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Global and non Global Solutions for Some Fractional Heat Equations With Pure Power Nonlinearity

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Abstract. The initial value problem for a semi-linear fractional heat equation is investigated. In the focusing case, global well-posedness and exponential decay are obtained. In the focusing sign, global and non global existence of solutions are discussed via the potential well method.

1 Introduction

Consider the Cauchy problem for a fractional nonlinear heat equation

$$\begin{cases} \dot{u} + (-\Delta)^{\alpha} u + cu = \epsilon |u|^{p-1} u; \\ u_{|t=0} = u_0, \end{cases} \tag{1.1}$$

which is a model of the so-called anomalous diffusion, a much-studied topic in physics, probability and finance. See [1, 13, 17, 20] and the references therein.

Henceforth, $N \geq 2$, $\alpha \in (0,1)$, $\epsilon = \pm 1$, the constant $c \in \{0,1\}$, and u is a real valued function of the variable $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$. The fractional Laplacian operator stands for $(-\Delta)^{\alpha} u := \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F} u)$.

The energy space $C([0,T], H^{\alpha}(\mathbb{R}^N))$ is naturally adapted to study the fractional heat problem (1.1) using, with minimal regularity, the following energy identity:

$$\begin{aligned} \partial_t E^c(t) &:= \partial_t E^c(u(t)) \\ &:= \partial_t \Big[\int_{\mathbb{R}^N} \Big(\frac{1}{2} |(-\Delta)^{\frac{\alpha}{2}} u(t)|^2 + \frac{c}{2} |u(t)|^2 - \frac{\epsilon}{1+p} |u(t)|^{1+p} \Big) \, dx \Big] \\ &= - \int_{\mathbb{R}^N} |\dot{u}(t,x)|^2 \, dx. \end{aligned}$$

If $\epsilon = -1$, the energy is positive and (1.1) is said to be defocusing. For $\epsilon = 1$, the energy no longer allows a control of the H^{α} norm of an eventual solution. In such a case, (1.1) is focusing.

In the classical case $\alpha = 1$, the equation (1.1) has been extensively studied in the scale of Lebesgue spaces $L^q(\mathbb{R}^N)$. The critical index $q_c := \frac{N(p-1)}{2}$ gives the following three different regimes.

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Case 1. *Subcritical case, $q > q_c \geq 1$*: Weissler [18] proved local well-posedness in $C([0,T); L^q(\mathbb{R}^N)) \cap L^\infty_{\mathrm{loc}}(]0,T]; L^\infty(\mathbb{R}^N))$. Then Brezis–Cazenave [4] showed unconditional uniqueness.

Case 2. *Critical case, $q = q_c$*: There are two cases.
2.1 $q_c > p+1$: local well-posedness holds [4, 18];

2.2 $q = q_c = p+1$: Weissler [19] proved a conditional well-posedness.

Case 3. *Supercritical case, $q < q_c$*: There is no solution in any reasonable weak sense [4, 18, 19]. Moreover, uniqueness is lost [8] for the initial data $u_0 = 0$ and $1 + \frac{1}{N} < p < \frac{N+2}{N-2}$.

See [9] for exponential type nonlinearity in two space dimensions.

This paper seems to be one of few works treating well-posedness issues of the semilinear fractional heat equation in the energy space [21]. The purpose of this paper is two-fold. First, global well posedness and exponential decay are established in the defocusing case. Second, in the focusing sign, the existence of global and non global solutions is discussed via a potential-well method. Compared with the classical case, we need to operate with various modifications due to the non-locality of the fractional Laplacian.

The rest of the paper is organized as follows. The second section is devoted to giving the main results and some tools needed in the sequel. Section three deals with local well posedness of (1.1). Section four contains a proof of the global existence of solutions and scattering in the critical case with small data. The fifth section deals with the associated stationary problem. Section six is about global existence of solutions with data in some stable sets in the spirit of Payne and Sattinger [15]. In the last section, the existence of infinitely many non global solutions near the ground state is proved.

We mention that C will be used to denote a constant that may vary from line to line; $A \lesssim B$ means that $A \leq CB$ for some absolute constant C. For simplicity, let $\int(\cdot)\,dx := \int_{\mathbb{R}^N}(\cdot)\,dx$, let $L^p := L^p(\mathbb{R}^N)$ be the Lebesgue space endowed with the norm $\|\cdot\|_p := \|\cdot\|_{L^p}$ and let $\|\cdot\| := \|\cdot\|_2$. The classical fractional Sobolev space is $H^{\alpha,p} := (I-\Delta)^{-\frac{\alpha}{2}}L^p$, and $H^\alpha := H^{\alpha,2}$ is the energy space. Using the Plancherel Theorem, the following norms are equivalent

$$\|u\|_{H^\alpha} := \Big(\int_{\mathbb{R}^N}(1+|\xi|^2)^\alpha |\hat{u}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \simeq \Big(\|u\|^2 + \|(-\Delta)^{\frac{\alpha}{2}}u\|^2\Big)^{\frac{1}{2}}.$$

We denote the real numbers $p_* := 1 + \frac{4\alpha}{N}$, $p^* := p_c - 1 := \frac{N+2\alpha}{N-2\alpha}$, and we assume henceforth that

$$c = 1 - \delta_p^{p^*} = \begin{cases} 0 & \text{if } p = p^*, \\ 1 & \text{if } p \neq p^*. \end{cases}$$

Finally, if $T > 0$ and X is an abstract functional space, we let

$$C_T(X) := C([0,T],X), \quad L^p_T(X) := L^p([0,T],X),$$

and let X_{rd} be the set of radial elements in X. Moreover, for an eventual solution to (1.1), we denote by $T^* > 0$ its lifespan.

2 Background and Main Results

In this section we give the main results and some technical tools needed in the sequel. Let us introduce some quantities to be used in this note. If a, b, and λ are three real numbers and $\phi \in H^\alpha$, we define the scaling $\phi^\lambda_{a,b} := \lambda^a \phi(\frac{\cdot}{\lambda^b})$, the so-called constraint $K^c_{a,b}(\phi) := \mathcal{L}_{a,b}E^c(\phi) := \partial_\lambda(E^c(\phi^\lambda_{a,b}))_{|\lambda=1}$, and the operator $H^c_{a,b} := E^c - \frac{1}{2a+Nb}K^c_{a,b}$. For ease of notation, we set $E := E^1$, $K_{a,b} := K^1_{a,b}$, and $H_{a,b} := H^1_{a,b}$.

2.1 Main Results

Results proved in this paper are listed in what follows. First, we deal with global well-posedness of the heat problem (1.1) in the energy space.

Theorem 2.1 *Let $N \geq 2$, $\alpha \in (0,1)$, $1 < p \leq p^*$, and $u_0 \in H^\alpha$. Then there exists a unique maximal solution to (1.1), $u \in C([0, T^*), H^\alpha)$. Moreover,*

(i) $u \in L^{1+p}_{\text{loc}}([0, T^*), L^{\frac{2N(1+p)}{2(N-2\alpha)+N(p-1)}})$;

(ii) $E(t) = E(0) - \int_0^t \int_{\mathbb{R}^N} |\dot{u}(s,x)|^2 \, dsdx$, *for any* $t \in [0, T^*)$;

(iii) *if* $c = 1$ *and* $\epsilon = -1$, *then* $T^* = \infty$ *and there exists* $\gamma > 0$ *such that* $\|u(t)\|_{H^\alpha} = O(e^{-\gamma t})$,*when* $t \to \infty$.

Remarks 2.2 • Local well-posedness for $c = 1$ was proved in a different way [21].
• With a classical time translation argument, any local solution to (1.1) in the energy space is equal to the maximal one (see [3] for uniqueness of free classical solutions).

In the critical case, for small data there exists a global solution to (1.1) which is asymptotic, as $t \to +\infty$, to a solution of the linear equation $\dot{v} + (-\Delta)^\alpha v = 0$. In other words, the effect of the nonlinearity is negligible for large times.

Theorem 2.3 *Let $N \geq 2$, $\alpha \in (0,1)$, and $c = 0 (p = p^*)$. Then there exists $\epsilon_0 > 0$ such that if $u_0 \in \dot{H}^\alpha$ satisfies $\|u_0\|_{\dot{H}^\alpha} \leq \epsilon_0$, the problem (1.1) possesses a unique global solution $u \in C(\mathbb{R}_+, \dot{H}^\alpha)$. Moreover, there exists $u_+ \in \dot{H}^\alpha$ such that*

$$\lim_{t\to\infty} \|u(t) - e^{-t(-\Delta)^\alpha} u_+\|_{\dot{H}^\alpha} = 0.$$

Second, we are interested in the focusing case. Using the potential well method due to Payne–Sattinger [15], we discuss global and non global existence of solutions to (1.1) when the data belongs to some stable sets. Here we are reduced to using the fact that the fractional elliptic problem $(-\Delta)^\alpha\phi + c\phi - |\phi|^{p-1}\phi = 0$, $0 \neq \phi \in H^\alpha_{rd}$ has a ground state in the sense that it has a nontrivial radial solution that minimizes the problem $m^c_{a,b} := \inf_{0\neq\phi\in H^\alpha}\{E^c(\phi) \mid K^c_{a,b}(\phi) = 0\}$. For ease of notation, we set $m_{a,b} := m^1_{a,b}$. The existence of the ground state in the subcritical case was partially known [16]. We extend this result as follows.

