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An arithmetic property of intertwining operators for p-adic groups

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Abstract. The main aim of this article is to show that normalised standard intertwining operator between induced representations of p-adic groups, at a very specific point of evaluation, has an arithmetic origin. This result has applications to Eisenstein cohomology and the special values of automorphic L-functions.

1 Introduction

If one proposes to use the theory of Eisenstein cohomology to prove algebraicity results for the special values of automorphic L-functions as in my work with Harder [8], or its generalizations in my recent papers: Raghuram [18], with Bhagwat [3] and Krishnamurthy [12], then in a key step, one needs to prove that the normalized standard intertwining operator between induced representations for p-adic groups has a certain arithmetic property. The principal aim of this article is to address this particular local problem in the generality of the Langlands–Shahidi machinery; the main result of this article is invoked in [3, 12], and I expect that it will be useful in future investigations on the arithmetic properties of automorphic L-functions.

Let F be a p-adic field, that is, a nonarchimedean local field of characteristic 0 with finite residue field k_F. Let $\mathbf{G}$ be a connected reductive group defined over F; assume that $\mathbf{G}$ is quasi-split over F. Fix a choice of Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ defined over F. Write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{T}$ is a maximal torus, and $\mathbf{U}$ the unipotent radical of $\mathbf{B}$; both defined over F. Suppose $\mathbf{P}$ is a maximal parabolic subgroup of $\mathbf{G}$ defined over F, assumed to be standard, i.e., containing $\mathbf{B}$, and with Levi decomposition $\mathbf{P} = \mathbf{MN}$. Let $\mathbf{A}$ denote the maximal central split torus of $\mathbf{M}$. The F-points of $\mathbf{G}, \mathbf{B}, \mathbf{T}, \mathbf{U}, \mathbf{P}, \mathbf{M}, \mathbf{N}$, and $\mathbf{A}$ are denoted by G, B, T, U, P, M, N, and A, respectively. To emphasize the dependence on P, we also denote $M = M_P$, $N = N_P$, and $A = A_P$. Let π be an irreducible admissible representation of M_P. (In the applications we have in mind, π will be a local component of a global cuspidal automorphic representation of cohomological type.) Let $I_P^G(s, \pi)$ be the induced representation as in the Langlands–Shahidi theory [23]; the precise definitions are recalled in the main body of this article; for the introduction suffice it to say that it is obtained by normalized parabolic induction from P to G of π with the complex variable s introduced in a delicate manner. Let Π be the set of simple roots of G with respect to B and α_P the unique simple root

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corresponding to P, and w_0 the unique element in the Weyl group of G such that $w_0(\Pi\backslash\{\alpha_P\}) \subset \Pi$ and $w_0(\alpha_P) < 0$; we also denote by w_0 a representative element in G. Let Q be the parabolic subgroup of G associate to P, and ${}^{w_0}\pi$ the corresponding representation of M_Q. We denote $T_{\rm st}(s,\pi) : I_P^G(s,\pi) \to I_Q^G(-s, {}^{w_0}\pi)$ for the standard intertwining operator. There is a choice of measure implicit in the integral that defines the intertwining operator. Consider the Langlands dual groups: let ${}^LP^\circ = {}^LM^\circ {}^LN^\circ$ be the Levi decomposition of the parabolic subgroup ${}^LP^\circ$ of ${}^LG^\circ$ corresponding to P. We write ${}^L\mathfrak{n} = \oplus_{j=1}^m r_j$ for the decomposition of ${}^L\mathfrak{n}$ under the adjoint action of ${}^LM^\circ$; it is a multiplicity free direct sum. Given π and r_j, the local aspects of the Langlands–Shahidi machinery attach a local L-factor $L(s,\pi,\tilde{r}_j)$ when π is generic, i.e., admits a Whittaker model. Denote by k the *point of evaluation*, which, by definition, is the point such that

$$I_P^G(s,\pi)|_{s=k} = {}^{\rm a}{\rm Ind}_P^G(\pi),$$

where the right-hand side is the algebraic (un-normalized) parabolic induction of π to a representation of G; see Definition 2.1.1. For brevity, let $\mathfrak{I} = I_P^G(k,\pi) = {}^{\rm a}{\rm Ind}_P^G(\pi)$ and $\tilde{\mathfrak{I}} = I_Q^G(-k, {}^{w_0}\pi)$.

Now, we impose an arithmetic context: suppose E is a "large enough" finite Galois extension of $\mathbb{Q}$, and suppose there is a smooth absolutely irreducible admissible representation $(\sigma, V_{\sigma,E})$ of M_P on an E-vector space $V_{\sigma,E}$ such that for some embedding $\iota : E \to \mathbb{C}$ we have ${}^\iota\sigma \cong \pi$. The induced modules $\mathfrak{I}_0 = {}^{\rm a}{\rm Ind}_P^G(\sigma)$ and $\tilde{\mathfrak{I}}_0 = I_Q^G(-k, {}^{w_0}\sigma)$ give E-structures on $\mathfrak{I}$ and $\tilde{\mathfrak{I}}$, i.e., the canonical $\mathfrak{I}_0 \otimes_{E,\iota} \mathbb{C} \to \mathfrak{I}$ and $\tilde{\mathfrak{I}}_0 \otimes_{E,\iota} \mathbb{C} \to \tilde{\mathfrak{I}}$ are isomorphisms. For the parabolic subgroup P, assume (i) the local Langlands correspondence to be known for M_P; this is a serious condition which is met in a lot of examples, and widely expected to hold in all generality with prescribed desiderata, and (ii) that P satisfies an integrality property: $\rho_P|_{A_P} \in X^*(A_P)$ – see Section 2 for notations not defined in the introduction. For the representation ${}^\iota\sigma$, motivated by global considerations, assume (i) ${}^\iota\sigma$ to be unitary up to a half-integral Tate twist, (ii) ${}^\iota\sigma$ to be essentially tempered, (iii) the point of evaluation $s = k$ to be "on the right of the unitary axis" (Definition 2.4.3) that guarantees absolute convergence of the integral defining the standard intertwining operator at $s = k$ and (iv) ${}^\iota\sigma$ is generic. The first main result (Theorem 3.3.7) of this article is an arithmeticity result for the standard intertwining operator at the point of evaluation $s = k$, i.e., there is an E-linear G-equivariant map $T_{\rm arith} : \mathfrak{I}_0 \to \tilde{\mathfrak{I}}_0$ such that

$$T_{\rm st}(s, {}^\iota\sigma)|_{s=k} = T_{\rm arith} \otimes_{E,\iota} \mathbb{C}.$$

The proof involves keeping track of arithmeticity in the proof of a rationality result for the standard intertwining operator for p-adic groups due to Waldspurger [25, Theorem IV.I.I].

The integrality property on P seems to tie up remarkably with motivic considerations; this is already very interesting in the example (see Section 5.1) of $G = {\rm GL}(N)$ and P maximal such that $M_P = {\rm GL}(n) \times {\rm GL}(n')$, in which case this integrality translates to $nn' \equiv 0 \pmod 2$ which is exactly the condition in [8] imposed therein due to motivic considerations.

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Consider the normalized standard intertwining operator defined as:

$$(1.0.1) \qquad T_{\mathrm{norm}}(s,\pi) = \left(\prod_{j=1}^{m} \frac{L(js,\pi,\tilde{r}_j)}{L(js+1,\pi,\tilde{r}_j)}\right)^{-1} T_{\mathrm{st}}(s,\pi), \quad \Re(s) \gg 0.$$

Continuing with all the hypotheses as above, at the point of evaluation $s = k$, the local L-values $L(jk, {}^{\iota}\sigma, \tilde{r}_j)$ and $L(jk+1, {}^{\iota}\sigma, \tilde{r}_j)$ are finite; and hence $T_{\mathrm{norm}}(s,\pi)|_{s=k}$ is convergent. We impose a "criticality" condition on $s = k$ that imposes a half-integrality property on k and is entirely a function of the parabolic subgroup P and the ambient group G. This condition has a global motivation in that the corresponding global L-values at $s = k$ are critical values in the sense of Deligne [6], and like the integrality condition on P it restricts the scope of global applications; see, for example, the interesting case of $G = \mathrm{Sp}(2n)$ and $M_P = \mathrm{GL}(n)$ in Section 5.3 that involves the exterior square L-functions for $\mathrm{GL}(n)$. The arithmeticity result on local critical L-values for Rankin–Selberg L-functions [17, Proposition 3.17] gives the impetus to hypothesize that

$$L(s_j, {}^{\iota}\sigma, \tilde{r}_j) \in \iota(E), \quad s_j \in \{jk,\ jk+1\},$$

and furthermore this L-value is Galois equivariant; see Hypothesis 4.2.1. Under such a hypothesis, which can be verified in many examples such as when G is a classical group, the main result in Theorem 3.3.7 can be strengthened to Theorem 4.3.1 that gives an arithmeticity result for the normalized standard intertwining operator at the point of evaluation.

The results of this article (Theorems 3.3.7 and 4.3.1) say that if we use Eisenstein cohomology to give a cohomological interpretation of Langlands's constant term theorem, and so attempt to prove a rationality result for ratios of critical values of automorphic L-functions, then at any given finite place, we do not pick up any possibly transcendental period. Suppose π is an unramified representation, i.e., has a vector fixed under the hyper-special maximal compact subgroup of M_P, then both $\mathfrak{I}$ and $\tilde{\mathfrak{I}}$ are also unramified; suppose $f_0 \in \mathfrak{I}$ (resp., $\tilde{f}_0 \in \tilde{\mathfrak{I}}$) is the normalized spherical vector; then Langlands's generalization of the classical Gindikin–Karpelevic formula says that $T_{\mathrm{norm}}(f_0) = \tilde{f}_0$. This implies the theorem because the E-structures are generated by these normalized spherical vectors. The real content of the theorem is that it works for any π whether or not it is unramified. Whereas the global theory of Eisenstein cohomology and the special values of automorphic L-functions provides the context, this article is purely local (p-adic) in nature, and does not need the reader to be familiar with such global aspects.

2 Local aspects of the Langlands–Shahidi machinery

2.1 Induced representations and "the point of evaluation"

Let δ_P be the modulus character of P; it is trivial on N_P and on M_P it is given by:

$$\delta_P(m) = |\det(\mathrm{Ad}_{N_P}(m))|, \quad m \in M_P,$$

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where $\mathrm{Ad}_{N_P} : M_P \to \mathrm{GL}(\mathrm{Lie}(N_P))$ is the adjoint representation of M_P on the Lie algebra of N_P, and $|\ |$ is the normalized absolute value on F. Let $Z(M_P)$ be the center of M_P, A_P the maximal split torus in $Z(M_P)$. Let $X^*(A_P) = \mathrm{Hom}(A_P, F^*)$ and $X^*(M_P) = \mathrm{Hom}(M_P, F^*)$ denote the group of rational characters of A_P and M_P. Restriction from M_P to A_P gives an inclusion $X^*(M_P) \hookrightarrow X^*(A_P)$, which induces an isomorphism $X^*(M_P) \otimes_{\mathbb{Z}} \mathbb{Q} \cong X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{Q}$. The modulus character δ_P is naturally an element of $\mathfrak{a}_P^* := X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a Weyl group invariant inner product $(,)$ on $X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let Δ_G be the set of all roots which are naturally in $X^*(T)$; for the choice of the Borel subgroup B, let Δ_G^+ be the set of positive roots, and Π_G the set of simple roots. Let ρ_P be half the sum of all positive roots whose root spaces are in $\mathrm{Lie}(N_P)$; via the restriction from T to A_P we have $\rho_P \in X^*(A_P) \otimes_{\mathbb{Z}} \mathbb{Q}$. We have the equality:

$$|2\rho_P(m)| = \delta_P(m), \quad \forall m \in M.$$

Let $\mathfrak{a}_P = \mathrm{Hom}(X^*(A_P), \mathbb{R}) = \mathrm{Hom}(X^*(M_P), \mathbb{R})$ denote the real Lie algebra of A_P, and $H_P : M_P \to \mathfrak{a}_P$ denote the Harish-Chandra homomorphism defined by:

$$\exp\langle \chi, H_P(m)\rangle \ := \ |\chi(m)|, \quad \forall \chi \in X^*(A_P),\ \forall m \in M.$$

In particular, taking $\chi = \rho_P$, we get:

$$\exp\langle \rho_P, H_P(m)\rangle \ = \ \delta_P(m)^{1/2}, \quad \forall m \in M.$$

