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DIRECTED PACKINGS OF PAIRS INTO QUADRUPLES

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Abstract

A directed packing of pairs into quadruples is a collection of 4-subsets of a set of cardinality v with the property that each ordered pair of elements appears at most once in a 4-subset (or block). The maximal number of blocks with this property is denoted by $DD(2, 4, v)$. Such a directed packing may also be thought of as a packing of transtivie tournaments into the complete directed graph on v points. It is shown that, for all but a finite number of values of v, $DD(2, 4, v)$ is maximal.

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1. Introduction

A directed packing is a collection of k-subsets (called blocks) of a set of cardinality v with the property that every ordered t-subset appears in at most one block. A t-set is contained in a k-set if its symbols appear in order, possibly interspersed with other symbols. Thus, the block *abcd* contains the six pairs: *ab*, *ac*, *ad*, *bc*, *bd* and *cd*. The maximal number of blocks with this property is denoted by $DD(t, k, v)$ for each choice of t, k and v. In this paper, directed packings of pairs into quadruples are considered.

A counting argument can be used to derive an upper bound on the value of $DD(2, 4, v)$. As no ordered pair can appear more than once, no symbol can appear in more than $2(v-1)$ pairs. Therefore its frequency cannot be greater than $\lfloor 2(v-1)/3 \rfloor$ since each time it appears in a block it appears in 3 pairs. Summing frequencies over all symbols gives

$$DD(2,4,v) \leqslant \left\lfloor \frac{v}{4} \left\lfloor \frac{2(v-1)}{3} \right\rfloor \right\rfloor. \tag{1}$$

Call the right-hand side of (1), $U(v)$.

A lower bound may be derived by considering ordinary packings. A packing may be made into a directed packing by writing each block of the packing twice, once forward and once reversed. This gives the lower bound

$$DD(2,4,v) \geqslant 2\,D(2,4,v). \tag{2}$$

2. The easy cases

D. J. Street and J. R. Seberry [4] have shown that, when $v \equiv 1 (\mathrm{mod}\ 3)$, a directed packing exists which contains every ordered pair exactly once. Such a structure is called a directed balanced incomplete block design and is of special interest having, for example, some of the statistical sampling properties of row complete latin squares. So we have

THEOREM 1. *If* $v \equiv 1(\mathrm{mod}\ 3)$ *then*

$$DD(2,4,v) = U(v).$$

When $v \equiv 2(\mathrm{mod}\ 3)$, A. E. Brouwer [1] has shown that

$$D(2,4,v) = \left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \right\rfloor \right\rfloor. \tag{3}$$

This result can be used to prove

THEOREM 2. *If* $v \equiv 2(\mathrm{mod}\ 3)$ *then*

$$DD(2,4,v) = U(v).$$

PROOF. (1) and (2) give, when $v \equiv 2(\mathrm{mod}\ 3)$

$$2\left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \right\rfloor \right\rfloor \leqslant DD(2,4,v) \leqslant \left\lfloor \frac{v}{4} \left\lfloor \frac{2(v-1)}{3} \right\rfloor \right\rfloor.$$

The left- and right-hand sides are equal, implying the result.

3. An indirect product construction

THEOREM 3. *If there are directed packings on w and v points such that $DD(2,4,w) = n$ and $DD(2,4,v) = m$ then*

$$DD(2,4,w(v-b)+b) \geqslant wm + n(v-b)^2$$

for all $b = 0, 1$ such that $v - b \in OA(4)$ (that is $v - b \neq 2$ or 6 for which there do not exist 2 mutually orthogonal latin squares).

PROOF. Take w sets of size v which are disjoint except for b points which are common to all of them. Call these sets C_i $(i = 1, 2, \ldots, w)$. From w copies of the packing on v points on these sets (since $b \leqslant 1$ there are no repeated pairs). The pairs from distinct sets are included as follows: take n orthogonal arrys of size 4 by $(v-b)^2$ and index the columns so that if $p_1 p_2 \cdots p_4$ is a block of the packing on w points then one of the orthogonal arrays has rows of the form $x_{p_1} x_{p_2} \cdots x_{p_4}$.

All ordered pairs from within each of the C_is may only appear in the packing written on C_i. Ordered pairs of the form $x_i y_j$ $(i \neq j)$ may only appear in the orthogonal array indexed by the block containing ij. The constructed object is therefore a directed packing on $w(v-b)+b$ points and has $wm + n(v-b)^2$ blocks.

This theorem is useful when the lower bound on the right-hand side equals the upper bound (1). This happens in the following cases.

COROLLARY 4. *If* $w \equiv 1 (\mathrm{mod}\, 3)$ *and* $v \equiv 0 (\mathrm{mod}\, 12)$ *and* $DD(2,4,v) = U(v)$ *then*

$$DD(2,4,wv) = U(wv).$$

COROLLARY 5. *If* $w = 4, 7$ *or* 10 *and* $v \equiv 0 (\mathrm{mod}\, 12)$ *and* $DD(2,4,v) = U(v)$ *then*

$$DD(2,4,w(v-1)+1) = U(w(v-1)+1).$$

To use these results, maximal packings with v in the congruence class $0 (\mathrm{mod}\, 12)$ must be constructed. The two smallest cases are shown below.

