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ON THE PRIMES IN FLOOR FUNCTION SETS

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Abstract

Let $[t]$ be the integral part of the real number t and let $\mathbb{1}_{\mathbb{P}}$ be the characteristic function of the primes. Denote by $\pi_{\mathcal{S}}(x)$ the number of primes in the floor function set $\mathcal{S}(x) := \{[x/n] : 1 \leqslant n \leqslant x\}$ and by $S_{\mathbb{1}_{\mathbb{P}}}(x)$ the number of primes in the sequence $\{[x/n]\}_{n\geqslant 1}$. Improving a result of Heyman ['Primes in floor function sets', *Integers* **22** (2022), Article no. A59], we show

$$\pi_{\mathcal{S}}(x) = \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)} + O(\sqrt{x}\,\mathrm{e}^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) \quad \text{and} \quad S_{\mathbb{1}_{\mathbb{P}}}(x) = C_{\mathbb{1}_{\mathbb{P}}} x + O_{\varepsilon}(x^{9/19+\varepsilon})$$

for $x \to \infty$, where $C_{\mathbb{1}_{\mathbb{P}}} := \sum_p 1/p(p+1)$, $c > 0$ is a positive constant and ε is an arbitrarily small positive number.

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1. Introduction

The distribution of prime numbers is one of the most important problems in number theory. Denote by $\pi(x)$ the number of primes $p \leqslant x$. The prime number theorem states that

$$\pi(x) = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big) \quad (x \to \infty).$$

A strong form of this theorem is

$$\pi(x) = \mathrm{Li}(x) + O(x\exp(-c(\log x)^{3/5}(\log_2 x)^{-1/5})) \quad (x \to \infty), \tag{1.1}$$

where c is a positive constant, $\log_2$ denotes the iterated logarithm function and

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

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The Riemann hypothesis is equivalent to the asymptotic formula

$$\pi(x) = \mathrm{Li}(x) + O_\varepsilon(x^{1/2+\varepsilon}) \quad (x \to \infty), \tag{1.2}$$

where ε is an arbitrarily small positive number. More generally, let $\mathcal{N}(x)$ be a set of integers of $[1, x]$ and let $\mathcal{N}_{\mathbb{P}}(x)$ be the set of prime numbers in $\mathcal{N}(x)$. We expect that

$$|\mathcal{N}_{\mathbb{P}}(x)| \sim \frac{|\mathcal{N}(x)|}{\log |\mathcal{N}(x)|} \quad (x \to \infty), \tag{1.3}$$

provided $\mathcal{N}(x)$ is rather regular and is not too sparse. Some well-known examples are

$$\{qn + a \leqslant x\}, \quad \{[n^c] \leqslant x\}, \quad \{m^2 + n^4 \leqslant x\}, \quad \{m^3 + 2n^3 \leqslant x\}, \quad \{x < n \leqslant x + x^{7/12+\varepsilon}\},$$

the respective densities for which are

$$\begin{aligned} x/\varphi(q) &\quad (q \leqslant (\log x)^A, \text{Walfisz–Siegel [3]}),\\ x^{1/c} &\quad \Big(c \leqslant \frac{2817}{2426}, \text{Rivat–Sargos [12]}\Big),\\ x^{3/4} &\quad (\text{Friedlander–Iwaniec [4]}),\\ x^{2/3} &\quad (\text{Heath-Brown [5]}),\\ x^{7/12+\varepsilon} &\quad (\text{Huxley [8]}), \end{aligned}$$

where $[t]$ is the integral part of the real number t, $\varphi(q)$ is the Euler function, A is any positive constant and $\varepsilon > 0$ is an arbitrarily small positive number.

Recently, Bordellès *et al.* [2] investigated the asymptotic behaviour of the summative function

$$S_f(x) := \sum_{n \leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big)$$

under some simple hypothesis on the growth of f and there are a number of further developments on this theme. If we use $\Lambda(n)$ to denote the von Mangoldt function, then [13, Theorem 1.2(i)] or [15, Theorem 1] give us immediately

$$S_\Lambda(x) = C_\Lambda x + O_\varepsilon(x^{1/2+\varepsilon}), \tag{1.4}$$

for any $\varepsilon > 0$ and $x \to \infty$, where $C_\Lambda := \sum_{n\geqslant 1} \Lambda(n)/n(n+1)$. Ma and Wu [11] applied the Vaughan identity and the technique of one-dimensional exponential sums to break the $\frac{1}{2}$-barrier by establishing

$$S_\Lambda(x) = C_\Lambda x + O_\varepsilon(x^{35/71+\varepsilon}). \tag{1.5}$$

This result seems rather interesting if we compare it with (1.2). The exponent 35/71 has been improved to 97/203 by Bordellès [1] and 9/19 by Liu *et al.* [10], using more sophisticated techniques of multiple exponential sums. Obviously, (1.5) is the prime number theorem for the floor function set

$$\mathcal{S}(x) := \Big\{\Big[\frac{x}{n}\Big] : 1 \leqslant n \leqslant x\Big\}$$