Proposition 2.4 *Take $N \geq 2$, $\alpha \in (0,1)$, $p_* < p \leq p^*$, and a pair of real numbers $(a,b) \in \mathbb{R}^*_+ \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\}$. Then*

(i) $m^c := m^c_{a,b}$ *is nonzero and independent of* (a,b);

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(ii) *there is a ground state solution to* (1.1) *in the sense that*

$$(-\Delta)^\alpha\phi + c\phi - |\phi|^{p-1}\phi = 0, \quad 0 \not\equiv \phi \in H^\alpha_{rd}, \quad \text{and} \quad m^c = E^c(\phi). \tag{2.1}$$

Remark 2.5 The previous result was proved in [16] for $p_* < p < p^*$ and $a, b \geq 0$.

Define the following spaces:

$$A^{c,+}_{a,b} := \{\phi \in H^\alpha \mid E^c(\phi) < m^c_{a,b} \text{ and } K^c_{a,b}(\phi) \geq 0\};$$
$$A^{c,-}_{a,b} := \{\phi \in H^\alpha \mid E(\phi) < m^c_{a,b} \text{ and } K^c_{a,b}(\phi) < 0\};$$
$$A^+_{a,b} := A^{1,+}_{a,b}, \quad A^-_{a,b} := A^{1,-}_{a,b}.$$

Let us discuss the existence of global and non global solutions to the heat problem (1.1).

Theorem 2.6 *Take $N \geq 2$, $\alpha \in (0,1)$, $\epsilon = 1$, $(a,b) \in \mathbb{R}^*_+ \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\}$, $p_* < p \leq p^*$, and let $u \in C([0,T^*), H^\alpha)$ be the maximal solution to* (1.1).

(i) *If $c = 1$ and $u_0 \in A^+_{a,b}$, then $T^* = \infty$ and $u(t) \in A^+_{a,b}$ for any time $t \geq 0$. Moreover for small $\|u_0\|$, there exists $\gamma > 0$ such that $\|u(t)\|_{\dot H^\alpha} = O(e^{-\gamma t})$,when $t \to \infty$.*
(ii) *If $u_0 \in A^{c,-}_{a,b}$, then u blows-up in finite time.*

The final result concerns instability by blow-up for stationary solutions to the heat problem (1.1). Indeed, near ground state, there exist infinitely many data giving non global solutions to (1.1).

Theorem 2.7 *Take $N \geq 2$, $\alpha \in (0,1)$, $\epsilon = 1$, and $p_* < p < p^*$. Let ϕ be a ground state solution to* (2.1). *Then for any $\varepsilon > 0$, there exists $u_0 \in H^\alpha$ such that $\|u_0 - \phi\|_{H^\alpha} < \varepsilon$ and the maximal solution to* (1.1) *with data u_0 is not global $(T^* < \infty)$.*

2.2 Tools

Let us collect some classical estimates needed later in this manuscript. We start with some technical results about the fractional heat equation. Some useful properties of the free fractional heat kernel are gathered in what follows.

Proposition 2.8 *Denoting the free operator associated with the fractional heat equation $T_\alpha(t)\phi := e^{-t(-\Delta)^\alpha}\phi := \mathcal{F}^{-1}(e^{-t|y|^{2\alpha}}) * \phi := K_\alpha(t) * \phi$, yields*

(i) *$T_\alpha(t)u_0$ is the solution to the linear problem associated with* (1.1)*;*
(ii) *$T_\alpha(t)u_0 - \epsilon\int_0^t T_\alpha(t-s)|u|^{p-1}u\,ds$ is the solution to the problem* (1.1)*;*
(iii) *$T_\alpha T_\beta = T_{\alpha+\beta}$, $T^*_\alpha = T_\alpha$.*

Let us recall the so-called Strichartz estimate [22].

Definition 2.9 A pair of real numbers (q,r) is said to be *admissible* if

$$q, r \geq 2 \quad \text{and} \quad \frac{2\alpha}{q} = N\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

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Proposition 2.10 *Let $N \geq 2$, $\alpha \in (0,1)$, $u_0 \in L^2$, and let (q,r), $(\tilde{q},\tilde{r})$ be two admissible pairs. Then there exists $C := C_{q,\tilde{q}}$ such that*

$$\|u\|_{L^q_t(L^r)} \leq C\Big(\|u_0\| + \|\dot{u} + (-\Delta)^\alpha u\|_{L^{\tilde{q}'}_t(L^{\tilde{r}'})}\Big).$$

Proof Write

$$\begin{aligned}(K_\alpha(t))(x) &= \mathcal{F}^{-1}(e^{-t|\cdot|^{2\alpha}})(x)\\ &= \frac{1}{t^{\frac{N}{2\alpha}}}\mathcal{F}^{-1}(e^{-|\cdot|^{2\alpha}})\Big(\frac{x}{t^{\frac{1}{2\alpha}}}\Big)\\ &= \frac{1}{t^{\frac{N}{2\alpha}}}K\Big(\frac{x}{t^{\frac{1}{2\alpha}}}\Big),\end{aligned}$$

where $K \in (L^1 \cap L^\infty)(\mathbb{R}^N)$; see [6]. Thus,

$$\|T_\alpha(t)\phi\| \lesssim \|\phi\|, \quad \|T_\alpha(t)T^*_\alpha(s)\phi\|_\infty \lesssim \frac{1}{|t-s|^{\frac{N}{2\alpha}}}\|\phi\|_1.$$

The proof is finished via [10, Theorem 1.2]. ∎

The existence of a ground state in the subcritical case is known [16].

Proposition 2.11 *Take a pair of real numbers $(a,b) \in \mathbb{R}^*_+ \times \mathbb{R}_+$ and $p_* < p < p^*$. Then*

(i) *$m := m_{a,b}$ is nonzero and independent of (a,b);*
(ii) *there is a ground state solution to (1.1) in the sense that*

$$(-\Delta)^\alpha\phi + \phi - |\phi|^{p-1}\phi = 0, \quad 0 \neq \phi \in H^\alpha_{rd}, \quad \textit{and} \quad m = E(\phi).$$

Now we list some general estimates about fractional derivative calculus. The next fractional chain rule (see [5, Proposition 3.1]) will be useful.

Lemma 2.12 *Let $G \in C^1(\mathbb{C})$, $\alpha \in (0,1]$, and let $1 < p, p_1, p_2 < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < \infty$. Then $\|(-\Delta)^{\frac{\alpha}{2}}G(u)\|_p \lesssim \|G'(u)\|_{p_1}\|(-\Delta)^{\frac{\alpha}{2}}u\|_{p_2}$.*

The following fractional Gagliardo–Nirenberg inequality [7, Corollary 1.5] holds.

Lemma 2.13 *Let $1 < p, p_1, p_2 < \infty$, $s, s_1 \in \mathbb{R}$, and $\mu \in [0,1]$. Then the fractional inequality $\|u\|_{\dot{H}^{s,p}} \lesssim \|u\|^{1-\mu}_{L^{p_0}}\|u\|^\mu_{\dot{H}^{s_1,p_1}}$ holds whenever*

$$\frac{N}{p} - s = (1-\mu)\frac{N}{p_0} + \mu\Big(\frac{N}{p_1} - s_1\Big) \quad \textit{and} \quad s \leq \mu s_1.$$

Corollary 2.14 *Let $2 \leq p \leq \frac{2N}{N-2\alpha}$ and $\mu = \frac{N}{\alpha}(\frac{1}{2} - \frac{1}{p})$. Then $\|u\|_p \lesssim \|u\|^{1-\mu}\|u\|^\mu_{\dot{H}^\alpha}$.*

The following Sobolev injections [2,12] give a meaning to the energy and several computations done in this note.

Lemma 2.15 *Let $N \geq 2$, $\alpha \in (0,1)$, and $p \in (1,\infty)$. Then*
(i) *$W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ whenever $1 < p < q < \infty$, $s > 0$, and $\frac{1}{p} \leq \frac{1}{q} + \frac{\alpha}{N}$;*

(ii) $H^\alpha(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *for any* $q \in [2, \frac{2N}{N-2\alpha}]$;
(iii) $H^\alpha_{rd}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ *for any* $q \in (2, \frac{2N}{N-2\alpha})$.

