq. (6) is different from zero and consists of terms involving σ^2 in their numerator. These terms are small for small values of $|\sigma^2|$ except near $r = R$. For g^+-modes, the right-hand member of Eq. (6) also displays a singularity at the point near $r = R$ where $S_\ell^2 = \sigma^2$.

In the standard asymptotic theory, one avoids the difficulties related to the large absolute value of the right-hand member of Eq. (6) near $r = R$ by passing on to a second-order differential equation in the function v for the construction of an asymptotic approximation of the solutions from $r = R$.

Eq. (5) can be used for the construction of an asymptotic approximation of the solutions from $r = R$ as well as from $r = 0$.

ASYMPTOTIC REPRESENTATION OF LOW-FREQUENCY G-MODES

For the sake of simplification, we assume that N^2 is different from zero everywhere inside the star except at $r = 0$ and keeps the same sign. Furthermore, we consider Γ_1 to be constant.

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In the frame of the subseismic approximation, the asymptotic representations of the function u constructed from $r = 0$ and from $r = R$ take the form

$$u_i = C_1 P^{-1/(2\Gamma_1)} \left(\epsilon \frac{N^2}{r^2}\right)^{-1/4} \xi_i^{1/2} J_{\ell+1/2}\left[\xi_i / \left(\epsilon\sigma^2\right)^{1/2}\right], \quad (8)$$

$$u_e = C_2 P^{-1/(2\Gamma_1)} \left(\epsilon \frac{N^2}{r^2}\right)^{-1/4} \left(2\xi_e^{1/2}\right)^{1/2} J_\nu \left[\left(2\xi_e^{1/2}\right) / \left(\epsilon\sigma^2\right)^{1/2}\right], \quad (9)$$

where

$$\epsilon = \frac{|N^2|}{N^2},\ \nu = \frac{n_e + 1}{\Gamma_1} + 1, \quad (10)$$

and

$$\xi_i(r) = [\ell(\ell+1)]^{1/2} \int_0^r \left[\epsilon \frac{N^2(r')}{r'^2}\right]^{1/2} dr', \quad (11)$$

$$2\xi_e^{1/2}(r) = [\ell(\ell+1)]^{1/2} \int_r^R \left[\epsilon \frac{N^2(r')}{r'^2}\right]^{1/2} dr'. \quad (12)$$

In these solutions, n_e is the effective polytropic index characterizing the surface layers of the star.

The degree ν of the Bessel function in the solution constructed from $r = R$ differs from the degree that is found in the standard asymptotic theory. This difference affects the equation for the eigenfrequencies. The ratio of the asymptotic eigenfrequencies determined on the basis of the subseismic approximation to the asymptotic eigenfrequencies determined by means of the standard asymptotic theory is given by

$$\frac{\left[(\epsilon\sigma^2)^{1/2}\right]_{subs.}}{\left[(\epsilon\sigma^2)^{1/2}\right]_{stan.}} = \frac{\ell + 1/2 + n_e + 2k}{\ell + 1/2 + (n_e + 1)/\Gamma_1 + 2k}. \quad (13)$$

The subseismic approximation leads to an inadequate asymptotic representation of the radial component of the Lagrangian displacement near the star's surface since this component becomes zero at $r = R$ and, for g^+-modes, displays one node less near $r = R$ than its regular number. Another consequence of the subseismic approximation is that the radial component of the Lagrangian displacement contains the factor $P^{-1/(2\Gamma_1)}$ instead of the factor $\rho^{-1/2}$.

Details and numerical results will be published in *Astronomy and Astrophysics.*

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