Let (π, V_π) be an irreducible admissible representation of M_P; the representation space V_π is a vector space over $\mathbb{C}$. For $\nu \in \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$, define the induced representation

$$I(\nu, \pi) \ := \ \mathrm{Ind}_P^G(\pi \otimes \exp(\langle \nu, H_P(\ )\rangle) \otimes 1_U),$$

where Ind means normalized parabolic induction. The representation space $V(\nu, \pi)$ is the vector space of all smooth (i.e., locally constant) functions $f : G \to V_\pi$ such that

$$f(mng) \ = \ \pi(m) \exp(\langle \nu + \rho_P, H_P(m)\rangle) f(g), \quad \forall g \in G,\ m \in M,\ n \in N.$$

Recall that P is a maximal parabolic subgroup, defined by a simple root α_P which is the unique simple root whose root space is in $\mathrm{Lie}(N_P)$. Set $\langle \rho_P, \alpha_P\rangle = 2\frac{(\rho_P, \alpha_P)}{(\alpha_P, \alpha_P)}$ and put

$$\gamma_P \ := \ \tilde{\alpha}_P \ := \ \frac{1}{\langle \rho_P, \alpha_P\rangle} \rho_P.$$

In the Langlands–Shahidi machinery the notation $\tilde{\alpha}$ is commonly used; elsewhere in the arithmetic theory of automorphic forms the notation γ_P is commonly used; it is the fundamental weight corresponding to the simple root α_P. For $s \in \mathbb{C}$, define ν_s as:

$$\nu_s \ := \ s\tilde{\alpha}_P \ = \ \frac{s}{\langle \rho_P, \alpha_P\rangle} \rho_P.$$

Let $I(s, \pi) := I(s\tilde{\alpha}_P, \pi)$, whose representation space $V(s, \pi) := V(s\tilde{\alpha}_P, \pi)$ consists of all locally constant functions $f : G \to V_\pi$ such that

$$f(mng) \ = \ \pi(m)\, \delta_P(m)^{\frac{1}{2} + \frac{s}{2\langle \rho_P, \alpha_P\rangle}} f(g), \quad \forall g \in G,\ m \in M,\ n \in N.$$

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Definition 2.1.1 (Point of evaluation) Define the point of evaluation k as

$$k := -\langle \rho_P, \alpha_P \rangle,$$

which depends only on P and G, and has the property that $I(k,\pi) = {}^{\mathrm{a}}\mathrm{Ind}_P^G(\pi)$ which is the algebraic (i.e., un-normalized) parabolic induction from P to G of the representation π.

The point of evaluation k is half-integral, i.e., $k \in \mathbb{Z}$ or $k \in \frac{1}{2} + \mathbb{Z}$, or more succinctly $2k \in \mathbb{Z}$, since $\langle \beta, \alpha_P \rangle = 2(\beta, \alpha_P)/(\alpha_P, \alpha_P) \in \mathbb{Z}$ for any root β. In general, k can be integral or a genuine half-integer; for example, if $G = \mathrm{GL}(N)$ with $N \geq 2$, and P is any maximal parabolic subgroup, then $k = -N/2$; see Section 5.1.

2.2 The standard intertwining operator: definition and analytic properties

For the maximal parabolic subgroup $P = P_\Theta$, where $\Theta = \Pi \backslash \{\alpha_P\}$, recall that w_0 is the unique element in the Weyl group such that $w_0(\Theta) \subset \Pi$ and $w_0(\alpha_P) < 0$; let $Q = P_{w_0(\Theta)}$ be the maximal parabolic subgroup associate to P. Then $M_Q = w_0 M_P w_0^{-1}$, and let ${}^{w_0}\pi$ be the representation of M_Q given by conjugation.

Let $f \in I_P^G(s,\pi)$ and $g \in G$. Suppose there exists a vector v in the inducing representation of $I_Q^G(-s, {}^{w_0}\pi)$, such that for all $\check{v}$ in the contragredient of this inducing representation the integral $\int_{N_Q} \langle f(w_0^{-1}ng), \check{v} \rangle dn$ converges absolutely to $\langle v, \check{v} \rangle$ then define $\int_{N_Q} \langle f(w_0^{-1}ng)\, dn = v$. If this is verified for all $f \in I_P^G(s,\pi)$ and all $g \in G$ then define an intertwining operator for G-modules

$$T_{\mathrm{st}}(s) : I_P^G(s,\pi) \longrightarrow I_Q^G(-s, {}^{w_0}\pi)$$

by the integral

$$T_{\mathrm{st}}(s)(f)(g) = \int_{N_Q} f(w_0^{-1}ng)\, dn; \quad f \in V(s,\pi),\ g \in G. \tag{2.2.1}$$

Assume, here and henceforth, that the measures in such intertwining integrals are chosen to be $\mathbb{Q}$-valued. The operator $T_{\mathrm{st}}(s)$ is denoted as $A(s,\pi,w_0)$ in Shahidi [23, Section 4.1] (see also Kim [11, Section 4.3]). That $T_{\mathrm{st}}(s)(f) \in I_Q^G(-s, {}^{w_0}\pi)$ is verified in *loc.cit.* The following convergence statement is a special case of [23, Proposition 4.1.2]:

Proposition 2.2.2 *If $\Re(s) \gg 0$ then $T_{\mathrm{st}}(s)(f)(g)$ converges absolutely for all $g \in G$ and all $f \in V(s,\pi)$.*

If $T_{\mathrm{st}}(s)(f)(g)$ converges absolutely for all $g \in G$ and all $f \in V(s,\pi)$, then we will simply say that $T_{\mathrm{st}}(s)$ converges absolutely. One can be more specific about the domain of convergence in the tempered case.

Proposition 2.2.3 *If π is a tempered (unitary) representation, then the standard intertwining operator $T_{\mathrm{st}}(s)$ converges absolutely for $\Re(s) > 0$.*

The above convergence statements are contained in Harish-Chandra's work on harmonic analysis on p-adic reductive groups; the reader is referred to [19, Section 2.2] and the references therein; see also [11, Proposition 12.3].

Without worrying about convergence, let us see the shape of the standard intertwining operator at the point of evaluation $s = k$. The domain of $T_{\mathrm{st}}(s)|_{s=k}$, as noted

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above, is $I(k,\pi) = {}^{\mathrm{a}}\mathrm{Ind}_P^G(\pi)$. The codomain is $I_Q^G(-s, {}^{w_0}\pi) := I(-s\tilde{\alpha}_Q, {}^{w_0}\pi)$, whose representation space consists of all locally constant functions $f' : G \to V_{w_0\pi} = V_\pi$ such that

$$f'(m'n'g') = {}^{w_0}\pi(m')\,|\delta_Q(m')|^{\frac{1}{2} - \frac{s}{2\langle\rho_Q,\alpha_Q\rangle}} f'(g'), \quad \forall g' \in G,\ m' \in M_Q,\ n' \in N_Q,$$

where ${}^{w_0}\pi(m') = \pi(w_0^{-1}m'w_0)$. Put $s = k = -\langle\rho_P, \alpha_P\rangle$; since $\langle\rho_Q, \alpha_Q\rangle = \langle\rho_P, \alpha_P\rangle$ we get:

$$f'(m'u'g') = {}^{w_0}\pi(m')\,\delta_Q(m')\,f'(g').$$

Hence, at the point of evaluation, in terms of un-normalized induction we get:

$$T_{\mathrm{st}}(s)|_{s=k} : {}^{\mathrm{a}}\mathrm{Ind}_P^G(\pi) \longrightarrow {}^{\mathrm{a}}\mathrm{Ind}_Q^G({}^{w_0}\pi \otimes \delta_Q). \tag{2.2.4}$$

2.3 Local factors and the local Langlands correspondence

A defining aspect of the Langlands program is Langlands's computation [14, Section 5] of the constant term of an Eisenstein series, which at a local unramified place boils down to computing the standard intertwining operator on "the" spherical vector which is a scalar multiple of the spherical vector on the other side, and this scalar multiple is an expression denoted $M(s)$ in *loc.cit.* Langlands says that Tits pointed out to him how to express $M(s)$ in a more convenient form. This is now an important ingredient in the Langlands–Shahidi machinery; see [20, Section 2].

Let ${}^LG^\circ$ be the complex reductive group which is the connected component of the Langlands dual ${}^LG$ of G; see [4, I.2]; and let ${}^LP$ be the parabolic subgroup of ${}^LG$ corresponding to P, and ${}^LN$ its unipotent radical. The Levi quotient ${}^LM^\circ$ of ${}^LP^\circ$ acts on the Lie algebra ${}^L\mathfrak{n}$ of ${}^LN^\circ$ by the adjoint action. There is a positive integer m such that the set $\{\langle\tilde{\alpha}_P, \beta\rangle\}$ – as β varies over positive roots such that the root space ${}^L\mathfrak{g}(\beta^\vee)$ of the dual root $\beta^\vee$ is in ${}^L\mathfrak{n}$ – is $\{1, \dots, m\}$. For each $1 \le j \le m$ put

$$V_j = \text{span of } {}^L\mathfrak{g}(\beta^\vee) \text{ for } \beta \text{ such that } \langle\tilde{\alpha}_P, \beta\rangle = j. \tag{2.3.1}$$

Then the action of ${}^LM^\circ$ on ${}^L\mathfrak{n}$ stabilizes each V_j and furthermore acts irreducibly on V_j. Denote r_j the action of ${}^LM^\circ$ on V_j, and ${}^L\mathfrak{n} = \oplus_{j=1}^m r_j$ is a multiplicity free decomposition as an ${}^LM^\circ$-representation. Let $\tilde{r}_j$ denote the contragredient of r_j.

Given a smooth irreducible admissible representation π that is *generic*, i.e., has a Whittaker model, and for $1 \le j \le m$, the local aspects of the Langlands–Shahidi machinery attaches a local L-factor $L(s, \pi, \tilde{r}_j)$ (see [21]) which is the inverse of a polynomial in q^{-s} of degree at most $d_j := \dim(V_j)$; when π is unramified this degree is d_j.

Let W_F be the Weil group of F, and $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil–Deligne group. The local Langlands correspondence for G says that to π corresponds its Langlands parameter which is an admissible homomorphism $\phi_\pi : W_F' \to {}^LM_P$; see Borel [4, Section 8] for the requirements on the parameter ϕ_π. Composing with $\tilde{r}_j$ gives $\tilde{r}_j \circ \phi_\pi : W_F' \to {}^L\mathrm{GL}_{d_j}$, an admissible homomorphism which parametrizes, via the local Langlands correspondence for $\mathrm{GL}_{d_j}(F)$, a smooth irreducible admissible representation of $\mathrm{GL}_{d_j}(F)$ that we denote $\tilde{r}_j(\pi)$. As in Shahidi [24, p. 3] we will impose the working

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hypothesis:

$$L(s,\pi,\tilde{r}_j) = L(s,\tilde{r}_j(\pi)), \tag{2.3.2}$$

that is known in a number of instances; see the references in *loc.cit.*

2.4 The notion of being on the right of the unitary axis

Recall that π is a smooth irreducible admissible representation of M_P, which is to be a local component of a globally generic cuspidal automorphic representation (needed by the context in which we can evoke the Langlands–Shahidi machinery), and keeping the generalized Ramanujan conjecture in the back of our minds (see, for example, [22]), we will impose the condition that π is essentially tempered, i.e., tempered mod the center.