$DD(2,4,12) = 21.$

$1_1\, 0_2\, 1_2\, 0_1$
$0_4\, 2_4\, 0_3\, 0_1$
$2_2\, 1_2\, 0_3\, 0_4$
$1_3\, 2_1\, 0_2\, 0_3$ all developed mod 3.
$2_1\, 0_1\, 2_3\, 0_4$
$2_4\, 0_2\, 1_1\, 0_4$
$2_3\, 1_4\, 0_3\, 0_2$

$DD(2,4,24) = 90$.

$0_2\, 0_1\, 0_3\, 0_4$	$4_2\, 5_4\, 2_1\, 0_4$	
$0_4\, 0_3\, 1_2\, 0_1$	$3_4\, 5_2\, 1_3\, 0_4$	
$0_1\, 1_4\, 2_3\, 0_2$	$1_1\, 3_1\, 4_3\, 0_4$	$5_3\, 1_3\, 2_3\, 0_4$
$1_3\, 3_1\, 4_2\, 0_1$	$3_2\, 5_2\, 2_4\, 0_1$	all developed
$0_3\, 4_3\, 1_1\, 0_2$	$1_4\, 5_4\, 4_4\, 0_1$	mod 6.
$2_1\, 1_3\, 4_4\, 0_3$	$3_1\, 4_1\, 2_1\, 0_2$	
$1_4\, 3_4\, 1_2\, 0_3$	$5_2\, 3_2\, 2_2\, 0_3$	

Construction of the remaining packings in this class depends on the following theorem.

THEOREM 6 [2, Brouwer, Hanani, Schrijver]. *Necessary and sufficient conditions for the existence of a group divisible design on v points with blocks of size 4 and groups of size m are that $v \equiv 0 \pmod{m}$, $v - m \equiv 0 \pmod{3}$ and $v = m$ or $v \geqslant 4m$ (except for two cases $v = 8$, $m = 2$ and $v = 24$, $m = 6$).*

THEOREM 7. *If $\Delta(v) = U(v) - DD(2,4,v)$ then*

$$\Delta(12v) \leqslant v\Delta(12) \quad \textit{for } v \geqslant 4.$$

PROOF. By Theorem 6, there exists a group divisible design on $12v$ points ($v \geqslant 4$) with v groups of size 12 and $24v^2 - 2v - 10$ directed blocks. Replacing each group by a directed packing on 12 points, the result follows.

In fact, since $DD(2,4,12) = 21$ it follows that

THEOREM 8. *If $v \equiv 0 \pmod{12}$, $v \neq 36$, then*

$$DD(2,4,v) = U(v).$$

Theorem 8, together with Corollaries 4 and 5, give

THEOREM 9. *For v in the following congruence classes, $DD(2,4,v) = U(v)$:*

$$v \equiv 45 \pmod{48}, \qquad v \neq 141,$$
$$v \equiv 78 \pmod{84}, \qquad v \neq 246,$$
$$v \equiv 111 \pmod{120}, \qquad v \neq 351.$$

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4. A recursive construction

THEOREM 10. *If* $m \in OA(10)$, $m \equiv 0 \pmod{12}$, $DD(2,4,m) = U(m)$ *and* $0 \leqslant t \leqslant m$ *then*

$$\Delta(10m + 3t) \leqslant \Delta(m + 3t).$$

PROOF. The following directed group divisible designs exist:

1. *DGDD* on 10 points with 10 groups of size 1 and 15 directed blocks.
2. *DGDD* on 13 points with 1 group of size 4, 9 groups of size 1 and 24 directed blocks.

Take a transversal design with 10 groups of size m. Use the construction of R. M. Wilson [5, Fundamental Construction] giving each point a weight of 1 except t points in a single group given a weight of 4. The resulting directed group divisible design has 9 groups of size m, 1 group of size $m + 3t$ and $m(15m + 9t)$ blocks. Replacing each of the groups by a directed packing of an appropriate size results in a directed packing on $10m + 3t$ points, the number of whose blocks falls short of the upper bound by at most as much as the directed packing on $m + 3t$ points.

THEOREM 11. *If* v *is sufficiently large then*

$$DD(2,4,v) = U(v).$$

PROOF. By Theorem 8, if $v \equiv 0 \pmod{12}$ and $v > 36$ then $DD(2,4,v) = U(v)$.

If $v \equiv 3 \pmod{12}$ and $0 \leqslant t \leqslant 399$ then v can be expressed as $10m + 3t$ in such a way that $m + 3t \equiv 111 \pmod{12}$. The largest m not known to be in $OA(10)$ is $m = 3576$ so that if $m > 3576$ the conditions of Theorem 10 are satisfied. Therefore if $v \geqslant 35778$ and $v \equiv 3 \pmod{12}$ then $DD(2,4,v) = U(v)$.

If $v \equiv 6 \pmod{12}$ and $0 \leqslant t \leqslant 279$ then v can be expressed as $10m + 3t$ with $m + 3t \equiv 78 \pmod{84}$. If $v \geqslant 35778$ the result follows.

Similarly if $v \equiv 9 \pmod{12}$ and $0 \leqslant t \leqslant 159$ then v can be written as $10m + 3t$ in such a way that $m + 3t \equiv 45 \pmod{48}$ and if $v \geqslant 35781$ the result follows.

Using Theorem 3, a computer program was run to determine which other packings are maximal using m values known to be in $OA(10)$. This showed that, in fact, if $v > 15579$ then $DD(2,4,v) = U(v)$. A list of values of v for which $DD(2,4,v)$ is not known to be maximal can be found in [3].

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