In the critical case, we recall some properties of the best constant [14].

Proposition 2.16 *Take* $\alpha \in (0,1)$, $N \geq 2$. *Then*

$$C^*_{N,\alpha} := \inf_{0 \neq u \in \dot{H}^\alpha} \frac{\|u\|^{p_c}_{p_c}}{\|(-\Delta)^{\frac{\alpha}{2}} u\|^2} = \frac{1}{2^{2\alpha}\pi^\alpha} \frac{\Gamma(\frac{N}{2} - \alpha)}{\Gamma(\frac{N}{2} + \alpha)} \frac{\Gamma(N)^{\frac{2\alpha}{N}}}{\Gamma(\frac{N}{2})^{\frac{2\alpha}{N}}}.$$

Moreover, u is such a minimizer if and only if there exist $c \in \mathbb{R}$, $\mu > 0$, and $x_0 \in \mathbb{R}^N$ such that $u(x) = c(\mu^2 + |x - x_0|^2)^{-\frac{N-2\alpha}{2}}$.

Let us give an abstract result.

Lemma 2.17 *Let $T > 0$ and $X \in C([0,T], \mathbb{R}_+)$ such that $X \leq a + bX^\theta$ on $[0,T]$, where a, $b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta})(\theta b)^{\frac{-1}{\theta}}$, and $X(0) \leq (\theta b)^{\frac{-1}{\theta-1}}$. Then $X \leq \frac{\theta}{\theta-1} a$ on $[0,T]$.*

Proof The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (b\theta)^{\frac{1}{1-\theta}}]$ and increasing on $[(b\theta)^{\frac{1}{1-\theta}}, \infty)$. The assumptions imply that $f((b\theta)^{\frac{1}{1-\theta}}) < 0$ and $f(\frac{\theta}{\theta-1} a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$, and $X(0) \leq (b\theta)^{\frac{1}{1-\theta}}$, we conclude the proof by a continuity argument. ∎

We close this subsection with a classical result about ordinary differential equations.

Proposition 2.18 *Let $\varepsilon > 0$. There is no real function $G \in C^2(\mathbb{R}_+)$ satisfying $G(0) > 0$, $G'(0) > 0$, and $GG'' - (1+\varepsilon)(G')^2 \geq 0$ on $\mathbb{R}_+$.*

Proof Assume by way of contradiction, the existence of such a function. Then

$$(G^{-(1+\varepsilon)}G')' \geq 0 \quad \text{and} \quad \frac{G'}{G^{1+\varepsilon}} \geq \frac{G'(0)}{G^{1+\varepsilon}(0)} > 0.$$

Integrating the previous inequality on $(0,T)$ yields

$$0 < \frac{1}{G^\varepsilon(T)} \leq \frac{1}{G^\varepsilon(0)} - \varepsilon \frac{G'(0)}{G^{1+\varepsilon}(0)} T,$$

which implies that $T < \frac{1}{\varepsilon} \frac{G(0)}{G'(0)}$. This contradiction achieves the proof. ∎

3 Local Well Posedness

In this section, we prove Theorem 2.1 about the existence of a solution to (1.1) in the energy space.

3.1 Local Existence and Uniqueness

Let the admissible pair (q,r) be defined as follows

$$r := 1+p, \quad q := \frac{4\alpha(1+p)}{N(p-1)}, \quad \text{and} \quad \theta := \frac{q(p-1)}{q-2}.$$

We use a standard fixed point argument and discuss two cases.

Subcritical case: $1 < p < p^*(c=1)$. For $T,\rho > 0$, denote the space

$$E_{T,\rho} := \left\{ u \in C_T(H^\alpha) \cap L^q_T(W^{\alpha,r}) \;\middle|\; \|u\|_{L^\infty_T(H^\alpha)\cap L^q_T(W^{\alpha,r})} \leq \rho \right\}$$

endowed with the complete distance $d(u,v) := \|u-v\|_{L^\infty_T(L^2)\cap L^q_T(L^r)}$. Define the function $\phi(u)(t) := T_\alpha(t)u_0 + \int_0^t T_\alpha(t-s)[\epsilon|u|^{p-1}u - cu]\,ds$. We prove the existence of some small $T,\rho > 0$ such that ϕ is a contraction of $E_{T,\rho}$. Take $u,v \in E_{T,\rho}$ and $w := u-v$. Using Strichartz and Hölder inequalities via the equality $\frac{1}{q'} = \frac{1}{q} + \frac{p-1}{\theta}$, we obtain

$$\begin{aligned}
d(\phi(u),\phi(v)) &\lesssim \| |u|^{p-1}u - |v|^{p-1}v\|_{L^{q'}_T(L^{r'})} + \|u-v\|_{L^1_T(L^2)} \\
&\lesssim \|w(|u|^{p-1} + |v|^{p-1})\|_{L^{q'}_T(L^{r'})} + T\|u-v\|_{L^\infty_T(L^2)} \\
&\lesssim \|w\|_{L^q_T(L^r)}\Big(\|u\|^{p-1}_{L^\theta_T(L^r)} + \|v\|^{p-1}_{L^\theta_T(L^r)}\Big) + Td(u,v) \\
&\lesssim T^{\frac{1}{\theta}}\|w\|_{L^q_T(L^r)}\Big(\|u\|^{p-1}_{L^\infty_T(H^\alpha)} + \|v\|^{p-1}_{L^\infty_T(H^\alpha)}\Big) + Td(u,v) \\
&\lesssim \Big(\rho^{p-1}T^{\frac{1}{\theta}} + T\Big)\, d(u,v).
\end{aligned} \tag{3.1}$$

On the other hand, thanks to Lemma 2.12,

$$\begin{aligned}
\|\phi(u)\|_{L^\infty_T(H^\alpha)\cap L^q_T(W^{\alpha,r})} &\lesssim \|u_0\|_{H^\alpha} + \| |u|^{p-1}u\|_{L^{q'}_T(H^{\alpha,r'})} + \|u\|_{L^1_T(H^\alpha)} \\
&\lesssim \|u_0\|_{H^\alpha} + \|(1+(-\Delta)^{\frac{\alpha}{2}})(|u|^{p-1}u)\|_{L^{q'}_T(L^{r'})} + T\|u\|_{L^\infty_T(H^\alpha)} \\
&\lesssim \|u_0\|_{H^\alpha} + \|u\|^{p-1}_{L^\theta_T(L^r)}\Big(\|u\|_{L^q_T(L^r)} + \|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^q_T(L^r)}\Big) + T\rho \\
&\lesssim \|u_0\|_{H^\alpha} + T^{\frac{1}{\theta}}\|u\|^{p-1}_{L^\infty_T(H^\alpha)}\|u\|_{L^q_T(H^{\alpha,r})} + T\rho \\
&\lesssim \|u_0\|_{H^\alpha} + T^{\frac{1}{\theta}}\rho^p + T\rho.
\end{aligned}$$

This implies that for $\rho := 2C\|u_0\|_{H^\alpha}$ (C given by the Strichartz estimate) and small $T > 0$, ϕ is a contraction of $E_{T,\rho}$. With a Picard fixed-point theorem, ϕ has a fixed point which is a local solution to (1.1). Moreover, uniqueness of such a solution is a direct consequence of (3.1) with a standard translation argument.

Critical case: $p = p^*(c=0)$. The proof follows like the subcritical case, where, rather than $E_{T,\rho}$, we take the complete space $F_{T,\rho} := \{u \in L^q_T(W^{\alpha,r}) \mid \|u\|_{L^q_T(W^{\alpha,r})} \leq \rho\}$ endowed with the complete distance $d(u,v) = \|u-v\|_{L^q_T(L^r)}$, via the fact that

$$\lim_{T\longrightarrow 0} \|T_\alpha(t)u_0\|_{L^q_T(L^r)} = 0$$

and the following lemma.

Lemma 3.1 *Let $u_0 \in H^\alpha$ and suppose $u \in L^q_T(W^{\alpha,r})$ is a solution of (1.1). Then there exists $0 < T' \le T$ such that $u \in C_{T'}(H^\alpha)$.*

Proof Using the previous computation via Duhamel's formula (Proposition 2.8 (ii)), yields $\|u\|_{L^\infty_T(H^\alpha)} \lesssim \|u_0\|_{H^\alpha} + \|u\|^{p-1}_{L^\infty_T(H^\alpha)}\|u\|_{L^q_T(H^{\alpha,r})}$. The proof is complete thanks to Lemma 2.17.