Let $M_P^1 = \bigcap_{\chi\in X^*(M_P)} \mathrm{Ker}(|\chi|_F)$, the subgroup of M_P generated by all compact subgroups. For the split center of M_P, say $A_P \cong F^* \times \cdots \times F^*$, and $\eta_i : A_P \to F^*$ is the projection to the ith copy. Define $A_P^1 \subset A_P$ similar to M_P^1; we have $A_P^1 = A_P \cap M_P^1$. Let $X(M_P) = \mathrm{Hom}(M_P/M_P^1, \mathbb{C}^*)$; similarly, $X(A_P)$. Restricting from M_P to A_P gives an isomorphism $X(M_P) \cong X(A_P)$. Given $\underline{z} = (z_1,\dots,z_l) \in \mathbb{C}^l$, we get an unramified character $A_P \to \mathbb{C}^*$ given by $|\eta_1|^{z_1} \otimes \cdots \otimes |\eta_l|^{z_l}$, and via $X(M_P) \cong X(A_P)$ an unramified character of M_P which we will denote as $\underline{\eta}^{\underline{z}}$. Given π as above, tempered modulo the center means that there exists an l-tuple of exponents $\underline{e} = (e_1,\dots,e_l) \in \mathbb{R}^l$ and a smooth irreducible unitary tempered representation π^t such that $\pi \cong \pi^t \otimes \underline{\eta}^{\underline{e}}$. (Keeping global applications in mind, we will impose later a hypothesis that the exponents e_i are [half-]integral.) The representation $\tilde{r}_j(\pi)$ of $\mathrm{GL}_{d_j}(F)$, obtained by functoriality, is also tempered modulo its center. A few words of explanation might be helpful. There is an exponent $\tilde{f}_j = f(\tilde{r}_j, e_1,\dots,e_l) \in \mathbb{R}$, that depends on the representation r_j and the exponents $e_1,\dots,e_l$, such that

$$\tilde{r}_j(\pi) \cong \tilde{r}_j(\pi)^t \otimes |\det|^{\tilde{f}_j}$$

with $\tilde{r}_j(\pi)^t = \tilde{r}_j(\pi^t)$ being a unitary tempered irreducible representation of $\mathrm{GL}_{d_j}(F)$. Local functoriality preserves temperedness, by the desiderata in [4, 10.4, (4)], and the one has to keep track of the central characters for which consider the diagram:

$$\begin{array}{ccccc} W_F' & \xrightarrow{\phi_\pi} & {}^L M_P^\circ & \xrightarrow{\tilde{r}_j} & {}^L(\mathrm{GL}_{d_j})^\circ = \mathrm{GL}_{d_j}(\mathbb{C}) \\ & & \uparrow & & \uparrow \\ & & {}^L A_P^\circ & \longrightarrow & {}^L(Z(\mathrm{GL}_{d_j}))^\circ = \mathbb{C}^* \end{array} \tag{2.4.1}$$

Since the representation $\tilde{r}_j$ is irreducible, the center ${}^L A_P^\circ \cong (\mathbb{C}^*)^l$ of ${}^L M_P^\circ$ acts via scalars, explaining the bottom horizontal arrow. The unramified character $\underline{\eta}^{\underline{e}}$ of A_P corresponds to its Satake parameter $\vartheta_{\underline{\eta}^{\underline{e}}}$ in ${}^L A_P^\circ$; we get $\tilde{r}_j(\vartheta_{\underline{\eta}^{\underline{e}}}) \in \mathbb{C}^*$, which corresponds to an unramified character $|\ |^{\tilde{f}_j}$ of F^*, or the character $|\det|^{\tilde{f}_j}$ of $\mathrm{GL}_{d_j}(F)$, for some exponent $\tilde{f}_j$ which, *a priori*, lives in $\mathbb{C}$, but since $e_j \in \mathbb{R}$ and $\tilde{r}_j$ is an algebraic representation, it is clear that $\tilde{f}_j \in \mathbb{R}$.

Lemma 2.4.2 *With notations as above we have $\tilde{f}_j = j \cdot \tilde{f}_1$.*

Proof The proof follows from the definition of V_j in (2.3.1) which is the representation space for r_j. (It is instructive to see this detail in the example discussed in Section 5.3.)

Definition 2.4.3 Let π be a smooth irreducible admissible generic representation of M_P that is tempered modulo the center with exponents $e_1, \dots, e_l \in \mathbb{R}$. We say π is on the right of the unitary axis with respect to the ambient group G, if

$$-\langle \rho_P, \alpha_P \rangle + \tilde{f}_1 > 0.$$

By Lemma 2.4.2 it follows that for each $1 \le j \le m$ we have: $-j\langle \rho_P, \alpha_P \rangle + \tilde{f}_j > 0$.

Corollary 2.4.4 *Let π be a smooth irreducible admissible generic representation of M_P that is tempered modulo the center with exponents $e_1, \dots, e_l \in \mathbb{R}$, and which is on the right of the unitary axis with respect to the ambient group G, then the local L-values*

$$L(jk, \pi, \tilde{r}_j) \text{ and } L(jk+1, \pi, \tilde{r}_j)$$

are finite for each $1 \le j \le m$, where, recall that $k = -\langle \rho_P, \alpha_P \rangle$ is the point of evaluation.

Proof If π is a unitary tempered representation of $\mathrm{GL}_d(F)$ then the standard local L-factor $L(s, \pi)$ is finite if $\Re(s) > 0$; this follows from Jacquet's classification of tempered representations of $\mathrm{GL}_d(F)$ and the well-known inductive recipe for local L-factors that is succinctly summarized in [13]. The proof follows from the equalities: $L(jk, \pi, \tilde{r}_j) = L(jk, \tilde{r}_j(\pi)) = L(jk + \tilde{f}_j, \tilde{r}_j(\pi)^t)$. ■

The condition of being on the right of the unitary axis is tailor-made to appeal to Shahidi's tempered L-functions conjecture that is now a theorem after the work of many authors (see [23, p. 147]) culminating in [9].

3 An arithmetic variation on a rationality result of Waldspurger

In this section, we recall a rationality result of Waldspurger [25, Theorem IV.1.1], and show how to reformulate it so that the statement works at an arithmetic level. Before that let us clarify some terminology that apparently causes some confusion.

3.1 Digression on the adjectives: rationality, algebraicity, and arithmeticity

First of all, even among experts, there seems to be no universal agreement on the precise meaning of these adjectives. In this article, all three words are used, and it might help the reader to clarify their meanings. The word rationality has two meanings and the context usually makes it clear. First of all, a result of the form "$(\mathrm{L-value})/(\mathrm{periods}) \in \mathbb{Q}$" is often called a *rationality result* for L-values. Then there is a common abuse of terminology and a result of the form "$(\mathrm{L-value})/(\mathrm{periods}) \in \overline{\mathbb{Q}}$" is also called a rationality result. We will refer to the latter as an *algebraicity result* for L-values. A second usage of rationality, as in the context of Waldspurger's result, comes from algebraic geometry and means that some function or operator at hand is a rational function on an algebraic variety. To explain our usage of the word arithmetic, suppose we have an L-value at hand, which is the value at $s = s_0$ of the L-function $L(s,\pi)$ attached to some object π defined over $\mathbb{C}$, for example, π can be the finite part of a cuspidal automorphic representation. We may set up our context for the object π to have an *arithmetic* origin, i.e., there is an object σ defined over some coefficient field E, such that for some embedding of fields $\iota : E \to \mathbb{C}$, the base-change ${}^\iota\sigma$ of σ via ι is the object π. In such a context, a result of the form "$L(s_0, {}^\iota\sigma)/(\text{periods}) \in \iota(E)$" is given the appellation of an *arithmetic result* for L-values. With this explanation of the words, the purpose of this section is to show that Waldspurger's rationality result for intertwining operator has an arithmetic origin. We will use the word *arithmeticity* for the behaviour of an arithmetic result upon changing ι, or equivalently, by replacing ι by $\tau \circ \iota$ for any $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$; this is compatible with the usage of arithmeticity as in [7].

3.2 A rationality result of Waldspurger

In this subsection, we will adumbrate the presentation in [25, IV.1]. Recall the notations M_P^1, A_P^1, $X(M_P)$, and $X(A_P)$ from Section 2.4. When P is fixed we drop the subscript P from M_P, A_P, etc.

Note that $X(M)$ has the structure of an algebraic variety over $\mathbb{C}$; denote by $\mathcal{B}$ the $\mathbb{C}$-algebra of polynomial functions on $X(M)$. Let (π, V) be a smooth admissible representation of M on a $\mathbb{C}$-vector space V, and let $\mathcal{O}_{\mathbb{C}} = \{\pi \otimes \chi : \chi \in X(M)\}$. A function $f : \mathcal{O}_{\mathbb{C}} \to \mathbb{C}$ is a polynomial if there exists $b \in \mathcal{B}$ such that $f(\pi \otimes \chi) = b(\chi)$. For an open set $\mathcal{U} \subset \mathcal{O}_{\mathbb{C}}$, a function $f : \mathcal{U} \to \mathbb{C}$ is a rational function if there exists $b_1, b_2 \in \mathcal{B}$ such that $b_1(\chi) f(\pi \otimes \chi) = b_2(\chi)$ for all $\chi \in X(M)$ with $\pi \otimes \chi \in \mathcal{U}$ and $b_1(\chi) \neq 0$.

Let $I_P^G(\pi \otimes \chi)$ be the normalized parabolically induced representation. Restriction from G to its maximal compact subgroup K sets up an isomorphism $I_P^G(\pi \otimes \chi) \cong I_{K\cap P}^K(\pi)$. Let P' be a maximal parabolic subgroup of G that has the same Levi subgroup $M = M_{P'} = M_P$. For each $\pi \otimes \chi \in \mathcal{O}_{\mathbb{C}}$ suppose we are given a G-equivariant operator $A(\pi \otimes \chi) : I_P^G(\pi \otimes \chi) \to I_{P'}^G(\pi \otimes \chi)$ that depends only on the equivalence class of $\pi \otimes \chi$. We say that the operator $A(\pi \otimes \chi)$ is a *polynomial* if for all $f \in I_{K\cap P}^K(\pi)$ there exist finitely many $f_1, \dots, f_r \in I_{K\cap P'}^K(\pi)$ and $b_1, \dots, b_r \in \mathcal{B}$ such that $A(\pi \otimes \chi)(f) = \sum_{i=1}^r b_i(\chi) f_i$ for all $\chi \in X(M)$. Furthermore, we say $A(\pi \otimes \chi)$ is *rational* if there exists $b \in \mathcal{B}$, such that for all $f \in I_{K\cap P}^K(\pi)$ there exist finitely many $f_1, \dots, f_r \in I_{K\cap P'}^K(\pi)$ and $b_1, \dots, b_r \in \mathcal{B}$ such that

$$b(\chi) A(\pi \otimes \chi)(f) = \sum_{i=1}^r b_i(\chi) f_i, \quad \text{for all } \chi \in X(M) \text{ with } b(\chi) \neq 0. \tag{3.2.1}$$

Rationality of the intertwining operators may be formulated in another way that is used in the proof of [25, Theorem IV.1.1], and which will allow us to descend the statement and proof to an arithmetic level to give us Theorem 3.3.7. For $m \in M$, let $b_m \in B$ be defined as $b_m(\chi) = \chi(m)$. Define $V_{\mathcal{B}} = V \otimes_{\mathbb{C}} \mathcal{B}$ on which M acts as: $\pi_{\mathcal{B}}(m)(v \otimes b) = \pi(m)v \otimes b_m b$. For $\chi \in X(M)$, let $\mathcal{B}_\chi$ be the maximal ideal $\{b \in \mathcal{B} : b(\chi) = 0\}$. Then the action of M on $V_{\mathcal{B}} \otimes \mathcal{B}/\mathcal{B}_\chi$ is the representation $\pi \otimes \chi$. Similarly, $I_P^G(V_{\mathcal{B}}) = I_P^G(V) \otimes_{\mathbb{C}} \mathcal{B}$. Let $\mathrm{sp}_\chi : \pi_{\mathcal{B}} \to \pi_{\mathcal{B}} \otimes \mathcal{B}/\mathcal{B}_\chi = \pi \otimes \chi$ denote the specialization map; same notation also for $\mathrm{sp}_\chi : I_P^G(\pi_{\mathcal{B}}) \to I_P^G(\pi \otimes \chi)$. The collection $\{A(\pi \otimes \chi)\}_{\pi\otimes\chi\in\mathcal{O}_{\mathbb{C}}}$ of operators is *polynomial* if and only if there exists a G-equivariant homomorphism of $\mathcal{B}$-modules $A_{\mathcal{B}} : I_P^G(\pi_{\mathcal{B}}) \to I_{P'}^G(\pi_{\mathcal{B}})$ such that the following diagram commutes

(3.2.2)
$$\begin{array}{ccc} I_P^G(\pi_{\mathcal{B}}) & \xrightarrow{A_{\mathcal{B}}} & I_{P'}^G(\pi_{\mathcal{B}}) \\ \downarrow \mathrm{sp}_\chi & & \downarrow \mathrm{sp}_\chi \\ I_P^G(\pi \otimes \chi) & \xrightarrow{A(\pi\otimes\chi)} & I_{P'}^G(\pi \otimes \chi) \end{array}$$

for all χ, i.e., $\mathrm{sp}_\chi \circ A_{\mathcal{B}} = A(\pi \otimes \chi) \circ \mathrm{sp}_\chi$. Similarly, the collection $\{A(\pi \otimes \chi)\}_{\pi\otimes\chi\in\mathcal{O}_{\mathbb{C}}}$ of operators is *rational* if and only if there exists $A_{\mathcal{B}}$ as above and an element $b \in \mathcal{B}$ such that

(3.2.3)
$$A(\pi \otimes \chi) \circ \mathrm{sp}_\chi \circ (1 \otimes b) \;=\; \mathrm{sp}_\chi \circ A_{\mathcal{B}}.$$

Suppose $\tilde{f} \in I_P^G(\pi_{\mathcal{B}})$ and $A_{\mathcal{B}}(\tilde{f}) = \sum_i \tilde{f}_i \otimes b_i$, and if sp_χ maps $\tilde{f}$ to f, and similarly, $\tilde{f}_i$ to f_i, then (3.2.3) becomes $b(\chi)A(\pi \otimes \chi)(f) = \sum_i b_i(\chi) f_i$ as in (3.2.1). We may and shall talk about the collection $\{A(\pi \otimes \chi)\}_{\chi\in\mathcal{U}}$ of operators being rational on an open subset $\mathcal{U} \subset X(M)$.