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considered as the weighted count of prime powers. Very recently, Heyman [7] examined the number of primes in the floor function set $\mathcal{S}(x)$ without the multiplicity. The principal result of Heyman [7, Theorem 1] is the asymptotic formula

$$\pi_{\mathcal{S}}(x) := \sum_{\substack{p\leqslant x\\ \exists\, n \text{ such that } [x/n]=p}} 1 = \frac{4\sqrt{x}}{\log x} + O\Big(\frac{\sqrt{x}}{(\log x)^2}\Big) \quad (x\to\infty). \tag{1.6}$$

Since Heyman [6, Theorems 1 and 2] proved that

$$|\mathcal{S}(x)| = 2\sqrt{x} + O(1) \quad (x\to\infty). \tag{1.7}$$

it follows from (1.6) that (1.3) holds for this sparse set $\mathcal{S}(x)$. This may be the first example of such a sparse subset of $[1,x]\cap\mathbb{N}$ (of density $x^{1/2}$) for which the prime number theorem holds.

It seems natural and interesting to establish an analogue of the strong form of the prime number theorem in (1.1) for the set $\mathcal{S}(x)$. We prove such a result.

THEOREM 1.1. (i) *For* $x\to\infty$,

$$\pi_{\mathcal{S}}(x) = \mathrm{Li}_{\mathcal{S}}(x) + O(\sqrt{x}\exp(-c'(\log x)^{3/5}(\log_2 x)^{-1/5})), \tag{1.8}$$

where $c' > 0$ *is a positive constant and*

$$\mathrm{Li}_{\mathcal{S}}(x) := \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)}.$$

(ii) *There is a real sequence* $\{a_n\}_{n\geqslant 1}$ *with* $a_1 = 4$ *such that for any positive integer* $N\geqslant 1$,

$$\pi_{\mathcal{S}}(x) = \sqrt{x}\sum_{n=1}^{N} \frac{a_n}{(\log x)^n} + O_N\Big(\frac{\sqrt{x}}{(\log x)^{N+1}}\Big) \quad (x\to\infty).$$

Let $\mathbb{P}$ be the set of all primes and let $\mathbb{P}_{\mathrm{ower}}$ be the set of all prime powers. Denote by $\mathbb{1}_{\mathbb{P}}$ and $\mathbb{1}_{\mathbb{P}_{\mathrm{ower}}}$ their characteristic functions. Define

$$S_{\mathbb{1}_{\mathbb{P}}}(x) := \sum_{n\leqslant x} \mathbb{1}_{\mathbb{P}}\Big(\Big[\frac{x}{n}\Big]\Big), \quad S_{\mathbb{1}_{\mathbb{P}_{\mathrm{ower}}}}(x) := \sum_{n\leqslant x} \mathbb{1}_{\mathbb{P}_{\mathrm{ower}}}\Big(\Big[\frac{x}{n}\Big]\Big).$$

Theorems 5 and 7 of [7] can be stated as follows:

$$S_{\mathbb{1}_{\mathbb{P}}}(x) = C_{\mathbb{1}_{\mathbb{P}}}x + O(x^{1/2}), \tag{1.9}$$

$$S_{\mathbb{1}_{\mathbb{P}_{\mathrm{ower}}}}(x) = C_{\mathbb{1}_{\mathbb{P}_{\mathrm{ower}}}}x + O(x^{1/2}), \tag{1.10}$$

where $C_{\mathbb{1}_{\mathbb{P}}} := \sum_p 1/p(p+1)$ and $C_{\mathbb{1}_{\mathbb{P}_{\mathrm{ower}}}} := \sum_{p,\,\nu\geqslant 1} 1/p^\nu(p^\nu+1)$. Similar to (1.4), these are immediate consequences of [13, Theorem 1.2(i)] or [15, Theorem 1]. Heyman [7, Theorem 6] also proved that there is a positive constant $B > 0$ such that the inequality

$$S_{\mathbb{1}_{\mathbb{P}}}(x) \geqslant C_{\mathbb{1}_{\mathbb{P}}}x - \frac{Bx^{1/2}}{\log x} \tag{1.11}$$

for $x\geqslant 2$. We improve these results by breaking the $\frac{1}{2}$-barrier in the error terms of (1.9), (1.10) and (1.11).