3.2 Global Existence in the Subcritical Defocusing Case

The global existence is a consequence of the energy decay and previous calculations. Let $u \in C([0,T^*),H^\alpha)$ be the unique maximal solution of (1.1). We prove that u is global. By contradiction, suppose that $T^* < \infty$. Consider for $0 < s < T^*$ the problem

$$(\mathcal{P}_s)\begin{cases} \dot v + (-\Delta)^\alpha v + v + |v|^{p-1}v = 0,\\ v(s,\cdot) = u(s,\cdot).\end{cases}$$

Using the same arguments of local existence, we can find a real $\tau > 0$ and a solution v to $(\mathcal{P}_s)$ on $C([s,s+\tau],H^\alpha)$. Thanks to the energy decay, we see that τ does not depend on s. Thus, if we let s be close to T^* such that $T^* < s+\tau$, this fact contradicts the maximality of T^*.

3.3 Exponential Decay

This subsection is devoted to proving that the global solution $u \in C(\mathbb{R}_+,H^\alpha)$ to (1.1) for $c = -\epsilon = 1$ and $1 < p < p^*$ satisfies an exponential decay in the energy space. Denoting the quantity $K(u(t)) := \|u(t)\|^2_{H^\alpha} + \int_{\mathbb{R}^N}|u(t)|^{1+p}\,dx$, yields

$$E(u(t)) \le K(u(t)) \le (p+1)E(u(t)).$$

On the other hand, for $T > 0$,

$$\int_t^T K(u(s))\,ds = \frac{1}{2}(\|u(t)\|^2 - \|u(T)\|^2) \le \frac{1}{2}\|u(t)\|^2 \le E(u(t)).$$

So $\int_t^T E(u(s))\,ds \lesssim \int_t^T K(u(s))\,ds \lesssim E(u(t))$. Thus, for some positive real number $T_0 > 0$,

$$y(t) := \int_t^\infty E(u(s))\,ds \lesssim E(u(t)) \le -T_0 y'(t).$$

This implies that, for $t \ge T_0$, $y(t) \le y(T_0)e^{1-\frac{t}{T_0}} \le T_0E(u(T_0))e^{1-\frac{t}{T_0}}$. Taking account the monotonicity of the energy for large $T > 0$,

$$\int_t^T E(u(s))\,ds \ge \int_t^{t+T_0} E(u(s))\,ds \ge T_0E(u(t+T_0)).$$

Then $E(u(t+T_0)) \le E(u(T_0))e^{1-\frac{t}{T_0}}$. Finally,

$$\|u(t+T_0)\|^2_{H^\alpha} \lesssim E(u(t+T_0)) \le E(u(T_0))e^{1-\frac{t}{T_0}}.$$

The proof is finished.

4 Global Existence and Scattering in the Critical Case

In this section we establish global existence of a solution to (1.1) in the critical case $p = p^*$ for small data, as claimed in Theorem 2.3. Several norms have to be considered in the analysis of the critical case. Letting $I \subset \mathbb{R}$ be a time slab, we define

$$W(I) := L^{\frac{2(N+2\alpha)}{N-2\alpha}}(I, L^{\frac{2N(N+2\alpha)}{N^2+4\alpha^2}}),$$
$$M(I) := L^{\frac{2(N+2\alpha)}{N-2\alpha}}(I, \dot{W}^{\alpha, \frac{2N(N+2\alpha)}{N^2+4\alpha^2}}) \cap C(I, \dot{H}^\alpha),$$
$$S(I) := L^{\frac{2(N+2\alpha)}{N-2\alpha}}(I, L^{\frac{2(N+2\alpha)}{N-2\alpha}}).$$

Remark 4.1 The continuous Sobolev embedding $M(I) \hookrightarrow S(I)$ is a direct consequence of Lemma 2.15.

Let us give an auxiliary result.

Proposition 4.2 *Take $p = p^*$ and $u_0 \in \dot{H}^\alpha$. There exists $\delta := \delta(A := \|u_0\|_{\dot{H}^\alpha}) > 0$ such that for any interval $I = [0, T)$, if $\|e^{-t(-\Delta)^\alpha} u_0\|_{S(I)} < \delta$, there exits a unique solution $u \in C(I, \dot{H}^\alpha)$ of (1.1), which satisfies $u \in M(I)$. Moreover, $\|u\|_{S(I)} \leq 2\delta$.*

Proof The proposition follows with a contraction mapping argument. We let the function $\phi(u)(t) := T_\alpha(t)u_0 - \int_0^t T_\alpha(t-s)|u|^{\frac{4\alpha}{N-2\alpha}} u\, ds$. Define the set

$$X_a := \left\{ u \in M(I) \mid \|u\|_{M(I)} \leq a \right\},$$

where $a > 0$ is sufficiently small to fix later. Using the Strichartz estimate, we get

$$\|\phi(u) - \phi(v)\|_{W(I)} \lesssim \||u|^{\frac{4\alpha}{N-2\alpha}} u - |v|^{\frac{4\alpha}{N-2\alpha}} v\|_{L_T^2(L^{\frac{2N}{N+2\alpha}})} := (\mathcal{I}).$$

Thanks to the Hölder inequality and the Sobolev embedding, this yields

$$\begin{aligned}
(\mathcal{I}) &\lesssim \||u-v|(|u|^{\frac{4\alpha}{N-2\alpha}} + |v|^{\frac{4\alpha}{N-2\alpha}})\|_{L_T^2(L^{\frac{2N}{N+2\alpha}})} \\
&\lesssim \|u-v\|_{L_T^{\frac{2(N+2\alpha)}{N-2\alpha}}(L^{\frac{2N(N+2\alpha)}{N^2+4\alpha^2}})} \Big(\|u\|^{\frac{4\alpha}{N-2\alpha}}_{L_T^{\frac{2(N+2\alpha)}{N-2\alpha}}(L^{\frac{2(N+2\alpha)}{N-2\alpha}})} + \|v\|^{\frac{4\alpha}{N-2\alpha}}_{L_T^{\frac{2(N+2\alpha)}{N-2\alpha}}(L^{\frac{2(N+2\alpha)}{N-2\alpha}})} \Big) \\
&\lesssim \|u-v\|_{W(I)} \big(\|u\|^{\frac{4\alpha}{N-2\alpha}}_{S(I)} + \|v\|^{\frac{4\alpha}{N-2\alpha}}_{S(I)} \big) \\
&\lesssim a^{\frac{4\alpha}{N-2\alpha}} \|u-v\|_{W(I)}.
\end{aligned}$$

Then $\|\phi(u) - \phi(v)\|_{W(I)} \lesssim a^{\frac{4\alpha}{N-2\alpha}} \|u-v\|_{W(I)}$. Using the fractional chain rule via the Strichartz estimate and the Hölder inequality, yields

$$\begin{aligned}
\|\phi(u)\|_{M(I)} &\lesssim \|u_0\|_{\dot{H}^\alpha} + \|(-\Delta)^{\frac{\alpha}{2}}(|u|^{\frac{4\alpha}{N-2\alpha}} u)\|_{L_T^2(L^{\frac{2N}{N+2\alpha}})} \\
&\lesssim \|u_0\|_{\dot{H}^\alpha} + \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L_T^{\frac{2(N+2\alpha)}{N-2\alpha}}(L^{\frac{2N(N+2\alpha)}{N^2+4\alpha^2}})} \|u\|^{\frac{4\alpha}{N-2\alpha}}_{L_T^{\frac{2(N+2\alpha)}{N-2\alpha}}(L^{\frac{2(N+2\alpha)}{N-2\alpha}})} \\
&\lesssim \|u_0\|_{\dot{H}^\alpha} + \|u\|_{M(I)} \|u\|^{\frac{4\alpha}{N-2\alpha}}_{S(I)} \\
&\lesssim A + a^{1+\frac{4\alpha}{N-2\alpha}}.
\end{aligned}$$

With a classical Picard argument, for small $0 < a \ll A$, there exists $u \in X_a$, a solution to (1.1). Moreover, arguing as previously,

$$\begin{aligned}\|\phi(u)\|_{S(I)} &\leq \|e^{-t(-\Delta)^{\alpha}}u_0\|_{S(I)} + C\|(-\Delta)^{\frac{\alpha}{2}}(|u|^{\frac{4\alpha}{N-2\alpha}}u)\|_{L^2_T(L^{\frac{2N}{N+2\alpha}})}\\ &\leq \delta + C\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L_T^{\frac{2(N+2\alpha)}{N-2\alpha}}(L^{\frac{2N(N+2\alpha)}{N^2+4\alpha^2}})}\|u\|^{\frac{4\alpha}{N-2\alpha}}_{L_T^{\frac{2(N+2\alpha)}{N-2\alpha}}(L^{\frac{2(N+2\alpha)}{N-2\alpha}})}\\ &\leq \delta + C\|u\|_{M(I)}\|u\|^{\frac{4\alpha}{N-2\alpha}}_{S(I)}\\ &\leq \delta + Ca^{1+\frac{4\alpha}{N-2\alpha}}.\end{aligned}$$