Let N' be the unipotent radical of P'. Let $f \in I_P^G(V)$ and $g \in G$. Suppose, there exists $v \in V$ such that for all $\check{v} \in \check{V}$—the representation space of the contragredient $\check{\pi}$ of π—the integral $\int_{N'\cap N\backslash N'} \langle f(n'g), \check{v}\rangle dn'$ converges absolutely to $\langle v, \check{v}\rangle$ then define $\int_{N'\cap N\backslash N'} f(n'g)\, dn' = v$. If this is verified for all $f \in I_P^G(\pi \otimes \chi)$ and all $g \in G$ then define an intertwining operator for G-modules $J(\pi \otimes \chi) : I_P^G(\pi \otimes \chi) \to I_{P'}^G(\pi \otimes \chi)$ as:

(3.2.4)
$$J(\pi \otimes \chi)(f)(g) \;=\; \int_{N'\cap N\backslash N'} f(n'g)\, dn'.$$

Note the similarities and differences between the integrals in (2.2.1) and (3.2.4).

Let $\Sigma(A_P)$ denote the set of roots of A_P in $\mathrm{Lie}(G)$; identify $\Sigma(A_P)$ with a subset of $\mathfrak{a}_M^*$. Denote by $\Sigma(P)$ the subset of $\Sigma(A_P)$ of those roots whose root spaces appear in $\mathrm{Lie}(P)$. For P' with the same Levi as P, let $\bar{P}'$ denote its opposing parabolic subgroup. (This $\bar{P}'$ is the Q from before.) The following theorem is contained in [25, Theorem IV.1.1].

Theorem 3.2.5 *Suppose π is an irreducible admissible smooth representation of M. Then there is an open cone $\mathcal{U} = \{\chi \in X(M) : \langle \Re(\chi), \alpha\rangle > 0,\ \forall \alpha \in \Sigma(P) \cap \Sigma(\bar{P}')\}$ of $X(M)$ such that $J(\pi \otimes \chi)$ is defined by the convergent integral (3.2.4) for $\chi \in \mathcal{U}$. The collection of intertwining operators $\{J(\pi \otimes \chi)\}_{\chi\in\mathcal{U}}$ on this cone is rational.*

It is the rationality assertion in the above theorem that we are particularly interested in, since convergence in our context is already guaranteed by Proposition 2.2.2. We summarize the key steps of its proof, and refer the reader to [25, IV.1] for all the details and also for some of the notations used below even if not defined here because it would take us too far to systematically define them.

(1) (*Reduction step.*) We need a G-equivariant homomorphism of $\mathcal{B}$-modules

$$J_{\mathcal{B}} : I_P^G(\pi_{\mathcal{B}}) \to I_{P'}^G(\pi_{\mathcal{B}})$$

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as in (3.2.2), that satisfies the requirement of (3.2.3). Frobenius reciprocity for Jacquet modules and parabolic induction gives:

$$\mathrm{Hom}_{G,\mathcal{B}}(I_P^G(\pi_{\mathcal{B}}),\, I_{P'}^G(\pi_{\mathcal{B}})) = \mathrm{Hom}_{M,\mathcal{B}}(I_P^G(\pi_{\mathcal{B}})_{P'},\, \pi_{\mathcal{B}}),$$

where, $I_P^G(\pi_{\mathcal{B}})_{P'}$ is the Jacquet module of $I_P^G(\pi_{\mathcal{B}})$ with respect to P' on which the action of M is the canonical action twisted by $\delta_{P'}^{-1/2}$ to account for normalized parabolic induction. It suffices then to construct

$$j_{\mathcal{B}} \in \mathrm{Hom}_{M,\mathcal{B}}(I_P^G(\pi_{\mathcal{B}})_{P'},\, \pi_{\mathcal{B}})$$

such that the associated map $J_{\mathcal{B}}$ via Frobenius reciprocity satisfies (3.2.3).

(2) (*Exponents in the Jacquet module of an induced representation.*) By the well-known results of Bernstein and Zelevinskii [2, 2.12] (see also [25, I.3]), the Jacquet module $I_P^G(\pi_{\mathcal{B}})_{P'}$ is filtered by $(M,\mathcal{B})$-submodules $\{\mathcal{F}_{w,P'}\}_{w\in{}^{P'}W^P}$, indexed by a certain totally ordered set ${}^{P'}W^P$ of representatives in the Weyl group, such that for the successive quotients we have an isomorphism

$$q_w : \mathcal{F}_{w,P'}/\mathcal{F}_{w^+,P'} \to I^M_{M\cap w\cdot P}(w\cdot V_{\mathcal{B},M\cap w^{-1}\cdot P'}),$$

the right-hand side being a parabolically induced module of M. Consider these successive quotients for the action of the split center A_P of M; and let $\mathcal{E}xp_w$ be the set of exponents, which are characters $A_P \to \mathcal{B}^\times$, that appear in the (co-)domain of q_w. We may suppose that $1 \in {}^{P'}W^P$; the image of q_1 is $V_{\mathcal{B}}$. For any $w \in {}^{P'}W^P$, if $w \neq 1$ then $\mathcal{E}xp_w \cap \mathcal{E}xp_1 = \varnothing$; see [25, p. 280].

(3) (*Killing all subquotients except one.*) Using the theory of resultants, Waldspurger constructs $R \in \mathcal{B}[A_P]$ and $b \in \mathcal{B}$ such that R maps the Jacquet module $I_P^G(\pi_{\mathcal{B}})_{P'}$ into $\mathcal{F}_{1,P'}$ and on each of the generalized eigenspace for $\mu \in \mathcal{E}xp_1$ appearing in $\mathcal{F}_{1,P'}/\mathcal{F}_{1^+,P'} \otimes \mathrm{Frac}(\mathcal{B})$ it acts as homothety by the element b. The required element $j_{\mathcal{B}}$ as in (1) is the composition of R followed by $\mathcal{F}_{1,P'} \to \mathcal{F}_{1,P'}/\mathcal{F}_{1^+,P'} \xrightarrow{q_1} V_{\mathcal{B}}$.

3.3 An arithmetic variant of Theorem 3.2.5

Let E be a "large enough" finite Galois extension of $\mathbb{Q}$. The meaning of large enough will be explained in context. Let $X_E(M) = \mathrm{Hom}(M/M^1, E^*)$; similarly, $X_E(A)$. Restriction from M to A gives an isomorphism $X_E(M) \cong X_E(A)$. If $A = F^* \times \cdots \times F^*$, l-copies, then $A/A^1 = \varpi_F^{\mathbb{Z}} \times \cdots \times \varpi_F^{\mathbb{Z}}$, with $\varpi_F^{\mathbb{Z}}$ being the multiplicative infinite cyclic group generated by the uniformizer ϖ_F. Also, $X_E(A) = E^* \times \cdots \times E^*$, where $\underline{t} = (t_1,\ldots,t_l) \in E^* \times \cdots \times E^*$ corresponds to the character $\chi_{\underline{t}}$ that maps $\underline{a} = (a_1,\ldots,a_l) \in A$ to $\prod t_i^{\mathrm{ord}_F(a_i)}$. An embedding of fields $\iota : E \to \mathbb{C}$, gives a map $\iota_* : X_E(M) \to X_{\mathbb{C}}(M)$ where $\iota_*\chi = \iota\circ\chi$. The following diagram might help the reader:

$$\begin{array}{ccccc}
X_{\mathbb{C}}(A) = \mathrm{Hom}(A/A^1,\mathbb{C}^*) & \longrightarrow & \mathbb{C}^*\times\cdots\times\mathbb{C}^* & \longleftarrow & \mathbb{C}/(\mathbb{Z}\cdot\frac{2\pi i}{\log(q)})\times\cdots\times\mathbb{C}/(\mathbb{Z}\cdot\frac{2\pi i}{\log(q)}) \\
\uparrow{\scriptstyle \iota_*} & & \uparrow{\scriptstyle \iota\times\cdots\times\iota} & & \\
X_E(A) = \mathrm{Hom}(A/A^1,E^*) & \longrightarrow & E^*\times\cdots\times E^* & &
\end{array}$$

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For $\underline{s} = (s_1, \ldots, s_l) \in \mathbb{C}/(\mathbb{Z} \cdot \frac{2\pi i}{\log(q)}) \times \cdots \times \mathbb{C}/(\mathbb{Z} \cdot \frac{2\pi i}{\log(q)})$ put $\underline{w} := q^{\underline{s}}$, i.e., $\underline{w} = (w_1, \ldots, w_l) = (q^{s_1}, \ldots, q^{s_l}) \in \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ that corresponds to the character $\chi_{\underline{w}}$ of A given by $\underline{a} \mapsto \prod_i w_i^{\mathrm{ord}_F(a_i)}$. Note that $X_E(M)$ has the structure of an algebraic variety over E; denote by $\mathcal{B}_E(M)$ the E-algebra of polynomial functions on $X_E(M)$; then $\mathcal{B}_E(M) = E[t_1, t_1^{-1}, \ldots, t_l, t_l^{-1}]$. Similarly, $\mathcal{B}_{\mathbb{C}}(M) = \mathbb{C}[w_1, w_1^{-1}, \ldots, w_l, w_l^{-1}] = \mathbb{C}[q^{s_1}, q^{-s_1}, \ldots, q^{s_l}, q^{-s_l}]$. Base-change via the embedding ι gives: $\mathcal{B}_E(M) \otimes_{E,\iota} \mathbb{C} = \mathcal{B}_{\mathbb{C}}(M)$. To homogenize with the notations of Waldspueger [25] as used in Section 3.2, abbreviate $X_{\mathbb{C}}(M)$ and $\mathcal{B}_{\mathbb{C}}(M)$ as $X(M)$ and $\mathcal{B}$, respectively.

Hypotheses we impose on a representation in the main result have an arithmetic origin

Let $(\sigma, V_{\sigma,E})$ be a smooth absolutely irreducible admissible representation of M over an E-vector space $V_{\sigma,E}$. For an embedding of fields $\iota : E \to \mathbb{C}$, we have the irreducible admissible representation ${}^\iota\sigma$ of M on the $\mathbb{C}$-vector space $V_{{}^\iota\sigma} := V_{\sigma,E} \otimes_{E,\iota} \mathbb{C}$. We may apply the considerations of Section 2 to $({}^\iota\sigma, V_{{}^\iota\sigma})$. We explicate below all the hypotheses we impose on the representation ${}^\iota\sigma$ in the main result Theorem 3.3.7; these hypotheses are motivated by our global applications, and are expected to have an arithmetic origin.

The global context of a cohomological cuspidal automorphic representation suggests, via purity considerations, the following hypothesis on σ. Recall that for exponents $\underline{e} = (e_1, \ldots, e_l) \in \mathbb{R}^l$, by $\underline{\eta}^{\underline{e}} \in X(A) = X(M)$ defined as: $\underline{\eta}^{\underline{e}}(\underline{a}) = \prod_i |a_i|^{e_i}$ for $\underline{a} = (a_1, \ldots, a_l) \in A$.

Hypothesis 3.3.1 (Arithmeticity for half-integral unitarity) *Let $(\sigma, V_{\sigma,E})$ be a smooth absolutely irreducible admissible representation of a reductive p-adic group M over an E-vector space $V_{\sigma,E}$. If for one embedding $\iota : E \to \mathbb{C}$, there exists an l-tuple of integers $\underline{\mathrm{w}} = (\mathrm{w}_1, \ldots, \mathrm{w}_l)$ such that the representation ${}^\iota\sigma \otimes \underline{\eta}^{\underline{\mathrm{w}}/2}$ is unitary then for every embedding $\iota : E \to \mathbb{C}$, the representation ${}^\iota\sigma \otimes \underline{\eta}^{\underline{\mathrm{w}}/2}$ is unitary.*

It makes sense to call a σ satisfying the above hypothesis as *half-integrally unitary*.

