From $V'(r) = V_r + h'(r)V_h$, we have

$$V''(r) = V_{rr} + h'(r)V_{rh} + h'(r)\left[V_{rh} + h'(r)V_{hh}\right] + h''(r)V_h$$
$$= \frac{S_h^2(S_hV_{rr} - V_hS_{rr}) + 2S_rS_h(V_hS_{rh} - S_hV_{rh}) + S_r^2(S_hV_{rr} - V_rS_{hh})}{S_h^3},$$

on substituting for $h'(r)$ and $h''(r)$.

In the same way, for the case $V\left(r, \tilde{h}(r)\right) = V_0$, fixed, we obtain

$$S''(r) = \frac{-\left[V_h^2(S_hV_{rr} - V_hS_{rr}) + 2V_rV_h(V_rS_{rh} - S_hV_{rh}) + V_r^2(S_hV_{rr} - V_rS_{hh})\right]}{V_h^3}.$$

These are general calculations. If we evaluate at the optimising value of r for which the Key Equation $\dfrac{S_r}{S_h} = \dfrac{V_r}{V_h}$ holds, we obtain $\dfrac{V''(r)}{S''(r)} = -\dfrac{V_h}{S_h} = -\dfrac{V_r}{S_r}$, as stated in the paragraph before Example 1.

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107.19 A quadratic harmonic approximation

Introduction

Some eight hundred years ago the French archbishop Nicholas Oresme developed his beautiful proof that the n-th *harmonic number*:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

satisfies the following growth inequality:

$$H_{2^k} > 1 + \frac{k}{2},$$

and thereby presented the first example in the history of mathematics, and the first seen by countless generations of calculus students, of an infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges although its n-th term decreases to zero.

Unfortunately H_n has no (known) simple closed formula representation and so its further study demanded that mathematicians find suitable

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approximation formulas. The great Leonhard Euler applied his famous Euler–Maclaurin sum formula to obtain the following asymptotic formula:

$$H_n \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} \pm \dots \qquad (1)$$

where $\gamma \approx 0.577\dots$ is Euler's constant. If one truncates this expansion after n terms, then the *error* E_n one commits in using the truncated series as an approximation to H_n is less than the first term truncated and has the same sign.

There is considerable interest in proving simplified versions of (1) without using the heavy analytical machinery employed by Euler. For example, Robert M. Young [1] used an elegant geometrical argument to prove the *linear* approximation:

$$H_n = \ln n + \gamma + \frac{1}{2(n + \theta_n)} \qquad (0 < \theta_n < 1).$$

In this Note we will modify his argument to prove the following *quadratic* approximation.

Theorem 1.

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \varepsilon_n \qquad \text{where} \qquad 0 < \varepsilon_n < \frac{1}{4n^3}.$$

Admittedly, the error in Euler's formula satisfies $0 < E_4 < 1/120n^4$ which is much sharper; but all known proofs require much more difficult analysis than ours, while our method still gives the dominant quadratic term $-1/12n^2$ and so is not too bad. The interest of our Note is the simplicity of method to obtain a rather difficult result.

Geometrical proof

We let T_n be the trapezoid with base the line segment $(n, 0)$ to $(n + 1, 0)$ on the x-axis, sides the lines $x = n$ and $x = n + 1$ and slanted top the line segment joining the point $\left(n, \frac{1}{n}\right)$ to the point $\left(n + 1, \frac{1}{n+1}\right)$. We decompose T_n into three parts, shown in Figure 1.

- The *rectangle* r_n, with vertices $(n, 0)$, $(n + 1, 0)$, $\left(n + 1, \frac{1}{n+1}\right)$, $\left(n, \frac{1}{n+1}\right)$; and area $\frac{1}{n+1}$.
- The *curvilinear right-angled triangle* with base the top of the rectangle r_n and side the segment joining $\left(n, \frac{1}{n+1}\right)$ to $\left(n, \frac{1}{n}\right)$ and curved 'hypotenuse' the portion of the curve $y = \frac{1}{x}$ joining the point $\left(n, \frac{1}{n}\right)$ to the point $\left(n + 1, \frac{1}{n+1}\right)$. We call its area δ_n.
- The '*sliver*' bounded below by the arc of $y = \frac{1}{x}$ and above by the top of the trapezoid. We call its area σ_n.

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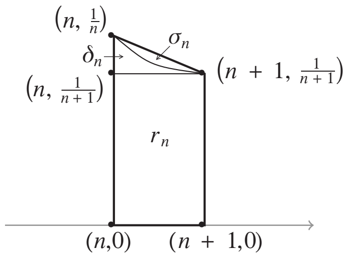

FIGURE 1

We define

$$\gamma_n = H_n - \ln n.$$

Then, as is well known [1] (see also [2]),

$$\sum_{p=n}^{\infty} \delta_p = \gamma_n - \gamma.$$

In the interest of completeness we reproduce Young's nice proof:

$$\sum_{p=n}^{N} \delta_p = \left[\int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1}\right] + \left[\int_{n+1}^{n+2} \frac{1}{x}\,dx - \frac{1}{n+2}\right] + \dots + \left[\int_{N-1}^{N} \frac{1}{x}\,dx - \frac{1}{N}\right]$$

$$= \int_n^N \frac{1}{x}\,dx - \sum_{r=1}^{N-n} \frac{1}{n+r} = \int_n^N \frac{1}{x}\,dx - \left[\sum_{r=1}^{N} \frac{1}{r} - \sum_{r=1}^{n} \frac{1}{r}\right]$$

$$= \left[\ln N - \sum_{r=1}^{N} \frac{1}{r}\right] - \left[\ln n - \sum_{r=1}^{n} \frac{1}{r}\right].$$