Then for small $a > 0$, $\|u\|_{S(I)} \leq 2\delta$. ∎

Proof of Theorem 2.3 We start by proving global well posedness. Using the previous proposition via the fact that

$$\|e^{-t(-\Delta)^{\alpha}}u_0\|_{S(I)} \lesssim \|e^{-t(-\Delta)^{\alpha}}u_0\|_{M(I)} \lesssim \|u_0\|_{\dot{H}^{\alpha}},$$

it suffices to prove that $\|u(t)\|_{\dot{H}^{\alpha}}$ remains small on the whole interval of existence of u. Write, using the decay of the energy,

$$\begin{aligned}\|u(t)\|^2_{\dot{H}^{\alpha}} &= 2E(u(t)) + \frac{1}{p_c}\int_{\mathbb{R}^N}|u(t,x)|^{p_c}\,dx\\ &\leq 2E(u_0) + \frac{1}{p_c}\int_{\mathbb{R}^N}|u(t,x)|^{p_c}\,dx\\ &\lesssim \big(\|u_0\|^2_{\dot{H}^{\alpha}} + \|u_0\|^{p_c}_{\dot{H}^{\alpha}}\big) + \|u(t)\|^{p_c}_{\dot{H}^{\alpha}}.\end{aligned}$$

So by Lemma 2.17, if $\|u_0\|_{\dot{H}^{\alpha}}$ is sufficiently small, then u stays small in the $\dot{H}^{\alpha}$ norm, and global existence is established.

We finish this section by proving scattering. Using Proposition 4.2, it follows that $u \in M(\mathbb{R}_+)$. Taking account of previous computations and denoting

$$v(t) := T_{\alpha}(-t)u(t),$$

we get for $t, t' \to \infty$,

$$\begin{aligned}\|v(t) - v(t')\|_{\dot{H}^{\alpha}} &\lesssim \Big\|\int_t^{t'} T_{\alpha}(-s)(|u|^{\frac{4\alpha}{N-2\alpha}}u)\,ds\Big\|_{\dot{H}^{\alpha}}\\ &\lesssim (\|u\|_{S(t,t')} + \|u\|_{M(t,t')})\|u\|^{\frac{4\alpha}{N-2\alpha}}_{S(t,t')} \to 0.\end{aligned}$$

Finally, taking $u_+ := \lim_{t\to\infty} v(t)$ in $\dot{H}^{\alpha}$, we have

$$\begin{aligned}\|u - T_{\alpha}(t)u_+\|_{\dot{H}^{\alpha}} &= \|T_{\alpha}(t)(T_{\alpha}(-t)u - u_+)\|_{\dot{H}^{\alpha}}\\ &\lesssim \|T_{\alpha}(-t)u - u_+\|_{\dot{H}^{\alpha}} \to 0, \quad \text{as} \quad t \to \infty.\end{aligned}$$

Scattering is proved. ∎

5 Existence of a Ground State

In this section we prove the existence of a ground state solution to (2.1) in the critical case and subcritical case for $(\alpha,\beta)=(1,-\frac{1}{N})$. Precisely, we establish Proposition 2.4. For $(a,b)\in\mathbb{R}_+^*\times\mathbb{R}_+\cup\{(1,-\frac{2}{N})\}$ and $\phi\in H^\alpha$, recall the quantities

$$K_{a,b}(\phi)=\frac{1}{2}(2a+Nb)\|\phi\|^2+\frac{1}{2}(2a+(N-2\alpha)b)\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2-\Big(a+\frac{Nb}{1+p}\Big)\|\phi\|_{1+p}^{1+p},$$

$$H_{a,b}(\phi)=\frac{\alpha b}{2a+Nb}\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2+\frac{a(p-1)}{(1+p)(2a+Nb)}\|\phi\|_{1+p}^{1+p},\quad 2a+Nb\neq 0.$$

First case: $c=0$, $p_*<p<p^*$ and $(\alpha,\beta)=(1,-\frac{2}{N})$. In this case we will use

$$T(\phi):=\Big(E-\frac{N}{4\alpha}K_{1,-\frac{2}{N}}\Big)(\phi)=\frac{1}{2}\|\phi\|^2+\frac{N(p-p_*)}{4\alpha(1+p)}\|\phi\|_{1+p}^{1+p}$$

rather then $H_{a,b}$ which is no longer defined. Let u_λ denote the scaling $u_\lambda:=\lambda^{\frac{N}{2}}u(\lambda\cdot)$.

Lemma 5.1 *Let $u\in H^\alpha$ such that $K_{1,-\frac{2}{N}}(u)\le 0$. Then there exists $\lambda_0\le 1$ such that*

(i) $K_{1,-\frac{2}{N}}(u_{\lambda_0})=0$,
(ii) $\lambda_0=1$ *if and only if* $K_{1,-\frac{2}{N}}(u)=0$,
(iii) $\frac{\partial}{\partial\lambda}E(u_\lambda)>0$ *for* $\lambda\in(0,\lambda_0)$ *and* $\frac{\partial}{\partial\lambda}E(u_\lambda)<0$ *for* $\lambda\in(\lambda_0,\infty)$,
(iv) $\lambda\to E(u_\lambda)$ *is concave on* (λ_0,∞),
(v) $\frac{\partial}{\partial\lambda}E(u_\lambda)=\frac{N}{2\lambda}K_{1,-\frac{2}{N}}(u_\lambda)$.

Proof With direct computations, we have

$$K_{1,-\frac{2}{N}}(u_\lambda)=\frac{2\alpha\lambda^{2\alpha}}{N}\|(-\Delta u)^{\frac{\alpha}{2}}\|^2-(1-\frac{2}{1+p})\lambda^{\frac{N}{2}(p-1)}\int_{\mathbb{R}^N}|u|^{1+p}\,dx,$$

$$\partial_\lambda E(u_\lambda)=\frac{N}{2\lambda}K_{1,-\frac{2}{N}}(u_\lambda),$$

which proves (v). Now

$$K_{1,-\frac{2}{N}}(u_\lambda)=\frac{2\alpha\lambda^{2\alpha}}{N}\Big[\|(-\Delta)^{\frac{\alpha}{2}}u\|^2-\frac{N}{\alpha}(\frac{1}{2}-\frac{1}{1+p})\lambda^{\frac{N}{2}(p-1)-2\alpha}\int_{\mathbb{R}^N}|u|^{1+p}\,dx\Big].$$

A monotonicity argument via the inequality $p_*<p$ closes the proof of (1), (ii), and (iii). For (iv), it is sufficient to compute using (iii). ∎

Lemma 5.2 *For $u\in H^\alpha$, the real function $\lambda\mapsto T(\lambda u)$ is increasing on $\mathbb{R}_+$.*

Proof Given $u\in H^\alpha$, we compute

$$T(\lambda u)=\frac{\lambda^2}{2}\Big(\|u\|^2+\lambda^{p-1}\frac{N(p-p_*)}{4\alpha(1+p)}\int_{\mathbb{R}^N}|u|^{1+p}\,dx\Big),$$

$$\partial_\lambda T(\lambda u)=\lambda\Big(\|u\|^2+\lambda^{p-2}\frac{N(p-p_*)}{4\alpha}\int_{\mathbb{R}^N}|u|^{1+p}\,dx\Big).$$

The proof is complete because $p>p_*$. ∎

We express the minimizing number $m_{1,-\frac{2}{N}}$ with a negative constraint.

Proposition 5.3 *We have* $m_{1,-\frac{2}{N}} = \inf_{0\neq u\in H^\alpha}\{T(u), K_{1,-\frac{2}{N}}(u)\leq 0\}$.