Hypothesis 3.3.2 (Arithmeticity for essential-temperedness) *Let $(\sigma, V_{\sigma,E})$ be a smooth, absolutely irreducible, admissible, half-integrally unitary representation of a reductive p-adic group M over an E-vector space $V_{\sigma,E}$. If for one embedding $\iota : E \to \mathbb{C}$ the representation ${}^\iota\sigma$ is essentially tempered, then for every embedding $\iota : E \to \mathbb{C}$ the representation ${}^\iota\sigma$ is essentially tempered.*

Proof of Hypothesis 3.3.2 for $\mathrm{GL}_n(F)$ For $\mathrm{GL}_n(F)$ this follows from the considerations in Clozel [5] while using Jacquet's classification of tempered representations [10]; such a proof is well-known to experts and so we will just sketch the details. The reader is also referred to [16, Section 9.2] for a summary of the classification of tempered representations that we will use below. For a representation π of $\mathrm{GL}_n(F)$ and $t \in \mathbb{R}$, $\pi(t)$ denotes $\pi \otimes |\ |^t$.

(1) Any tempered representation π of $G = \mathrm{GL}_n(F)$ is fully induced from discrete series representations; it is of the form:

$$\pi = \mathrm{Ind}^G_{P_{n_1,\dots,n_r}(F)}(\pi_1 \otimes \cdots \otimes \pi_r),$$

where π_i is a discrete series representation of $\mathrm{GL}_{n_i}(F)$; $\sum_i n_i = n$; $P_{n_1,\dots,n_r}(F)$ is the parabolic subgroup of G with Levi subgroup $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$.

(2) A discrete series representation π_i of $\mathrm{GL}_{n_i}(F)$ is of the form

$$\pi_i = Q(\Delta(\sigma_i, b_i)),$$

where $n_i = a_i b_i$, with $a_i, b_i \in \mathbb{Z}_{\geq 1}$; σ_i is a supercuspidal representation of $\mathrm{GL}_{a_i}(F)$ such that $\sigma_i(\frac{b_i-1}{2})$ is unitary, and $Q(\Delta(\sigma_i, b_i))$ is the unique irreducible quotient of a parabolically induced representation:

$$\mathrm{Ind}^{\mathrm{GL}_{n_i}(F)}_{P_{b_i,\dots,b_i}(F)}(\sigma_i \otimes \sigma_i(1) \otimes \cdots \otimes \sigma_i(b_i - 1)) \twoheadrightarrow Q(\Delta(\sigma_i, b_i)).$$

In both the steps, the parabolic induction used is normalized induction which is not, in general, Galois equivariant. As in [5], we may force Galois equivariance by considering a half-integral Tate twisted version of induction. Using the notations of (1), but letting for the moment π_i be any irreducible admissible representation of $\mathrm{GL}_{n_i}(F)$, define:

$$\begin{aligned} &{}^T\mathrm{Ind}^G_{P_{n_1,\dots,n_r}(F)}(\pi_1 \otimes \cdots \otimes \pi_r) \\ &:= \mathrm{Ind}^G_{P_{n_1,\dots,n_r}(F)}\left(\pi_1(\frac{1-n_1}{2}) \otimes \cdots \otimes \pi_r(\frac{1-n_r}{2})\right)\left(\frac{n-1}{2}\right). \end{aligned}$$

Suppose $\tau \in \mathrm{Aut}(\mathbb{C})$; then one may verify that

$${}^\tau({}^T\mathrm{Ind}^G_{P_{n_1,\dots,n_r}(F)}(\pi_1 \otimes \cdots \otimes \pi_r)) = {}^T\mathrm{Ind}^G_{P_{n_1,\dots,n_r}(F)}({}^\tau\pi_1 \otimes \cdots \otimes {}^\tau\pi_r).$$

Define a quadratic character of F^* as $\varepsilon_\tau := (\tau \circ |\,|^{1/2})/|\,|^{1/2}$; it is trivial if and only if τ fixes $q^{1/2}$, where q is the cardinality of the residue field of F. Then:

$${}^\tau\mathrm{Ind}^G_{P_{n_1,\dots,n_r}(F)}(\pi_1 \otimes \cdots \otimes \pi_r) = \mathrm{Ind}^G_{P_{n_1,\dots,n_r}(F)}((\pi_1 \otimes \varepsilon_\tau^{n-n_1}) \otimes \cdots \otimes (\pi_r \otimes \varepsilon_\tau^{n-n_r})).$$

Similarly, using Lemma 3.2.1 and the few lines following that lemma in [5], we have:

$${}^\tau Q(\Delta(\sigma_i, b_i)) = Q(\Delta({}^\tau\sigma_i \otimes \varepsilon_\tau^{a_i(b_i-1)}, b_i)).$$

Of course each ${}^\tau\sigma_i$ is supercuspidal and so also is any of its quadratic twists; furthermore, $({}^\tau\sigma_i \otimes \varepsilon_\tau^{a_i(b_i-1)})(\frac{b_i-1}{2})$ is unitary. Hence, the τ-conjugate of the tempered representation π is tempered.

Using the notations in the hypothesis, take $\pi = {}^\iota\sigma$, and if $\iota' : E \to \mathbb{C}$ is any other embedding then take $\tau \in \mathrm{Aut}(\mathbb{C})$ such that $\iota' = \tau \circ \iota$. By assumption $\pi = \pi^t \otimes |\,|^{\mathrm{w}/2}$ for a unitary tempered representation and an integral exponent w. Then

$$\text{(3.3.3)} \qquad {}^\tau\pi = {}^\tau\pi^t \otimes (\tau \circ |\,|^{\mathrm{w}/2}) = ({}^\tau\pi^t \otimes \varepsilon_\tau^{\mathrm{w}}) \otimes |\,|^{\mathrm{w}/2}.$$

By the above argument ${}^\tau\pi^t$ is tempered, and hence so also is ${}^\tau\pi^t \otimes \varepsilon_\tau^{\mathrm{w}}$. ■

Remarks on the proof of Hypothesis 3.3.2 for classical groups For classical groups, using a similar argument as in the case of $\mathrm{GL}_n(F)$, and also the result for $\mathrm{GL}_n(F)$, a proof follows from Mœglin and Tadic's classification [15] for discrete series and tempered representations. The proof is tedious. We will sketch the argument for even-orthogonal groups.

Consider $G = \mathrm{O}_{2n}(F) = \{g \in \mathrm{GL}_{2n}(F) : {}^t g \cdot J \cdot g = J\}$ the split even orthogonal group of rank n, where $J_{i,j} = \delta(i, 2n - j + 1)$. Suppose $n = a_1 + \cdots + a_q + n_0$, with $a_1, \ldots, a_q \geq 1$ and $n_0 \geq 0$, and $P_{(a_1,\ldots,a_q;n_0)}$ is the parabolic subgroup of $\mathrm{O}_{2n}(F)$ with Levi subgroup

$$M_{(a_1,\ldots,a_q;n_0)} = \mathrm{GL}_{a_1}(F) \times \cdots \times \mathrm{GL}_{a_q}(F) \times \mathrm{O}_{2n_0}(F).$$

Let π_0 be a discrete series representation of $\mathrm{O}_{2n_0}(F)$ and θ_j an essentially discrete series representation of $\mathrm{GL}_{a_j}(F)$. For brevity, let

$$\theta_1 \times \cdots \times \theta_q \rtimes \pi_0 \ := \mathrm{Ind}^G_{P_{(a_1,\ldots,a_q;n_0)}}(\theta_1 \otimes \cdots \otimes \theta_q \otimes \pi_0).$$

This induced representation is a multiplicity-free direct sum of tempered representations (see Mœglin–Tadic [15, Theorem 13.1] and Atobe–Gan [1, Desideratum 3.9, (6)]). Suppose π is one such tempered representation: $\pi \hookrightarrow \theta_1 \times \cdots \times \theta_q \rtimes \pi_0$. Suppose $m_{h_1,\ldots,h_q,x} \in M$ with $h_j \in \mathrm{GL}_{a_j}(F)$ and $x \in \mathrm{O}_{2n_0}(F)$, then the absolute-value of the determinant of the adjoint action of $m_{h_1,\ldots,h_q,x}$ on the Lie algebra of the unipotent radical of P is given by:

$$\delta_P(m_{h_1,\ldots,h_q,x}) = \left(|\det(h_1)|^{2n-2a_1}|\det(h_2)|^{2n-2(a_1+a_2)}\cdots|\det(h_q)|^{2n_0}\right)\cdot \\ \cdot\left(|\det(h_1)|^{a_1-1}|\det(h_2)|^{a_2-1}\cdots|\det(h_q)|^{a_q-1}\right).$$

Using this, for $\tau \in \mathrm{Aut}(\mathbb{C})$, one may verify that:

$$(3.3.4) \qquad {}^\tau\pi \hookrightarrow {}^\tau(\theta_1 \times \cdots \times \theta_q \rtimes \pi_0) = ({}^\tau\theta_1 \otimes \varepsilon_\tau^{a_1-1}) \times \cdots \times ({}^\tau\theta_q \otimes \varepsilon_\tau^{a_q-1}) \rtimes {}^\tau\pi_0.$$

By appealing to the above proof for $\mathrm{GL}_{n_0}(F)$, we know that each ${}^\tau\theta_j$, and so also its quadratic twist ${}^\tau\theta_1 \otimes \varepsilon_\tau^{a_1-1}$, is an essentially discrete series representation. Hence, proof of arithmeticity for tempered representation of $\mathrm{O}_{2n}(F)$ boils down to proving arithmeticity for discrete series representation of $\mathrm{O}_{2n_0}(F)$.

For the discrete series representation π_0 of $\mathrm{O}_{2n_0}(F)$, there exist a and n_1 such that $n_0 = a + n_1$, and there exist an essentially discrete series representation θ of $\mathrm{GL}_a(F)$ and a discrete series representation π_1 of the smaller even-orthogonal group $\mathrm{O}_{2n_1}(F)$ such that π_0 is one of two possible subrepresentations of $\theta \rtimes \pi_1$; and both these subrepresentations are in the discrete series; this being the crux of [15]. Then, as above ${}^\tau\pi \hookrightarrow {}^\tau\theta \otimes \varepsilon_\tau^{a-1} \otimes {}^\tau\pi_1$. An induction argument (see [15, p. 721]) concludes the proof as the reduction to a smaller even-orthogonal group ends with the case of π_1 being a supercuspidal representation (in *loc.cit.* called the weak cuspidal support of π_0), and clearly conjugation by τ preserves supercuspidality as it leaves the support of a matrix coefficient unchanged.

It is an interesting problem to prove this for a general p-adic group. Assuming that Hypothesis 3.3.2 is true, we can then formulate another hypothesis:

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Hypothesis 3.3.5 (Arithmeticity for being on the right of the unitary axis) *Let $(\sigma, V_{\sigma,E})$ be a smooth, absolutely irreducible, admissible, half-integrally unitary, essentially tempered representation of a reductive p-adic group M over an E-vector space $V_{\sigma,E}$. If for one embedding $\iota : E \to \mathbb{C}$ the representation ${}^{\iota}\sigma$ is to the right of the unitary axis, then for every embedding $\iota : E \to \mathbb{C}$ the representation ${}^{\iota}\sigma$ is to the right of the unitary axis.*

For $\mathrm{GL}_n(F)$ this follows from (3.3.3) since the exponent for π and ${}^{\tau}\pi$ are equal. Similarly, the above hypothesis will follow from Hypothesis 3.3.2 that the half-integral exponents $\underline{w}/2$ for ${}^{\iota}\sigma$ are independent of ι; in particular the exponent f_1 of $\tilde{r}_1({}^{\iota}\sigma)$ would be independent of ι.