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THEOREM 1.2. *For any $\varepsilon > 0$,*

$$S_{\mathbb{1}_{\mathbb{P}}}(x) = C_{\mathbb{1}_{\mathbb{P}}}x + O_\varepsilon(x^{9/19+\varepsilon}), \tag{1.12}$$

$$S_{\mathbb{1}_{\mathbb{P}\mathrm{ower}}}(x) = C_{\mathbb{1}_{\mathbb{P}\mathrm{ower}}}x + O_\varepsilon(x^{9/19+\varepsilon}), \tag{1.13}$$

as $x \to \infty$, where the implied constants depend on ε.

REMARK 1.3. It is possible to improve the error terms in (1.12) and (1.13). It seems interesting to prove Ω-results for the error terms in (1.8), (1.12) and (1.13). We shall return to this problem in forthcoming work.

Very recently, Yu and Wu [14] generalised Heyman's (1.7) by showing

$$\mathcal{S}(x;q,a) := \sum_{\substack{m\in\mathcal{S}(x)\\ m\equiv a\,(\mathrm{mod}\,q)}} 1 = \frac{2\sqrt{x}}{q} + O((x/q)^{1/3}\log x)$$

uniformly for $x \geqslant 3$, $1 \leqslant q \leqslant x^{1/4}/(\log x)^{3/2}$ and $1 \leqslant a \leqslant q$, where the implied constant is absolute. This confirms a numerical test of Heyman.

2. Proof of Theorem 1.1

We begin by following the argument of [7]. First, we note that

$$\mathcal{S}(x) = \Big\{p \in \mathbb{P} : \exists\, n \in [1, x] \text{ such that } \Big[\frac{x}{n}\Big] = p\Big\}.$$

Further, if $[x/n] = p \in \mathbb{P}$, then $x/(p+1) < n \leqslant x/p$. Thus, we can write

$$\pi_{\mathcal{S}}(x) = \sum_{p\leqslant x} \mathbb{1}\Big(\Big[\frac{x}{p}\Big] - \Big[\frac{x}{p+1}\Big] > 0\Big) = G_1(x) + G_2(x), \tag{2.1}$$

where $\mathbb{1}(Q) = 1$ if the statement Q is true and 0 otherwise, and

$$G_1(x) := \sum_{p\leqslant\sqrt{x}} \mathbb{1}\Big(\Big[\frac{x}{p}\Big] - \Big[\frac{x}{p+1}\Big] > 0\Big),$$

$$G_2(x) := \sum_{\sqrt{x}<p\leqslant x} \mathbb{1}\Big(\Big[\frac{x}{p}\Big] - \Big[\frac{x}{p+1}\Big] > 0\Big).$$

For $p \leqslant \sqrt{x} - 1$,

$$\Big[\frac{x}{p}\Big] - \Big[\frac{x}{p+1}\Big] > \frac{x}{p(p+1)} - 1 > 0.$$

Thus, the prime number theorem (1.1) gives us

$$G_1(x) = \pi(\sqrt{x}) + O(1) = \mathrm{Li}(\sqrt{x}) + O(\sqrt{x}\exp(-c'(\log x)^{3/5}(\log_2 x)^{-1/5})) \tag{2.2}$$

for $x \geqslant 3$, where $c' > 0$ is a positive constant.

Next, we treat $G_2(x)$. Noticing that

$$0 < \frac{x}{p} - \frac{x}{p+1} = \frac{x}{p(p+1)} < 1$$

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for $p > \sqrt{x}$, the quantity $[x/p] - [x/p+1]$ can only equal 0 or 1. However, for $p > x^{10/19}$, we have $p = [x/n]$ for some $n \leqslant x^{9/19}$. Thus, we can write