Proof Letting m_1 be the right-hand side, it is sufficient to prove that $m_{1,-\frac{2}{N}} \leq m_1$. Take $u\in H^\alpha$ such that $K_{1,-\frac{2}{N}}(u)<0$. Then by Lemma 5.1 and the fact that $\lambda\mapsto T(\lambda u)$ is increasing, there exists $\lambda\in(0,1)$ such that $K_{1,-\frac{2}{N}}(\lambda u)=0$ and $m_{1,-\frac{2}{N}}\leq T(\lambda u)\leq T(u)$. This finishes the proof. ∎

Now we prove Proposition 2.4. Let (ϕ_n) a minimizing sequence, namely

$$0\neq\phi_n\in H^\alpha,\quad K_{1,-\frac{2}{N}}(\phi_n)=0,\quad\text{and}\quad \lim_n E(\phi_n)=m_{1,-\frac{2}{N}}. \tag{5.1}$$

With a rearrangement argument [11], we can assume that ϕ_n is radial decreasing and satisfies

$$0\neq\phi_n\in H^\alpha,\quad K_{1,-\frac{2}{N}}(\phi_n)\leq 0,\quad\text{and}\quad \lim_n E(\phi_n)\leq m_{1,-\frac{2}{N}}.$$

We can suppose that ϕ_n is radial decreasing and satisfies (5.1). Indeed, by Lemmas 5.1–5.2, there exists $\lambda\in(0,1)$ such that $K_{1,-\frac{2}{N}}(\lambda\phi_n)=0$ and $T(\lambda\phi_n)\leq m_{1,-\frac{2}{N}}$. Then

$$\frac{2\alpha}{N}\|(-\Delta)^{\frac{\alpha}{2}}\phi_n\|^2 = \Big(1-\frac{2}{1+p}\Big)\int_{\mathbb{R}^N}|\phi_n|^{1+p}\,dx,$$
$$\|\phi_n\|^2_{H^\alpha}-\frac{2}{1+p}\int_{\mathbb{R}^N}|\phi_n|^{1+p}\,dx\to 2m_{1,-\frac{2}{N}}.$$

So for any real number $a\neq 0$,

$$\Big[\Big(1-\frac{2a\alpha}{N}\Big)\|(-\Delta)^{\frac{\alpha}{2}}\phi_n\|^2+\|\phi_n\|^2+\Big(a-2\frac{1+a}{1+p}\Big)\int_{\mathbb{R}^N}|\phi_n|^{1+p}\Big]\,dx\to 2m_{1,-\frac{2}{N}}.$$

Letting $a\in(\frac{1}{p-1},\frac{N}{4\alpha})$, gives that (ϕ_n) is bounded in H^α. Taking account of the compact injections in Lemma 2.15, we take $\phi_n\rightharpoonup\phi$ in H^α and $\phi_n\to\phi$ in L^{1+p}. Assume, by contradiction, that $\phi=0$. The equality $K_{1,-\frac{2}{N}}(\phi_n)=0$, via the Hölder inequality and the fact that $p_*<p<p^*$, implies that for large n,

$$\frac{2\alpha}{N}\|\phi_n\|^2_{\dot H^\alpha}=\Big(1-\frac{2}{1+p}\Big)\|\phi_n\|^{1+p}_{1+p}\lesssim\|\phi_n\|^{1+p}_{\dot H^\alpha}=o(\|\phi_n\|^2_{\dot H^\alpha}).$$

This contradiction implies that $\phi\neq 0$. Thanks to the lower semicontinuity of $\|\cdot\|_{H^\alpha}$, we have $K_{1,-\frac{2}{N}}(\phi)\leq 0$ and $E(\phi)\leq m_{1,-\frac{2}{N}}$. Using Lemmas 5.2–5.1, we can assume that $K_{1,-\frac{2}{N}}(\phi)=0$ and $T(\phi)\leq m_{1,-\frac{2}{N}}$. So ϕ is a minimizer satisfying

$$0\neq\phi\in H^\alpha_{rd},\quad K_{1,-\frac{2}{N}}(\phi)=0,\quad\text{and}\quad T(\phi)=m_{1,-\frac{2}{N}}.$$

This implies that $0<\|\phi\|^2\leq T(\phi)=m_{1,-\frac{2}{N}}$. Now there is a Lagrange multiplier $\eta\in\mathbb{R}$ such that $E'(\phi)=\eta K'_{1,-\frac{2}{N}}(\phi)$. Thus

$$\begin{aligned}0=K_{1,-\frac{2}{N}}(\phi)&=\mathcal{L}_{1,-\frac{2}{N}}E(\phi)=\langle E'(\phi),\mathcal{L}_{1,-\frac{2}{N}}(\phi)\rangle\\&=\eta\langle K'_{1,-\frac{2}{N}}(\phi),\mathcal{L}_{1,-\frac{2}{N}}(\phi)\rangle\\&=\eta\mathcal{L}_{1,-\frac{2}{N}}K_{1,-\frac{2}{N}}(\phi)=\eta\mathcal{L}^2_{1,-\frac{2}{N}}E(\phi).\end{aligned}$$

With a direct computation, we have $\mathcal{L}_{1,-\frac{2}{N}}(\|\phi\|^2) = (\mathcal{L}_{1,-\frac{2}{N}} - \frac{4\alpha}{N})(\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2) = 0$ and $\mathcal{L}_{1,-\frac{2}{N}}(|\phi|^{1+p}) = (p-1)|\phi|^{1+p}$. So

$$-\mathcal{L}_{1,-\frac{2}{N}}(\mathcal{L}_{1,-\frac{2}{N}} - \frac{4\alpha}{N})E(\phi) = \frac{p-1}{p+1}(p-p_*)\int_{\mathbb{R}^N}|\phi|^{1+p}\,dx > 0.$$

Then $-\mathcal{L}^2_{1,-\frac{2}{N}}E(\phi) > 0$, so $\eta = 0$, and $E'(\phi) = 0$. Finally, ϕ is a ground-state solution to (2.1).

Second case: $p = p^*$. Define the massless action

$$\begin{aligned} K^0_{a,b}(\phi) &:= \mathcal{L}_{a,b}E^0(\phi) \\ &= \frac{1}{2}(2a + (N-2\alpha)b)\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 - (a + \frac{Nb}{p_c})\|\phi\|^{p_c}_{p_c} \\ &= (a + \frac{Nb}{p_c})\Big(\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 - \|\phi\|^{p_c}_{p_c}\Big) \end{aligned}$$

and the operator

$$H^0_{a,b}(\phi) := \Big(E^0 - \frac{1}{ap_c + Nb}K^0_{a,b}\Big)(\phi) = \frac{\alpha}{N}\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2.$$

Let the real number $d^0_{a,b} := \inf_{0\neq\phi\in H^\alpha}\{H^0_{a,b}(\phi) \mid K^0_{a,b}(\phi) < 0\}$.

Claim 1 $m^0_{a,b} = d^0_{a,b}$.

Since $K^0_{a,b} = 0$ implies that $E^0 = H^0_{a,b}$, it follows that $m^0_{a,b} \geq d^0_{a,b}$. Conversely, take $0 \neq \phi \in H^\alpha$ such that $K^0_{a,b}(\phi) < 0$. Thus, when $0 < \lambda \to 0$, we get

$$\begin{aligned} K^0_{a,b}(\lambda\phi) &= \frac{1}{2}(2a + (N-2\alpha)b)\lambda^2\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 - \Big(a + \frac{Nb}{p_c}\Big)\lambda^{p_c}\|\phi\|^{p_c}_{p_c} \\ &\simeq \frac{1}{2}(2a + (N-2\alpha)b)\lambda^2\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 > 0. \end{aligned}$$

So there exists $\lambda \in (0,1)$ satisfying $K^0_{a,b}(\lambda\phi) = 0$ and

$$m^0_{a,b} \leq H^0_{a,b}(\lambda\phi) = \lambda^2 H^0_{a,b}(\phi) \leq H^0_{a,b}(\phi).$$

Thus, $m^0_{a,b} \leq d^0_{a,b}$. So $m^0_{a,b} = d^0_{a,b}$, proving the claim.

Because of the definitions of $K^0_{a,b}$ and $H^0_{a,b}$, it is clear that $m^0_{a,b}$ is independent of (a,b) and

$$m := m^0_{a,b} = \inf_{0\neq\phi\in H^\alpha}\Big\{\frac{\alpha}{N}\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 \mid \|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 < \|\phi\|^{p_c}_{p_c}\Big\}.$$

Taking the scaling $\lambda\phi$,

$$\begin{aligned} m &= \inf_{0\neq\phi\in H^\alpha}\Big\{\frac{\alpha}{N}\lambda^2\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 \quad \text{s. t} \quad \lambda^{2-p_c}\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2 < \|\phi\|_{p_c}^{p_c}\Big\} \\ &= \inf_{0\neq\phi\in H^\alpha}\Big\{\frac{\alpha}{N}\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2\Big(\frac{\|\phi\|_{p_c}^{p_c}}{\|(-\Delta)^{\frac{\alpha}{2}}\phi\|^2}\Big)^{\frac{2}{2-p_c}}\Big\} \\ &= \frac{\alpha}{N}\inf_{0\neq\phi\in H^\alpha}\Big\{\Big(\frac{\|(-\Delta)^{\frac{\alpha}{2}}\phi\|}{\|\phi\|_{p_c}}\Big)^{\frac{N}{\alpha}}\Big\} \\ &= \frac{\alpha}{N}(C^*)^{-\frac{N}{\alpha}}. \end{aligned}$$

Here C^* denotes the best constant of the Sobolev injection $\|\phi\|_{p_c} \le C^*\|(-\Delta)^{\frac{\alpha}{2}}\phi\|$, which is known [14] to be attained by the explicit $Q \in \dot{H}^\alpha$,

$$Q(x) := \frac{a}{(1+|x|^2)^{\frac{N}{2}-\alpha}},$$

which solves the massless equation $(-\Delta)^\alpha Q = Q^{p_c-1}$. ∎

6 Invariant Sets and Applications

This section is devoted to establishing Theorem 2.6. The proof is based on two auxiliary results.