Lemma 3.3.6 (Arithmeticity for genericity) *Let $(\sigma, V_{\sigma,E})$ be a smooth absolutely irreducible admissible representation of a reductive quasi-split p-adic group M over an E-vector space $V_{\sigma,E}$. If for one embedding $\iota : E \to \mathbb{C}$ the representation ${}^{\iota}\sigma$ is generic, then for every embedding $\iota : E \to \mathbb{C}$ the representation ${}^{\iota}\sigma$ is generic.*

Proof Suppose $\ell : {}^{\iota}\sigma \to \mathbb{C}$ is a Whittaker functional with respect to a character $\psi : U \to \mathbb{C}^*$ (that is nontrivial on all the root spaces corresponding to simple roots). Given another embedding $\iota' : E \to \mathbb{C}$, there exists $\tau \in \mathrm{Aut}(\mathbb{C})$ such that $\iota' = \tau \circ \iota$. Then $\tau \circ \ell$ is a Whittaker functional for ${}^{\iota'}\sigma$ with respect to the character $\tau \circ \psi$ of U. ■

After the above hypotheses and lemma, it makes sense to say that a smooth absolutely-irreducible admissible representation $(\sigma, V_{\sigma,E})$ of a reductive p-adic group M is half-integrally unitary, essentially tempered, to the right of the unitary axis, or generic, if for some, and hence any, embedding $\iota : E \to \mathbb{C}$ the representation ${}^{\iota}\sigma$ is half-integrally unitary, essentially tempered, to the right of the unitary axis, or generic, respectively. The first main theorem of this article is the following result.

An arithmetic variant of Theorem 3.2.5

Theorem 3.3.7 *Let $P = MN$ be a maximal parabolic subgroup of a connected reductive p-adic group G. Let $(\sigma, V_{\sigma,E})$ be a smooth absolutely-irreducible admissible representation of M over an E-vector space $V_{\sigma,E}$. Assume that E is large enough to contain the values of the exponents of A that appear in the Jacquet module of ${}^{\mathrm{a}}\mathrm{Ind}_P^G(\sigma)$ with respect to the associate parabolic subgroup Q. Assuming Hypotheses 3.3.1, 3.3.2, and 3.3.5, we suppose that σ is half-integrally unitary, essentially tempered, to the right of the unitary axis, and generic. Suppose that P satisfies the integrality condition: $\rho_P|_{A_P} \in X^*(A_P)$, then so does Q and the modular character δ_Q takes values in $\mathbb{Q}^*$. There exists an E-linear G-equivariant map*

$$T_{st,E} : {}^{\mathrm{a}}\mathrm{Ind}_P^G(\sigma) \longrightarrow {}^{\mathrm{a}}\mathrm{Ind}_Q^G(\sigma \otimes \delta_Q)$$

such that for any embedding $\iota : E \to \mathbb{C}$ we have:

$$T_{\mathrm{st},E} \otimes_{E,\iota} 1_{\mathbb{C}} = T_{\mathrm{st},\iota},$$

where $T_{\mathrm{st},\iota} = T_{\mathrm{st}}(s, {}^{\iota}\sigma)|_{s=k} : {}^{\mathrm{a}}\mathrm{Ind}_P^G({}^{\iota}\sigma) \to {}^{\mathrm{a}}\mathrm{Ind}_Q^G({}^{\iota}\sigma \otimes \delta_Q)$ is the standard intertwining operator at the point of evaluation.

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Proof Fix an $\iota : E \to \mathbb{C}$. For $\chi \in X(M)$, we have the standard intertwining operator

$$T_{\rm st}({}^\iota\sigma, \chi) : I_P^G({}^\iota\sigma \otimes \chi) \to I_Q^G({}^{w_0}({}^\iota\sigma \otimes \chi))$$

given by an integral where it converges. We will ultimately specialize to the point χ_k corresponding to the point of evaluation $k = -\langle \rho_P, \alpha_P \rangle$; note that $\chi_k = -\rho_P$; at this point our hypotheses guarantee convergence. Consider Theorem 3.2.5 with the small variation that we take the associate parabolic Q and not P' which required $M_{P'} = M_P$; for Q we have M_Q is the w_0-conjugate of M_P. This causes no problem as long as we use the correct integral, i.e., we use (2.2.1) instead of (3.2.4). From Theorem 3.2.5 we get a $(G, \mathcal{B})$-module map

$$T_{\mathcal{B}} : I_P^G({}^\iota\sigma \otimes_{\mathbb{C}} \mathcal{B}) \longrightarrow I_Q^G({}^{w_0\iota}\sigma \otimes_{\mathbb{C}} \mathcal{B})$$

that satisfies (3.2.3) with a homothety element $b \in \mathcal{B}$.

The main steps in the proof of Theorem 3.2.5 ((i) reduction via Frobenius reciprocity, (ii) Jacquet module calculation, and (iii) construction of an M-equivariant map using an element R in the group ring of A via the theory of resultants) are all purely algebraic in nature. The same proof, but now working with modules over E, gives us an E-linear map of $(G, \mathcal{B}_E)$-modules:

$$T_{\mathcal{B},E} : I_P^G(\sigma \otimes_E \mathcal{B}_E) \longrightarrow I_Q^G({}^{w_0}\sigma \otimes_E \mathcal{B}_E)$$

with a homothety $b_0 \in \mathcal{B}_E$ such that for any $\iota : E \to \mathbb{C}$ we have: $T_{\mathcal{B},E} \otimes_{E,\iota} \mathbb{C} = T_{\mathcal{B}}$, and $b_0 \otimes_{E,\iota} 1 = b$. Specialize at the point of evaluation in (3.2.3), i.e., take $\chi = \chi_k = -\rho_P \in X_E(A)$; hence, $b(\chi_k) = b_0(-\rho_P) \in E^*$; note that we have used $\rho_P|_{A_P}$ is an integral weight. We have:

$$\iota(b_0(-\rho_P))T_{\rm st}({}^\iota\sigma, \chi_k) = {\rm sp}_{\chi_k} \circ (T_{\mathcal{B},E} \otimes_{E,\iota} 1_{\mathbb{C}}).$$

For $\chi_0 \in X_E(A)$, if ${\rm sp}_{\chi_0,E} : \mathcal{B}_E \to E$ denotes the specialization map at an arithmetic level, then clearly, ${\rm sp}_{\chi_0,E} \otimes_{E,\iota} 1_{\mathbb{C}} = {\rm sp}_{\chi_0}$. Hence, $T_{\rm st}({}^\iota\sigma, \chi_k) = (b_0(\rho_P){\rm sp}_{\chi_k,E} \circ T_{\mathcal{B},E}) \otimes_{E,\iota} 1_{\mathbb{C}}$. ■

4 Arithmeticity of local critical *L*-values

The purpose of this section is to formulate an arithmeticity hypothesis on local critical L-values for automorphic L-functions. It is a generalization of [17, Proposition 3.17] which was in the context of Rankin–Selberg L-functions and was a crucial ingredient in the proof of the main theorem of that article. Using this hypothesis, we may strengthen Theorem 3.3.7 to give an arithmeticity result for the normalized standard intertwining operator.

4.1 Criticality condition on the point of evaluation

In the context of Rankin–Selberg L-functions one takes $M_P = {\rm GL}_n \times {\rm GL}_{n'}$ as a Levi subgroup of an ambient $G = {\rm GL}_N$, where $N = n + n'$. The integrality condition on P forces nn' to be even. For an inducing data $\pi \times \pi'$ of M_P, the critical set for the L-function $L(s, \pi \times \pi'^{\vee})$ consists of integers if $n \equiv n' \pmod 2$ and consists of half-integers, i.e., elements of $\frac{1}{2} + \mathbb{Z}$ if $n \not\equiv n' \pmod 2$; see [8, Definition 7.3]. The purpose

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of this subsection is to formalize such parity constraints in the context of Langlands–Shahidi machinery.

Suppose $A_i = \eta_i(A) = F^*$ and A is the internal product $A_1 \times \cdots \times A_l$; correspondingly, suppose $M = M_1 \cdots M_l$ an almost direct product of reductive subgroups, with A_i in the center of M_i. Let ρ_{M_i} be half the sum of positive roots for M_i. If ρ_{M_i} is integral, then put $\varepsilon_{M_i} = 0$. If ρ_{M_i} is not integral, then necessarily $2\rho_{M_i}$ is integral, and put $\varepsilon_{M_i} = 1$. Fix an unramified character $\chi_P^{\varepsilon_P/2} \in \mathrm{Hom}(M/M^1, \mathbb{C}^*) = \mathrm{Hom}(A/A^1, \mathbb{C}^*)$, defined by

$$\chi_P^{\varepsilon_P/2}(a_1, \dots, a_l) := |a_1|^{\varepsilon_{M_1}/2} \dots |a_l|^{\varepsilon_{M_l}/2}, \quad (a_1, \dots, a_l) \in A = F^* \times \cdots \times F^*.$$

Let $\vartheta_P \in {}^L A_P^\circ$ be the Satake parameter of $\chi_P^{\varepsilon_P/2}$. Using (2.4.1), there exists $h_j \in \frac{1}{2}\mathbb{Z}$ such that $\tilde{r}_j(\vartheta_P) = q^{-h_j}$ or that $\tilde{r}_j(\chi_P^{\varepsilon_P/2}) = |\ |^{h_j}$. Let π be an irreducible admissible half-integrally unitary, essentially tempered, generic representation of M_P. Consider $\pi \otimes \chi_P^{\varepsilon_P/2}$; we have:

$$L(s, \pi, \tilde{r}_j) = L(s - h_j, \pi \otimes \chi_P^{\varepsilon_P/2}, \tilde{r}_j).$$

The idea is that given π, we *algebrize* it by considering the twist $\pi \otimes \chi_P^{\varepsilon_P/2}$. For GL_n this is equivalent to replacing π by $\pi \otimes |\ |^{\varepsilon_n/2}$, where $\varepsilon_n \in \{0, 1\}$ and $\varepsilon_n \equiv n - 1 \pmod 2$. Any point of evaluation of a global L-function attached to an algebraic data (think of a motivic L-function) should be an integer for the L-value to be critical in the sense of Deligne [6]. This motivates the following definition which is independent of π and depends only on (G, P).

Definition 4.1.1 Let G be a connected reductive p-adic group and P a maximal parabolic subgroup. We say that P is critical for G if the point of evaluation $k = -\langle \rho_P, \alpha_P \rangle$ satisfies the condition:

$$jk \in h_j + \mathbb{Z}, \quad \forall\, 1 \le j \le m.$$

4.2 Hypothesis on local critical *L*-values

We can now formulate the arithmeticity hypothesis for local critical L-values.

Hypothesis 4.2.1 *Let G be a connected reductive p-adic group and P a maximal parabolic subgroup. Assume that P satisfies the following two conditions:*

(i) *the integrality condition:* $\rho_P|_{A_P} \in X^*(A_P)$ *and*

(ii) *the criticality condition: P is critical for G.*

Let σ *be a smooth, absolutely irreducible, half-integrally unitary, essentially-tempered, admissible, generic representation of* M_P *over a field of coefficients E. Let* $k = -\langle \rho_P, \alpha_P \rangle$ *be the point of evaluation, and let* $s_j \in \{jk, jk+1\}$ *for any* $1 \le j \le m$. *Then for any embedding* $\iota : E \to \mathbb{C}$ *we have:*

(1) $L(s_j, {}^\iota\sigma, \tilde{r}_j) \in \iota(E)$, *and furthermore*

(2) *for any* $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *we have* $\tau(L(s_j, {}^\iota\sigma, \tilde{r}_j)) = L(s_j, {}^{\tau\circ\iota}\sigma, \tilde{r}_j)$.

As already mentioned, this hypothesis can be verified in various concrete examples of interest. We briefly mention two examples below; these contexts are amplified in Section 5; the reader will readily appreciate that such examples may be generalized.

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Example 4.2.2 (Local L-functions for $\mathrm{GL}_n(F)$) If π is an irreducible admissible representation of $\mathrm{GL}_n(F)$ then it follows from Clozel [5, Lemma. 4.6] that for any $k_0 \in \mathbb{Z}$ and any $\tau \in \mathrm{Aut}(\mathbb{C})$ one has:

$$\tau\left(L(k_0 + \tfrac{1-n}{2}, \pi)\right) = L(k_0 + \tfrac{1-n}{2}, {}^\tau\pi).$$

The reader can check that Hypothesis 4.2.1 follows from this Galois equivariance after appealing to the details in Section 5.1. Such a Galois equivariance can be reformulated as

$$\tau(L(k_0, \pi)) = L(k_0, {}^\tau\pi \otimes \varepsilon_\tau^{n-1}),$$

which is useful in other situations; see the next example below.