$$\begin{aligned} G_2(x) &= \sum_{x^{1/2}<p\leqslant x^{10/19}} \Big(\Big[\frac{x}{p}\Big] - \Big[\frac{x}{p+1}\Big]\Big) + O(x^{9/19}) \\ &= \sum_{x^{1/2}<p\leqslant x^{10/19}} \Big(\frac{x}{p} - \frac{x}{p+1} - \psi\Big(\frac{x}{p}\Big) + \psi\Big(\frac{x}{p+1}\Big)\Big) + O(x^{9/19}) \\ &= G_{2,1}(x) - G_{2,2}^{\langle 0\rangle}(x) + G_{2,2}^{\langle 1\rangle}(x) + O(x^{9/19}), \end{aligned} \tag{2.3}$$

where $\psi(t) := t - [t] - \frac{1}{2}$ and

$$G_{2,1}(x) := \sum_{x^{1/2}<p\leqslant x^{10/19}} \Big(\frac{x}{p} - \frac{x}{p+1}\Big),$$

$$G_{2,2}^{\langle\delta\rangle}(x) := \sum_{x^{1/2}<p\leqslant x^{10/19}} \psi\Big(\frac{x}{p+\delta}\Big) \quad (\delta = 0, 1).$$

With the help of the prime number theorem (1.1), a simple partial integration gives

$$\begin{aligned} G_{2,1}(x) &= \sum_{x^{1/2}<p\leqslant x/2} \frac{x}{p^2} + O(x^{9/19}) = x\int_{\sqrt{x}}^{x/2} \frac{d\pi(t)}{t^2} + O(x^{9/19}) \\ &= x\int_{\sqrt{x}}^{x/2} \frac{dt}{t^2\log t} + O(\sqrt{x}\exp(-c'(\log x)^{3/5}(\log_2 x)^{-1/5}), \end{aligned}$$

where $c' > 0$ is a positive constant. Making the change of variables $t \to x/t$ in the last integral, it follows that

$$G_{2,1}(x) = \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)} + O(\sqrt{x}\exp(-c'(\log x)^{3/5}(\log_2 x)^{-1/5}) \tag{2.4}$$

for $x \to \infty$.

It remains to bound $G_{2,2}^{\langle\delta\rangle}(x)$. Similar to [10], define

$$\mathfrak{S}_\delta(x; D, D') := \sum_{D<d\leqslant D'} \Lambda(d)\psi\Big(\frac{x}{d+\delta}\Big).$$

According to [10, (4.3)], for any $\varepsilon > 0$,

$$\mathfrak{S}_\delta(x; D, 2D) \ll_\varepsilon (x^2D^7)^{1/12}x^\varepsilon$$

uniformly for $x \geqslant 3$ and $x^{6/13} \leqslant D \leqslant x^{2/3}$. The same proof shows that for any $\varepsilon > 0$,

$$\mathfrak{S}_\delta(x; D, D') \ll_\varepsilon (x^2D^7)^{1/12}x^\varepsilon \tag{2.5}$$

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uniformly for $x \geqslant 3$, $x^{6/13} \leqslant D \leqslant x^{2/3}$ and $D < D' \leqslant 2D$. Since we have trivially

$$\sum_{D<p^\nu\leqslant D',\, \nu\geqslant 2} \Lambda(p^\nu)\psi\Big(\frac{x}{p^\nu+\delta}\Big) \ll \sum_{p\leqslant (2D)^{1/2}} \sum_{\nu\leqslant (\log 2D)/\log p} \log p \ll D^{1/2},$$

the inequality (2.5) implies that the bound

$$\sum_{D<p\leqslant D'} (\log p)\psi\Big(\frac{x}{p+\delta}\Big) \ll_\varepsilon (x^2D^7)^{1/12}x^\varepsilon \tag{2.6}$$

holds uniformly for $x \geqslant 3$, $x^{6/13} \leqslant D \leqslant x^{2/3}$ and $D < D' \leqslant 2D$. Using (2.6),

$$\begin{aligned} G_{2,2}^{\langle\delta\rangle}(x) &\ll \max_{x^{1/2}<D\leqslant x^{10/19}} \sum_{D<p\leqslant 2D} \psi\Big(\frac{x}{p+\delta}\Big) \\ &\ll \max_{x^{1/2}<D\leqslant x^{10/19}} \int_D^{2D} \frac{1}{\log t}\, d\Big(\sum_{D<p\leqslant t} (\log p)\psi\Big(\frac{x}{p+\delta}\Big)\Big) \\ &\ll_\varepsilon \max_{x^{1/2}<D\leqslant x^{10/19}} (x^2D^7)^{1/12}x^\varepsilon \\ &\ll_\varepsilon x^{9/19+\varepsilon}. \end{aligned} \tag{2.7}$$

Inserting (2.4) and (2.7) into (2.3), we find that

$$G_2(x) = \int_2^{\sqrt{x}} \frac{dt}{\log(x/t)} + O(\sqrt{x}\exp(-c'(\log x)^{3/5}(\log_2 x)^{-1/5})). \tag{2.8}$$

Now the required result (1.8) follows from (2.1), (2.2) and (2.8).