Lemma 6.1 *The sets $A^{c,+}_{a,b}$ and $A^{c,-}_{a,b}$ are independent of the pair (a,b).*

Proof Take (a,b) and (a',b') in $\mathbb{R}^*_+ \times \mathbb{R}_+ \cup \{(1,-\frac{2}{N})\}$. By Propositions 2.4, 2.8, 2.10, and 2.11, the reunion $A^{c,+}_{a,b} \cup A^{c,-}_{a,b}$ is independent of (a,b). So it is sufficient to prove that $A^{c,+}_{a,b}$ is independent of (a,b). If $E^0(v) < m$ and $K^0_{a,b}(v) = 0$, then $v = 0$. So $A^{c,+}_{a,b}$ is open. The rescaling $v^\lambda := \lambda^a v(\frac{\cdot}{\lambda^b})$ implies that a neighborhood of zero is in $A^{c,+}_{a,b}$. Moreover, this rescaling with $\lambda \to 0$ gives that $A^{c,+}_{a,b}$ is contracted to zero, and so it is connected. Now write

$$A^{c,+}_{a,b} = A^{c,+}_{a,b} \cap (A^{c,+}_{a',b'} \cup A^{c,-}_{a',b'}) = (A^{c,+}_{a,b} \cap A^{c,+}_{a',b'}) \cup (A^{c,+}_{a,b} \cap A^{c,-}_{a',b'}).$$

Since by the definition, $A^{c,-}_{a,b}$ is open and $0 \in A^{c,+}_{a,b} \cap A^{c,+}_{a',b'}$, using a connectivity argument, we have $A^{c,+}_{a,b} = A^{c,+}_{a',b'}$. The proof is complete. ∎

Lemma 6.2 *The sets $A^{c,+}_{a,b}$ and $A^{c,-}_{a,b}$ are invariant under the flow of (1.1).*

Proof Take $(a,b) \in \mathbb{R}^*_+ \times \mathbb{R}_+ \cup \{(1,-\frac{2}{N})\}$. Let $u_0 \in A^{c,+}_{a,b}$ and $u \in C_{T^*}(H^\alpha)$ be the maximal solution to (1.1). The proof follows with contradiction. Assume that for some time $t_0 \in (0,T^*)$, $u(t_0) \notin A^{c,+}_{a,b}$, and $u(t) \in A^{c,+}_{a,b}$ for all $t \in (0,t_0)$. Since the energy is decreasing and $E(u(t_0)) < m$, then, with a continuity argument, there exists a positive time $t_1 \in (0,t_0)$ such that $K_{a,b}(u(t_1)) = 0$. This contradicts the definition of m and finishes the proof in this case. The proof for $A^{c,+}_{a,b}$ is similar. ∎

Proof of Theorem 2.6 (i) Using the two previous lemmas via a translation argument, we can assume that $u(t) \in A^+_{1,1}$ for any $t \in [0, T^*)$. Taking account of the definition of m, we get

$$\begin{aligned} m &> E(u(t)) \\ &> E(u(t)) - \frac{1}{2+N} K_{1,1}(u(t)) \\ &= \frac{\alpha}{2+N} \|(-\Delta)^{\frac{\alpha}{2}} u(t)\|^2 + \frac{p-1}{(1+p)(2+N)} \|u(t)\|^{1+p}_{1+p}. \end{aligned}$$

This implies, via decay of the equality, $\partial_t(\|u(t)\|^2) = 2K_{1,0}(u(t)) < 0$. It follows that $\sup_{[0,T^*]} \|u(t)\|_{H^\alpha} < \infty$. Then u is global.

Now we prove an exponential decay. For small $\|u_0\|$, since $\sup_t \|u(t)\|_{\dot{H}^\alpha} \lesssim 1$, we get using Corollary 2.14,

$$\begin{aligned} K_{1,0}(u(t)) &= \|u(t)\|^2_{H^\alpha} - \int_{\mathbb{R}^N} |u(t)|^{1+p}\, dx \\ &\geq \|u(t)\|^2 + \|u(t)\|^2_{\dot{H}^\alpha} - C\|u(t)\|^{p+1-\frac{N(p-1)}{2\alpha}} \|u(t)\|^{\frac{N(p-1)}{2\alpha}}_{\dot{H}^\alpha} \\ &\geq \|u(t)\|^2 + \|u(t)\|^2_{\dot{H}^\alpha}\Big(1 - C\|u_0\|^{p+1-\frac{N(p-1)}{2\alpha}} \|u(t)\|^{\frac{N(p-1)}{2\alpha}}_{\dot{H}^\alpha}\Big) \\ &\geq C\|u(t)\|^2_{H^\alpha} \\ &\geq CE(u(t)). \end{aligned}$$

On the other hand,

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u(t)\|^2_{H^\alpha} - \frac{1}{1+p} \int_{\mathbb{R}^N} |u(t)|^{1+p}\, dx \\ &= \frac{1}{2}\|u(t)\|^2_{H^\alpha} - \frac{1}{1+p}\big(\|u(t)\|^2_{H^\alpha} - K_{1,0}(u(t))\big) \\ &= \Big(\frac{1}{2} - \frac{1}{1+p}\Big)\|u(t)\|^2_{H^\alpha} + \frac{1}{1+p} K_{1,0}(u(t)) \\ &\geq C \max\{K_{1,0}(u(t)), \|u(t)\|^2_{H^\alpha}\}. \end{aligned}$$

Moreover, for $T > 0$,

$$\begin{aligned} \int_t^T K_{1,0}(u(s))\, ds &= \frac{1}{2}(\|u(t)\|^2 - \|u(T)\|^2) \\ &\leq \frac{1}{2}\|u(t)\|^2 \\ &\leq CE(u(t)). \end{aligned}$$

So $\int_t^T E(u(s))\, ds \lesssim \int_t^T K_{1,0}(u(s))\, ds \lesssim E(u(t))$. Thus, for some positive real number $T_0 > 0$, $y(t) := \int_t^\infty E(u(s))\, ds \lesssim E(u(t)) \leq -T_0 y'(t)$. This implies that for $t \geq T_0$, $y(t) \leq y(T_0) e^{1-\frac{t}{T_0}} \leq T_0 E(u(T_0)) e^{1-\frac{t}{T_0}}$. Taking account of the monotonicity of the energy, for large $T > 0$,

$$\int_t^T E(u(s))\, ds \geq \int_t^{t+T_0} E(u(s))\, ds \geq T_0 E(u(t+T_0)).$$

Then $E(u(t+T_0)) \le E(u(T_0))e^{1-\frac{t}{T_0}}$. Finally,

$$\|u(t+T_0)\|_{H^\alpha}^2 \lesssim E(u(t+T_0)) \le E(u(T_0))e^{1-\frac{t}{T_0}}.$$

The proof is complete. ∎

Proof of Theorem 2.6 (ii) Using the two previous lemmas via a translation argument, we can assume that $u(t) \in A_{1,\lambda}^{c,-}$ for any $t \in [0, T^*)$ and any $\lambda > 0$. Take the real function $L(t) := \frac{1}{2}\int_0^t \|u(s)\|^2\,ds, \quad t \in [0, T^*)$. Using equation (1.1), a direct computation gives

$$L''(t) = \int_{\mathbb{R}^N} \dot{u}u\,dx = -\|u(t)\|_{\dot{H}^\alpha}^2 - c\|u(t)\|^2 + \int_{\mathbb{R}^N} |u|^{1+p}\,dx.$$

We discuss two cases.