Example 4.2.3 (Local L-functions for $\mathrm{O}_{2n}(F)$) Suppose π is an irreducible tempered representation of $\mathrm{O}_{2n}(F)$ as in the proof of Hypothesis 3.3.2 for orthogonal groups; in particular, $\pi \hookrightarrow \theta_1 \times \cdots \times \theta_q \rtimes \pi_0$, with notations as therein. Then, the L-parameters are related as: $\phi_\pi = \phi_{\theta_1} + \cdots + \phi_{\theta_q} + \phi_{\pi_0} + \phi^\vee_{\theta_1} + \cdots + \phi^\vee_{\theta_q}$ (see [1, Desideratum 3.9]). In particular, for L-functions, evaluating at $s = k \in \mathbb{Z}$ (see Section 5.2 for the fact that the point of evaluation is an integer), we get: $L(k, \pi) = L(k, \theta_1) \cdots L(k, \theta_q) \cdot L(k, \pi_0) \cdot L(k, \theta_1^\vee) \cdots L(k, \theta_q^\vee)$. Apply $\tau \in \mathrm{Aut}(\mathbb{C})$ to both sides while using Example 4.2.2 to get $\tau(L(k, \pi))$ is equal to

$$L(k, {}^\tau\theta_1 \otimes \varepsilon_\tau^{a_1 - 1}) \cdots L(k, {}^\tau\theta_q \otimes \varepsilon_\tau^{a_q - 1}) \cdot \tau(L(k, \pi_0)) \\ \cdot L(k, {}^\tau\theta_1^\vee \otimes \varepsilon_\tau^{a_1 - 1}) \cdots L(k, {}^\tau\theta_q^\vee \otimes \varepsilon_\tau^{a_q - 1}).$$

Now, use (3.3.4) for ${}^\tau\pi$; then take its L-function evaluated at $s = k$ to get:

$$\tau(L(k, \pi)) = L(k, {}^\tau\pi),$$

assuming by induction that $\tau(L(k, \pi_0)) = L(k, {}^\tau\pi_0)$ holds for a discrete series representation π_0 of $\mathrm{O}_{2n_0}(F)$. The proof for a discrete series representation follows the same reduction strategy as in the proof of Hypothesis 3.3.2 for even orthogonal groups.

I expect that a proof of Hypothesis 4.2.1 in the general case should come from an arithmetic-analysis of Shahidi's theory of local factors.

4.3 An arithmetic variant of Theorem 3.2.5 for normalized intertwining operator

We can now strengthen Theorem 3.3.7 for the normalized intertwining operator.

Theorem 4.3.1 *Let the notations and hypotheses be as in Theorem 3.3.7. Assume furthermore that Hypothesis 4.2.1 holds. Let $T_{\mathrm{norm}} = T_{\mathrm{norm}}(s, {}^\iota\sigma)|_{s=k}$ be the normalized standard intertwining operator (see (1.0.1)) at the point of evaluation $s = k$. Then there exists an E-linear G-equivariant map*

$$T_{\mathrm{norm},E} : {}^{\mathrm{a}}\mathrm{Ind}_P^G(\sigma) \longrightarrow {}^{\mathrm{a}}\mathrm{Ind}_Q^G(\sigma \otimes \delta_Q)$$

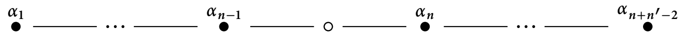
such that for any embedding $\iota : E \to \mathbb{C}$ we have:

$$T_{\mathrm{norm},E} \otimes_{E,\iota} 1_{\mathbb{C}} = T_{\mathrm{norm}}.$$

5 Examples

5.1 Rankin–Selberg *L*-functions

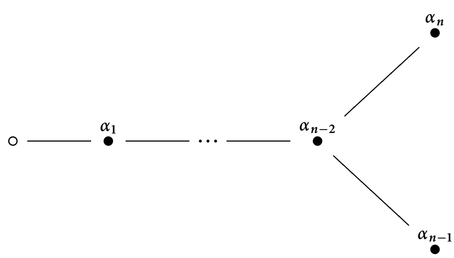
See case (A_{N-1}) in [23, Appendix B].



(1) Ambient group: $G = \mathrm{GL}(N)/F$ with $N \geq 2$.
(2) Maximal parabolic subgroup: take $N = n + n'$ and let P be the maximal parabolic subgroup with Levi $M_P = \mathrm{GL}(n) \times \mathrm{GL}(n')$; the deleted simple root $\alpha_P = e_n - e_{n+1}$.
(3) The integrality condition $\rho_P \in X^*(A_P)$ holds if and only if $nn' \equiv 0 \pmod 2$. The set of roots with roots spaces appearing in the Lie algebra $\mathfrak{n}_P$ of the unipotent radical of P is $\{e_i - e_j : 1 \leq i \leq n,\ \ n+1 \leq j \leq n+n'\}$. Hence $\rho_P = \frac{n'}{2}(e_1 + \cdots + e_n) - \frac{n}{2}(e_{n+1} + \cdots + e_{n+n'})$. Whence, ρ_P as a character of A_P is given by: $\mathrm{diag}(t1_n, t'1_{n'}) \mapsto (tt')^{nn'/2}$, which is integral if and only if nn' is even. It is curious that this very condition was imposed in [8] due to motivic considerations (the tensor product motive therein needed to be of even rank).
(4) At the level of dual groups, ${}^L M_P^\circ \cong \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n'}(\mathbb{C})$ acts irreducibly on the Lie algebra ${}^L\mathfrak{n}_P \cong M_{n\times n'}(\mathbb{C})$ of ${}^L N_P$; $m = 1$.
(5) Inducing data consists of π and π' which are essentially tempered irreducible generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_{n'}(F)$ then $L(s, \pi \otimes \pi', \tilde{r}_1) = L(s, \pi \times \pi^\vee)$ is the local Rankin–Selberg L-function attached to $\mathrm{GL}_n(F) \times \mathrm{GL}_{n'}(F)$.
(6) The point of evaluation is $k = -\langle \rho_P, \alpha_P \rangle = -N/2$.
(7) P is critical for G: $M = M_1 M_2$ with $M_1 = \mathrm{GL}_n(F)$ and $M_2 = \mathrm{GL}_{n'}(F)$; $\varepsilon_{M_1} = \varepsilon_n$ and $\varepsilon_{M_2} = \varepsilon_{n'}$ (recall: $\varepsilon_n \in \{0,1\}$ by $\varepsilon_n \equiv n-1 \pmod 2$); $h_1 = (\varepsilon_n - \varepsilon_{n'})/2$; and $k \in h_1 + \mathbb{Z}$.

5.2 *L*-functions for orthogonal groups

See case $(D_{n,i})$ in [23, Appendix A]; the corresponding global context is studied in [3].



(1) Ambient group $G = \mathrm{O}(n+1, n+1) = \{g \in \mathrm{GL}_{2n+2}(F) : {}^t g \cdot J_{2n+2} \cdot g = J_{2n+2}\}$, where $J_{2n+2}(i,j) = \delta(i, 2n+3-j)$; this is the split even orthogonal group of rank $n+1$; the maximal torus consists of all diagonal matrices $\mathrm{diag}(t_0, t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}, t_0^{-1})$.

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(2) Let P be the maximal parabolic subgroup described by the above Dynkin diagram; deleted simple root $\alpha_P = e_0 - e_1$; Levi subgroup is

$$M_P = \left\{ m_{t,h} = \begin{pmatrix} t & & \\ & h & \\ & & t^{-1} \end{pmatrix} : t \in \mathrm{GL}_1(F),\ h \in \mathrm{O}(n,n) \right\};$$

clearly, $A_P = \{m_{t,1} \in M_P : t \in \mathrm{GL}_1\}$; the unipotent radical of P is:

$$N_P = \left\{ u_{y_1,y_2,\dots,y_{2n}} = \begin{pmatrix} 1 & y_1 & y_2 & \dots & y_{2n} & 0 \\ & 1 & & & & -y_{2n} \\ & & 1 & & & -y_{2n-1} \\ & & & \ddots & & \vdots \\ & & & & \ddots & -y_1 \\ & & & & & 1 \end{pmatrix} \mid y_1,\dots,y_{2n} \in F \right\}.$$

(3) The integrality condition on ρ_P holds for all n. The set of roots with root spaces in the Lie algebra of N_P is $\{e_0 - e_1,\ e_0 - e_2, \dots, e_0 - e_{2n}\}$. Hence

$$\rho_P = ne_0 - \tfrac{1}{2}(e_1 + e_3 + \dots + e_{2n});$$

from the maximal torus one has $e_{n+1} = -e_n, e_{n+2} = -e_{n-1}, \dots, e_{2n} = -e_1$ from which it follows that $\rho_P = ne_0$. Whence, $\rho_P|_{A_P}$ is the integral character $t = m_{t,1} \mapsto t^n$.

(4) At the level of dual groups, ${}^LM_P^\circ = \{m_{t,h} : t \in \mathbb{C}^*,\ h \in \mathrm{O}(n,n)(\mathbb{C})\}$ acts irreducibly on the Lie algebra ${}^L\mathfrak{n}_P$ of ${}^LN_P^\circ$; $m = 1$ and r_1 is the standard representation of $\mathrm{O}(n,n)(\mathbb{C})$ twisted by the $\mathbb{C}^*$ in the obvious way.

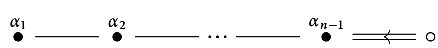
(5) Inducing data is of the form $\chi \otimes \pi$ for a character $\chi : F^* \to \mathbb{C}^*$, and a tempered, irreducible, generic representation π of $\mathrm{O}(n,n)(F)$. The local L-function $L(s, \chi \otimes \pi, \tilde{r}_1)$ is the local Rankin–Selberg L-function $L(s, \chi \otimes \tilde{r}_1(\pi))$ for $\mathrm{GL}_1 \times \mathrm{GL}_{2n}$.

(6) The point of evaluation is $k = -\langle \rho_P, \alpha_P \rangle = -n$.

(7) P is critical for G, since $M = M_1 M_2$ with $M_1 = \mathrm{GL}_1(F)$ and $M_2 = \mathrm{O}(n,n)(F)$; ρ_{M_i} is integral; $h_1 = 0$; $k \in \mathbb{Z}$.

5.3 Exterior square L-functions

See case $(C_{n-1,ii})$ in [23, Appendix A].



(1) Ambient group $G = \mathrm{Sp}_{2n}(F) = \left\{ g \in \mathrm{GL}_{2n}(F) : {}^tg \cdot \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} \cdot g = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} \right\}$, where $J_n(i,j) = \delta(i, r - j + 1)$ and ${}^tg$ is the transpose of g.

(2) Maximal parabolic subgroup as depicted by the above Dynkin diagram has Levi subgroup: $M_P = \left\{ \begin{pmatrix} h & \\ & {}^{(t)}h^{-1} \end{pmatrix} : h \in \mathrm{GL}_n(F) \right\}$ where ${}^{(t)}h = J_n \cdot {}^th \cdot J_n$ is the "other-transpose" of h defined by $({}^{(t)}h)_{i,j} = h_{n-j+1,n-i+1}$. The deleted simple root $\alpha_P = 2e_n$.

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(3) The integrality condition $\rho_P \in X^*(A_P)$ holds if and only if $n \equiv 0, 3 \pmod 4$. The Lie algebra of the unipotent radical of P is of the form

$$\mathfrak{n}_P = \left\{ \begin{pmatrix} 0_n & X \\ 0_n & 0_n \end{pmatrix} : X \in M_n(F),\ {}^{(t)}X = X \right\}.$$

The set of roots with root spaces appearing in $\mathfrak{n}_P$ is $\{e_1 - e_{n+1}, \dots, e_1 - e_{2n}, e_2 - e_{n+1}, \dots, e_2 - e_{2n-1}, \dots, e_n - e_{n+1}\}$. Keeping in mind that $e_j = -e_{2n-j+1}$ we get

$$\rho_P = \frac{n+1}{2}(e_1 + \cdots + e_n).$$

Whence, $\rho_P|_{A_P}$ is given by: $\mathrm{diag}(t1_n, t^{-1}1_n) \mapsto t^{n(n+1)/2}$, which is integral if and only if $n(n+1)/2$ is even, i.e., $n \equiv 0$ or $3 \pmod 4$.