Now we let $N \to \infty$ in the last expression and use the definitions of γ_n and γ to obtain

$$\sum_{p=n}^{\infty} \delta_p = -\gamma + \gamma_n = \gamma_n - \gamma,$$

as asserted. But the area of the right-angled triangle at the top of the trapezoid equals

$$\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \delta_n + \sigma_n,$$

and summing from n to infinity we obtain

$$\frac{1}{2n} = H_n - \ln n - \gamma + \sum_{p=n}^{\infty} \sigma_p,$$

that is,

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{p=n}^{\infty} \sigma_p.$$

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Since σ_n is the area of the trapezoid decreased by the area under the curve $y = \frac{1}{x}$, we obtain

$$\sigma_n = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) - \int_n^{n+1} \frac{1}{x}dx = \frac{1}{2n} + \frac{1}{2n\left(1 + \frac{1}{n}\right)} - \ln\left(1 + \frac{1}{n}\right)$$

$$= \frac{1}{2n} + \left(\frac{1}{2n} - \frac{1}{2n^2} + \frac{1}{2n^3} - \frac{1}{2n^4} \pm \ldots\right) - \left(\frac{1}{n} + \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} \mp \ldots\right)$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right)\frac{1}{n^3} - \left(\frac{1}{2} - \frac{1}{4}\right)\frac{1}{n^4} + \left(\frac{1}{2} - \frac{1}{5}\right)\frac{1}{n^5} - \left(\frac{1}{2} - \frac{1}{6}\right)\frac{1}{n^6} \pm \ldots$$

$$= \frac{1}{6n^3} - \frac{2}{8n^4} + \frac{3}{10n^5} - \frac{4}{12n^6} \pm \ldots$$

which is an alternating series whose terms decrease monotonically to zero. A well-known theorem due to Leibniz states that if

$$S = a_1 - a_2 + a_3 - a_4 \pm \ldots$$

is an alternating series such that $a_n \geqslant 0$ and a_n decreases monotonically to 0, then the series converges to a sum S; and if

$$S_n = a_1 - a_2 + a_3 - a_4 \pm \ldots + (-1)^{n-1} a_n$$

is the n-th partial sum, then the absolute value of the remainder R_n satisfies:

$$|R_n| = |S - S_n| \leqslant a_{n+1},$$

and the sign of R_n is $(-1)^n$. Therefore, by this Leibniz error estimate,

$$\frac{1}{6n^3} - \frac{1}{4n^4} < \sigma_n < \frac{1}{6n^3}.$$

The standard estimate for the remainder from the integral test is:

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_n^{\infty} f(x)\,dx$$

where R_n is the remainder

$$R_n = f(n+1) + f(n+2) + \ldots$$

in the series $\sum_{n=1}^{\infty} f(n)$. If we apply it to the series $\sum_{n=1}^{\infty} \frac{1}{6n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{4n^4}$ we obtain

$$\frac{1}{12(n+1)^2} - \frac{1}{12n^3} < \sum_{n=1}^{\infty} \sigma_n < \frac{1}{12n^2}.$$

But,

$$\frac{1}{12(n+1)^2} - \frac{1}{12n^3} = \frac{1}{12n^2} - 2\frac{1}{12n^3} + 3\frac{1}{12n^4} - 4\frac{1}{12n^5} + 5\frac{1}{12n^6} \mp \ldots - \frac{1}{12n^3}$$

$$= \frac{1}{12n^2} - 3\frac{1}{12n^3} + 3\frac{1}{12n^4} - 4\frac{1}{12n^5} + 5\frac{1}{12n^6} \mp \ldots$$

$$= \frac{1}{12n^2} - \frac{1}{4n^3} + 3\frac{1}{12n^4} \mp \ldots$$

$$> \frac{1}{12n^2} - \frac{1}{4n^3}$$

since the series is alternating and the terms converge monotonically to zero. Therefore, if we define

$$\varepsilon_n = \frac{1}{12n^2} - \sum_{n=1}^{\infty} \sigma_n$$

we conclude that

$$0 < \varepsilon_n < \frac{1}{4n^3}$$

as stated in the theorem. This completes the proof.

Concluding remarks

Our method does not lead to an error term $O\left(\frac{1}{n^4}\right)$ since the terms of order $\frac{1}{n^3}$ for σ_n do not cancel. It would be desirable to modify this geometric reasoning to achieve such a cancellation (perhaps using telescopic cancellation, if necessary).

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107.20 Euler's constant and the speed of convergence

Introduction

Inspired in part by [1, 2], we present an elementary and unified approach to defining Euler's constant γ, and to obtaining bounds on the associated speed of convergence. These bounds give a modest refinement (with entirely different proof) of those obtained in the much-cited paper [3]. See the Proposition below.