First case: $E^c(u_0) > 0$. For any $\lambda > 0$,

$$H_{1,\lambda}(u) = \frac{1}{2+N\lambda}\Big[\alpha\lambda\|(-\Delta)^{\frac{\alpha}{2}}u\|^2 + \frac{p-1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx\Big] > m.$$

Thus, for any $\varepsilon > 0$,

$$\begin{aligned}
L'' &= \varepsilon\|(-\Delta)^{\frac{\alpha}{2}}u\|^2 - (1+\varepsilon)\|(-\Delta)^{\frac{\alpha}{2}}u\|^2 - c\|u(t)\|^2 + \int_{\mathbb{R}^N}|u|^{p+1}\,dx\\
&> \frac{\varepsilon}{\alpha}\Big[\Big(\frac{2}{\lambda}+N\Big)m - \frac{1}{\lambda}\frac{p-1}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx\Big]\\
&\quad - 2(1+\varepsilon)\Big[E^c(u_0) + \frac{1}{2(1+p)}\int |u|^{p+1}\,dx\Big]\\
&\quad + 2(1+\varepsilon)\int_0^t \|\dot{u}(s)\|^2\,ds + \int_{\mathbb{R}^N}|u|^{p+1}\,dx\\
&> \Big[\frac{\varepsilon}{\alpha}\Big(\frac{2}{\lambda}+N\Big)m - 2(1+\varepsilon)E^c(u_0)\Big] + \Big(1 - \frac{1+\varepsilon}{1+p} - \frac{\varepsilon(p-1)}{\alpha\lambda(p+1)}\Big)\int_{\mathbb{R}^N}|u|^{p+1}\,dx\\
&\quad + 2(1+\varepsilon)\int_0^t\|\dot{u}(s)\|^2\,ds\\
&:= (\mathrm{I}) + \frac{(\mathrm{II})}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx + 2(1+\varepsilon)\int_0^t\|\dot{u}(s)\|^2\,ds.
\end{aligned}$$

Taking $\lambda := a\varepsilon$ and $\gamma := m - E^c(u_0)$, we get

$$(\mathrm{I}) = 2\gamma(1+\varepsilon) + m\Big[\frac{2}{\alpha a} - 2 + \varepsilon\Big(-2+\frac{N}{\alpha}\Big)\Big] = \varepsilon\Big(2\gamma - 2m + \frac{Nm}{\alpha}\Big) + 2m\Big(\frac{1}{\alpha a} - 1\Big) + 2\gamma.$$

On the other hand,

$$(\mathrm{II}) = p + 1 - (1+\varepsilon) - \frac{p-1}{\alpha a} = (p-1)(1 - \frac{1}{\alpha a}) + 1 - \varepsilon.$$

The choice $\frac{1}{\alpha}\frac{p-1}{p-\varepsilon} < a < \frac{1}{\alpha}$, via $\varepsilon > 0$ near to zero implies that the terms (I) and (II) are non negative. Thus, $L'' > 2(1+\varepsilon)\int_0^t\|\dot{u}(s)\|^2\,ds$. Thanks to the Cauchy–Schwarz inequality, it follows that

$$LL'' > (1+\varepsilon)\|\dot{u}\|_{L_t^2(L^2)}^2\|u\|_{L_t^2(L^2)}^2 > (1+\varepsilon)\|u\dot{u}\|_{L_t^1(L^1)}^2 > (1+\varepsilon)L'^2.$$

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In fact, if $L(t)=0$ for some positive time, we get $u_0=E(u_0)=0$, which is a contradiction. Thus $(L^{-\varepsilon})''=-\varepsilon L^{-\varepsilon-2}[L''L-(1+\varepsilon)(L')^2]>0$. Taking account of Proposition 2.18, for some finite time $T>0$, $\limsup_{t\to T}\int_0^T\|u(s)\|^2\,ds=\infty$. Thus, $T^*<\infty$, and u is not global. This ends the proof. ∎

Second case: $E^c(u_0)\le 0$. Compute

$$\begin{aligned}L''&=-\|u\|_{\dot H^\alpha}^2-c\|u\|^2+\int_{\mathbb{R}^N}|u|^{p+1}\,dx\\&\ge(2+\varepsilon)\Big(\int_{\mathbb{R}^N}\frac{|u|^{p+1}}{p+1}\,dx-\frac12\|u\|_{\dot H^\alpha}^2-\frac c2\|u\|^2\Big)\\&\ge-(2+\varepsilon)E^c(u).\end{aligned}$$

So, thanks to the identity $\dot E^c(u)=-\|\dot u\|^2$, we get

$$L''\ge(2+\varepsilon)\big(\|\dot u\|_{L_t^2(L^2)}^2-E^c(u_0)\big).\tag{6.1}$$

Now the proof goes by contradiction, assuming that $T^*=\infty$.

Claim 2 There exists $t_1>0$ such that $\int_0^{t_1}\|\dot u(s)\|^2\,ds>0$. Indeed, otherwise $u(t)=u_0$ almost everywhere and solves the elliptic stationary equation $(-\Delta)^\alpha u+cu=|u|^{p-1}u$. Therefore, $\|u\|_{\dot H^\alpha}^2+c\|u\|^2=\int_{\mathbb{R}^N}|u|^{p+1}dx$ and

$$\begin{aligned}\|u_0\|_{\dot H^\alpha}^2+c\|u_0\|^2-\frac{2}{p+1}\int_{\mathbb{R}^N}|u_0|^{p+1}\,dx&=(1-\frac{2}{p+1})\int_{\mathbb{R}^N}|u_0|^{p+1}\,dx\\&=2E(u_0)\\&\le 0.\end{aligned}$$

Then $u_0=0$, which contradicts the fact that $K_{0,1}(u_0)<0$.

Claim 3 For any $0<\alpha<1$, there exists $t_\alpha>0$ such that $(L'-L'(0))^2\ge\alpha L'^2$, on (t_α,∞). The claim immediately follows from the first one and (6.1), observing that

$$\lim_{t\to\infty}L(t)=\lim_{t\to\infty}L'(t)=+\infty.$$

Claim 4 One can choose $\alpha=\alpha(\varepsilon)$ such that $LL''\ge(1+\alpha)L'^2$, on (t_α,∞). Indeed, we have

$$\begin{aligned}LL''&\ge\frac{2+\varepsilon}{2}\|u\|_{L_t^2(L^2)}^2\|\dot u\|_{L_t^2(L^2)}^2\\&\ge\frac{2+\varepsilon}{2}\|u\dot u\|_{L_t^1(L^1)}^2\\&\ge\frac{2+\varepsilon}{2}(L'-L'(0))^2\\&\ge\frac{(2+\varepsilon)\alpha}{2}L'^2,\end{aligned}$$

where we used (6.1) in the first estimate, Cauchy–Schwarz inequality in the second, and Claim 2 in the last one. Now choosing α such that $1<\frac{(2+\varepsilon)\alpha}{2}:=1+\varepsilon$, we get $LL''>(1+\varepsilon)L'^2$, for large time.

Thanks to Proposition 2.18, this ordinary differential inequality blows up in finite time and contradicts our assumption that the solution is global. This ends the proof. ∎

7 Strong Instability

This section is devoted to prove Theorem 2.7 about strong instability of stationary solutions to (1.1). Henceforth $c = \epsilon = 1$. Denote the scaling $u_\lambda := \lambda^{\frac{N}{2}} u(\lambda .)$. Let us write an auxiliary result.

Lemma 7.1 *Let ϕ to be a ground-state solution of (2.1), $\lambda > 1$ a real number close to one, and $u_\lambda \in C([0, T^*), H^\alpha)$ the solution to (1.1) with data ϕ_λ. Then for any $t \in (0, T^*)$,*

$$E(u_\lambda(t)) < E(\phi) \quad \textit{and} \quad K_{1,-\frac{2}{N}}(u_\lambda(t)) < 0.$$

Proof By Lemma 5.1, we have $E(\phi_\lambda) < E(\phi)$ and $K_{1,-\frac{2}{N}}(\phi_\lambda) < 0$. Moreover, thanks to the decay of energy, it follows that for any $t > 0$, $E(u_\lambda(t)) \le E(\phi_\lambda(t)) < E(\phi)$. Then $K_{1,-\frac{2}{N}}(u_\lambda(t)) \neq 0$ because ϕ is a ground state. Finally $K_{1,-\frac{2}{N}}(u_\lambda(t)) < 0$ with a continuity argument. ∎

Now we are ready to prove the instability result. Take $u_\lambda \in C_{T^*}(H^\alpha)$, the maximal solution to (1.1) with data ϕ_λ, where $\lambda > 1$ is close to one and ϕ is a ground-state solution to (2.1). With the previous lemma, we get $u_\lambda(t) \in A^-_{1,-\frac{2}{N}}$, for any $t \in (0, T^*)$. Then using Theorem 2.7, it follows that $\limsup_{t\to T^*} \|u_\lambda(t)\|_{H^\alpha} = \infty$. The proof is finished via the fact that $\lim_{\lambda\to 1} \|\phi_\lambda - \phi\|_{H^\alpha} = 0$.

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