(4) Dual groups: ${}^LG^\circ = \mathrm{SO}(2n+1, \mathbb{C}) = \{g \in \mathrm{SL}_{2n+1}(F) : {}^tg \cdot J_{2n+1} \cdot g = J_{2n+1}\}$,

$${}^LM_P^\circ = \left\{ m_g = \begin{pmatrix} g & & \\ & 1 & \\ & & {}^{(t)}g^{-1} \end{pmatrix} : g \in \mathrm{GL}_n(\mathbb{C}) \right\} \cong \mathrm{GL}_n(\mathbb{C});$$

$${}^L\mathfrak{n}_P = \left\{ n_{y,X} = \begin{pmatrix} 0_n & y & X \\ 0_{1\times n} & 0 & -{}^tyJ_n \\ 0_n & 0_{n\times 1} & 0_n \end{pmatrix} : y \in M_{n\times 1}(\mathbb{C}),\ X \in M_{n\times n}(\mathbb{C}),\ {}^{(t)}X = -X \right\}.$$

The adjoint action of ${}^LM_P^\circ$ on ${}^L\mathfrak{n}_P$ is the direct sum of two irreducible representations with representation spaces $V_1 = \{n_{y,0} \in {}^L\mathfrak{n}_P\}$ and $V_2 = \{n_{0,X} \in {}^L\mathfrak{n}_P\}$ of dimensions n and $n(n-1)/2$, respectively; r_1 is the standard representation and r_2 is the exterior square representation; $m = 2$. The center ${}^LA_P^\circ$ of ${}^LM_P^\circ$ consists of elements $a_t = m_{t\cdot I_n}$ for $t \in \mathbb{C}^\times$; then a_t acts on V_1 by the scalar t and on V_2 by the scalar t^2.

(5) The inducing data is a half-integrally unitary, irreducible, essentially tempered, generic representation π of $\mathrm{GL}_n(F)$; for the L-functions we have:
   (a) $L(s, \pi, \tilde{r}_1) = L(s, \pi)$, the standard L-function for $\mathrm{GL}(n)$ and
   (b) $L(s, \pi, \tilde{r}_2) = L(s, \pi, \wedge^2)$, the exterior square L-function for $\mathrm{GL}(n)$.

(6) The point of evaluation is $k = -\langle \rho_P, \alpha_P \rangle = -\frac{n+1}{2}$.

(7) P is critical for G. Since $\varepsilon_M = \varepsilon_n$, $h_1 = \varepsilon_n/2$, and $h_2 = \varepsilon_n$; hence $jk \in h_j + \mathbb{Z}$ holds for $j = 1, 2$.

5.4 Explicit intertwining calculation for the case of GL(2)

Some essential features of main results are already visible for the example of $\mathrm{GL}(2)$ from first principles; although the reader is warned of the well-known dictum that $\mathrm{GL}(2)$ is misleadingly simple and it is difficult to carry out a straightforward generalization of such calculations.

Let $E/\mathbb{Q}$ be a finite extension, and for $i = 1, 2$, let $\chi_i : F^\times \to E^\times$ be a smooth character, and χ_i° its restriction to $\mathcal{O}_F^\times$. Let $\iota : E \to \mathbb{C}$ be an embedding of fields, and ${}^\iota\chi_i = \iota \circ \chi_i$ be the corresponding $\mathbb{C}$-valued character of $F^\times$. Let $G = \mathrm{GL}_2(F)$, $K = \mathrm{GL}_2(\mathcal{O}_F)$, and for $m \geq 0$ let $K(m)$ be the principal congruence subgroup of K of level m; $K(0) = K$. The standard intertwining operator $T_{\mathrm{st}}(s)$ at the point of evaluation

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$s = -1$ between the $K(m)$-invariants of algebraically induced representations has the shape:

$$T_{\rm st}(s)|_{s=-1} : {}^{\rm a}{\rm Ind}_B^G({}^\iota\chi_1 \otimes {}^\iota\chi_2)^{K(m)} \longrightarrow {}^{\rm a}{\rm Ind}_B^G({}^\iota\chi_2(1) \otimes {}^\iota\chi_1(-1))^{K(m)}.$$

The standing assumptions that ${}^\iota\chi_i$ is half-integrally unitary, essentially tempered, and $\pi = {}^\iota\chi_1 \otimes {}^\iota\chi_2$ is on the right of the unitary axis with respect to G implies that $T := T_{\rm st}(s)|_{s=-1}$ is finite. A function in ${}^{\rm a}{\rm Ind}_B^G({}^\iota\chi_1 \otimes {}^\iota\chi_2)^{K(m)}$ is completely determined by its restriction to K. This gives us the following diagram:

$$\begin{array}{ccc} {}^{\rm a}{\rm Ind}_B^G({}^\iota\chi_1 \otimes {}^\iota\chi_2)^{K(m)} & \xrightarrow{\quad T \quad} & {}^{\rm a}{\rm Ind}_B^G({}^\iota\chi_2(1) \otimes {}^\iota\chi_1(-1))^{K(m)} \\ {\scriptstyle f \mapsto f|_K}\downarrow & & \downarrow{\scriptstyle f \mapsto f|_K} \\ {}^{\rm a}{\rm Ind}_{K\cap B}^K({}^\iota\chi_1^\circ \otimes {}^\iota\chi_2^\circ)^{K(m)} & \xrightarrow{\quad T^\circ \quad} & {}^{\rm a}{\rm Ind}_{K\cap B}^K({}^\iota\chi_2^\circ \otimes {}^\iota\chi_1^\circ)^{K(m)} \end{array}$$

Working with $K(m)$-invariants is not strictly necessary; it has the virtue of making the spaces finite-dimensional and G-action is replaced by action of the Hecke-algebra $\mathcal{C}_c^\infty(G/\!/K(m))$. Let $f^\circ \mapsto \tilde{f}^\circ$ denote the inverse of $f \mapsto f|_K$. Let $f \in {}^{\rm a}{\rm Ind}_B^G({}^\iota\chi_1 \otimes {}^\iota\chi_2)^{K(m)}$ and for brevity let $f^\circ = f|_K$. Since $T(f)$ is determined by its restriction to K, we have:

$$T^\circ(f^\circ)(k) = T(f)(k) = \int_F f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}k\right)dx, \quad k \in K.$$

Break up the integral over $x \in \mathcal{P}^{-m}$ and $x \notin \mathcal{P}^{-m}$. Note that

$$\int_{\mathcal{P}^{-m}} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}k\right)dx = \sum_{a\in\mathcal{P}^{-m}/\mathcal{P}^m}\int_{y\in\mathcal{P}^m} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & a+y \\ & 1 \end{pmatrix}k\right)dy.$$

We write

$$\begin{pmatrix} 1 & a+y \\ & 1 \end{pmatrix}k = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}k = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}k \cdot k^{-1}\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}k$$

and use that $K(m)$ is a normal subgroup of K and $\widetilde{f^\circ}$ is right $K(m)$-invariant to get

(5.4.1)

$$\int_{\mathcal{P}^{-m}} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}k\right)dx = {\rm vol}(\mathcal{P}^m)\sum_{a\in\mathcal{P}^{-m}/\mathcal{P}^m} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}k\right),$$

which is a finite-sum. For the integral over $x \notin \mathcal{P}^{-m}$ use:

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & \\ & x \end{pmatrix}\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix};$$

break up $\int_{x\notin\mathcal{P}^{-m}}$ as $\sum_{r=m}^\infty \int_{\varpi^{-r}\mathcal{O}^\times}$ to get

$$\begin{aligned} &\int_{x\notin\mathcal{P}^{-m}} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}k\right)dx \\ &= \sum_{r=m}^\infty \int_{\varpi^{-r}\mathcal{O}^\times} f\left(\begin{pmatrix} x^{-1} & \\ & x \end{pmatrix}\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}k\right)dx. \end{aligned}$$

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Since $x^{-1} \in \mathcal{P}^m$, using the equivariance of *f*, the right hand side simplifies to:

$$\sum_{r=m}^{\infty} \int_{\varpi^{-r}\mathcal{O}^\times} {}^\iota\chi_1(x^{-1})\,{}^\iota\chi_2(x) f(k)\, dx.$$

Make the substitution $x = \varpi^{-r}u$ with $u \in \mathcal{O}^\times$; then $dx = q^r du = q^r d^\times u$, and one gets:

$$f(k) \sum_{r=m}^{\infty} {}^\iota\chi_1(\varpi^r)\,{}^\iota\chi_2(\varpi^{-r}) q^r \int_{\mathcal{O}^\times} {}^\iota\chi_1(u^{-1})\,{}^\iota\chi_2(u)\, d^\times u.$$

The inner integral is nonzero if and only if ${}^\iota\chi_1(u) = {}^\iota\chi_2(u)$ for all $u \in \mathcal{O}^\times$; assuming this to be the case we get:

$$\begin{aligned} &\int_{x \notin \mathcal{P}^{-m}} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k\right) dx \\ (5.4.2) \qquad &= \mathrm{vol}(\mathcal{O}^\times) \cdot {}^\iota\chi_1(\varpi^m)\,{}^\iota\chi_2(\varpi^{-m}) q^m \cdot \left(1 - {}^\iota\chi_1(\varpi)\,{}^\iota\chi_2(\varpi^{-1}) q\right)^{-1} \cdot f(k). \end{aligned}$$

For $G = \mathrm{GL}(2)$, the point of evaluation $k = -1$, and $\left(1 - {}^\iota\chi_1(\varpi)\,{}^\iota\chi_2(\varpi^{-1}) q\right)^{-1}$ is nothing but $L(s, {}^\iota\chi_1 \otimes {}^\iota\chi_2, \tilde{r}) = L(s, {}^\iota\chi_1 \otimes {}^\iota\chi_2^{-1})$ evaluated at this point of evaluation. Putting (5.4.1) and (5.4.2) together, one sees that $T(f)(k)$ is a finite-sum:

$$\begin{aligned} &\mathrm{vol}(\mathcal{P}^m) \sum_{a \in \mathcal{P}^{-m}/\mathcal{P}^m} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} k\right) \\ (5.4.3) \qquad &+ \delta(\chi_1^\circ, \chi_2^\circ) \cdot \mathrm{vol}(\mathcal{O}^\times) \cdot {}^\iota\chi_1(\varpi^m)\,{}^\iota\chi_2(\varpi^{-m}) q^m \cdot L(-1, {}^\iota\chi_1 \otimes {}^\iota\chi_2^{-1}) \cdot f(k). \end{aligned}$$

For brevity, let $\mathfrak{I} = {}^{\mathrm{a}}\mathrm{Ind}_B^G({}^\iota\chi_1 \otimes {}^\iota\chi_2)^{K(m)}$ and $\tilde{\mathfrak{I}} = {}^{\mathrm{a}}\mathrm{Ind}_B^G({}^\iota\chi_2(1) \otimes {}^\iota\chi_1(-1))^{K(m)}$; these induced representations admit a natural E-structures; define

$$\mathfrak{I}_0 := {}^{\mathrm{a}}\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)^{K(m)}, \quad \tilde{\mathfrak{I}}_0 := {}^{\mathrm{a}}\mathrm{Ind}_B^G(\chi_2(1) \otimes \chi_1(-1))^{K(m)}.$$

If $\chi : F^\times \to E^\times$ is a locally constant homomorphism then for any integer n, we denote $\chi(n) = \chi \otimes |\ |^n$ the E-valued character: $u \mapsto \chi(u)$ for all $u \in \mathcal{O}^\times$ and $\varpi \mapsto q^{-n}\chi(\varpi)$. It is clear then that $\mathfrak{I} = \mathfrak{I}_0 \otimes_{E,\iota} \mathbb{C}$ and $\tilde{\mathfrak{I}} = \tilde{\mathfrak{I}}_0 \otimes_{E,\iota} \mathbb{C}$. Note that $\mathfrak{I}_0$ consists of all E-valued functions in $\mathfrak{I}$; similarly, $\tilde{\mathfrak{I}}_0$. The local L-value that appears in (5.4.3) is E-rational, i.e., $L(-1, {}^\iota\chi_1 \otimes {}^\iota\chi_2^{-1}) = \left(1 - {}^\iota\chi_1(\varpi)\,{}^\iota\chi_2(\varpi^{-1}) q\right)^{-1} \in \iota(E)$, and furthermore, if $L_0(-1, \chi_1 \otimes \chi_2^{-1}) = \left(1 - \chi_1(\varpi)\chi_2(\varpi^{-1}) q\right)^{-1} \in E$ then $\iota(L_0(-1, \chi_1 \otimes \chi_2^{-1})) = L(-1, {}^\iota\chi_1 \otimes {}^\iota\chi_2^{-1})$. It is clear now from (5.4.3) that $T(\mathfrak{I}_0) \subset \tilde{\mathfrak{I}}_0$; also that if $T_0 = T|_{\mathfrak{I}_0}$ then $T = T_0 \otimes_{E,\iota} \mathbb{C}$.

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