The second assertion is an immediate consequence of the first one thanks to a simple partial integration.

3. Proof of Theorem 1.2

We begin by following the argument of [9]. Let $f = \mathbb{1}_{\mathbb{P}}$ or $\mathbb{1}_{\mathbb{P}_{\text{ower}}}$ and let $N \in [x^{1/3}, x^{1/2})$ be a parameter which can be chosen later. First, we write

$$S_f(x) = \sum_{n\leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big) = S_f^\dagger(x) + S_f^\sharp(x) \tag{3.1}$$

with

$$S_f^\dagger(x) := \sum_{n\leqslant N} f\Big(\Big[\frac{x}{n}\Big]\Big), \quad S_f^\sharp(x) := \sum_{N<n\leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big).$$

We have trivially

$$S_f^\dagger(x) \ll N. \tag{3.2}$$

To bound $S_f^\sharp(x)$, we put $d = [x/n]$. Noticing that

$$x/n - 1 < d \leqslant x/n \Leftrightarrow x/(d+1) < n \leqslant x/d,$$

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we see that

$$\begin{aligned} S_f^\sharp(x) &= \sum_{d\leqslant x/N} f(d) \sum_{x/(d+1)<n\leqslant x/d} 1 \\ &= \sum_{d\leqslant x/N} f(d)\Big(\frac{x}{d} - \psi\Big(\frac{x}{d}\Big) - \frac{x}{d+1} + \psi\Big(\frac{x}{d+1}\Big)\Big) \\ &= x\sum_{d\geqslant 1} \frac{f(d)}{d(d+1)} + \mathcal{R}_1^f(x,N) - \mathcal{R}_0^f(x,N) + O(N), \end{aligned} \tag{3.3}$$

where we have used the bounds

$$x\sum_{d>x/N} \frac{f(d)}{d(d+1)} \ll N, \quad \sum_{d\leqslant N} f(d)\Big(\psi\Big(\frac{x}{d+1}\Big) - \psi\Big(\frac{x}{d}\Big)\Big) \ll N$$

and

$$\mathcal{R}_\delta^f(x,N) = \sum_{N<d\leqslant x/N} f(d)\psi\Big(\frac{x}{d+\delta}\Big).$$

Combining (3.1), (3.2) and (3.3), it follows that

$$S_f(x) = x\sum_{d\geqslant 1} \frac{f(d)}{d(d+1)} + O_\varepsilon(|\mathcal{R}_1^f(x,N)| + |\mathcal{R}_0^f(x,N)| + N).$$

However,

$$\mathcal{R}_\delta^{\mathbb{1}_{\mathbb{P}\mathrm{ower}}}(x,N) = \sum_{N<p^\nu\leqslant x/N} \psi\Big(\frac{x}{p^\nu+\delta}\Big) = \mathcal{R}_\delta^{\mathbb{1}_{\mathbb{P}}}(x,N) + O((x/N)^{1/2}).$$

Thus, to prove Theorem 1.2, it suffices to show that

$$\mathcal{R}_\delta^{\mathbb{1}_{\mathbb{P}}}(x,N) \ll_\varepsilon Nx^\varepsilon \quad (x\geqslant 1)$$

for $N = x^{9/19}$. This can be done exactly as for (2.7) by using (2.6):

$$\begin{aligned} \mathcal{R}_\delta^{\mathbb{1}_{\mathbb{P}}}(x,N) &\ll x^\varepsilon \max_{x^{9/19}<D\leqslant x^{10/19}} \sum_{D<p\leqslant 2D} \psi\Big(\frac{x}{p+\delta}\Big) \\ &\ll x^\varepsilon \max_{x^{9/19}<D\leqslant x^{10/19}} \int_D^{2D} \frac{1}{\log t}\, d\Big(\sum_{D<p\leqslant t} (\log p)\psi\Big(\frac{x}{p+\delta}\Big)\Big) \\ &\ll_\varepsilon \max_{x^{9/19}<D\leqslant x^{10/19}} (x^2D^7)^{1/12} x^\varepsilon \\ &\ll_\varepsilon x^{9/19+\varepsilon}. \end{aligned}$$

This completes the proof.

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