Ergod. Th. & Dynam. Sys., (2021), **41**, 3336–3362
doi:10.1017/etds.2020.105

Unstable pressure and u-equilibrium states for partially hyperbolic diffeomorphisms

HUYI HU†‡, WEISHENG WU§ and YUJUN ZHU§

† *Department of Mathematics, Southern University of Science and Technology of China, Shenzhen, P. R. China*
(e-mail: huhy@sustc.edu.cn)

‡ *Department of Applied Mathematics, College of Science, China Agricultural University, Beijing, 100083, P. R. China*
(e-mail: hhu@math.msu.edu)

§ *School of Mathematical Sciences, Xiamen University, Xiamen, 361005, P. R. China*
(e-mail: wuweisheng@cau.edu.cn; yjzhu@xmu.edu.cn)

(*Received* 21 *December* 2019 *and accepted in revised form* 3 *August* 2020)

Abstract. Unstable pressure and u-equilibrium states are introduced and investigated for a partially hyperbolic diffeomorphism f. We define the unstable pressure $P^u(f,\varphi)$ of f at a continuous function φ via the dynamics of f on local unstable leaves. A variational principle for unstable pressure $P^u(f,\varphi)$, which states that $P^u(f,\varphi)$ is the supremum of the sum of the unstable entropy and the integral of φ taken over all invariant measures, is obtained. U-equilibrium states at which the supremum in the variational principle attains and their relation to Gibbs u-states are studied. Differentiability properties of unstable pressure, such as tangent functionals, Gateaux differentiability and Fréchet differentiability and their relations to u-equilibrium states, are also considered.

Key words: Unstable pressure, u-equilibrium state, partial hyperbolicity
2020 Mathematics Subject Classification: 37D35 (Primary); 37D30 (Secondary)

1. *Introduction*

Entropy and pressure are important invariants in the study of dynamical systems and ergodic theory. Entropies, including topological entropy and measure-theoretic entropy, are measurements of complexity of the orbit structure of the system from different points of view. As a generalization of entropy, the concept of pressure was introduced by Ruelle [18] and studied in the general case in Walters [20]. In fact, the theory of pressure and its related topics, such as Gibbs measures and equilibrium states, are the main constituent components of mathematical statistical mechanics.

The main purpose of this paper is to introduce unstable topological pressure $P^u(f, \varphi)$ for a C^1-partially hyperbolic diffeomorphism $f : M \to M$ and any continuous function φ on M, obtain a variational principle for this pressure and investigate the corresponding so-called u-equilibria and differentiability properties.

Let f be a $C^{1+\epsilon}$-diffeomorphism on a closed Riemannian manifold M, where $\epsilon > 0$. The well-known entropy formula in [11] shows that if μ is an SRB (after Sinai–Ruelle–Bowen) measure, then the corresponding metric entropy $h_\mu(f)$ is the integration of the summation of the positive Lyapunov exponents. It tells us that positive exponents have a contribution to the metric entropy. In particular, when f is uniformly hyperbolic, both of the metric entropy and the topological entropy are caused by the dynamics on the unstable foliations. However, when f is (uniformly) partially hyperbolic, things become delicate. The presence of the center direction makes the dynamics much more complicated.

In recent years, the entropy theory for partially hyperbolic diffeomorphisms has been increasingly investigated. We can see the progress in this research topic in [9, 10, 23, 24], etc. In particular, for any C^1-partially hyperbolic diffeomorphism f, Hu, Hua and Wu [9] introduced the definitions of unstable metric entropy $h^u_\mu(f)$ for any invariant measure μ and unstable topological entropy $h^u_{\text{top}}(f)$. Precisely, $h^u_\mu(f)$ is defined by using $H_\mu(\bigvee_{i=0}^{n-1} f^{-i}\alpha|\eta)$, where α is a finite measurable partition and η is a measurable partition subordinate to unstable manifolds that can be obtained by refining a finite partition into pieces of unstable leaves; $h^u_{\text{top}}(f)$ is defined by the topological entropy of f on local unstable manifolds. Similar to that in the classical entropy theory, the corresponding versions of the Shannon–McMillan–Breiman theorem and the local entropy formula for $h^u_\mu(f)$, and the variational principle relating $h^u_\mu(f)$ and $h^u_{\text{top}}(f)$ are given. The main feature of these unstable entropies is to rule out the complexity caused by the center directions and focus on that caused by the unstable directions. In fact, $h^u_\mu(f)$ is equal to $h_\mu(f, \xi) := H_\mu(\xi|f\xi)$ (where ξ is an increasing partition subordinate to the unstable leaves), which was introduced by Ledrappier and Young [11]. Comparing the above two types of definition for the unstable metric entropy, we can see that the former one is more natural and easy to understand than the latter one.

Similar to the way by which the unstable entropy is defined in [9], the unstable pressure $P^u(f, \varphi)$ is defined via the information of the potential φ as iterating f on local unstable leaves (see Definition 2.2). It is well known that the variational principle for the classical pressure was first given by Ruelle [18] for the system with the expansiveness and specification assumptions and then was obtained by Walters [21] for the general case. It shows that

$$P(f, \varphi) = \sup\left\{h_\mu(f) + \int_M \varphi\, d\mu : \mu \in \mathcal{M}_f(M)\right\},$$

where $\mathcal{M}_f(M)$ is the set of all f-invariant probability measures on M. We will combine the elegant method in Walters [21] and the technique in Hu, Hua and Wu [9] to obtain the variational principle for unstable pressure (Theorem A), i.e. the equality as the above in which $P(f, \varphi)$ and $h_\mu(f)$ are replaced by $P^u(f, \varphi)$ and $h^u_\mu(f)$, respectively. In particular, if $\varphi \equiv 0$, then we get the variational principle for unstable entropy [9, Theorem D].

The measure at which the supremum attains in the variational principle for unstable pressure $P^u(f, \varphi)$ is called a *u-equilibrium state for f at φ* (see Definition 2.3). Some fundamental properties for the set of u-equilibrium states are considered (Theorem B). Among these properties, we show that there always exists a u-equilibrium state for a C^1-partially hyperbolic diffeomorphism, in contrast to the case for the classical equilibrium state (cf. [4, 12], etc). This is essentially due to the upper semicontinuity of the unstable entropy map $\mu \mapsto h^u_\mu(f)$. For the particular potential $\varphi^u = -\log|\det Df|_{E^u}|$, we relate the u-equilibrium states at φ^u to the Gibbs u-states of f (Theorem C). In [21, 22], some properties about the classical pressure and equilibrium states were investigated. We can consider the corresponding properties for unstable pressure and u-equilibrium states. We show that unstable pressure determines invariant measures (Theorem D) and there is a close relation between u-equilibrium states and tangent functionals (Theorem E). To study the uniqueness of the u-equilibrium state, we define two types, Gateaux type and Frechét type, of differentiability of unstable pressure of f at φ and obtain several properties of them (Theorems F and G).

The paper is organized as follows. In §2, we give the definitions of unstable pressure and u-equilibrium state, and formulate the main results. We provide some properties of unstable pressure in §3. Section 4 is for the proof of the variational principle of unstable pressure. In §5 and §6, we consider the properties of u-equilibrium states and study how the unstable pressure determines invariant measures. In §7, differentiability properties of unstable pressure are investigated.

2. *Definitions and statements of results*

Let M be an n-dimensional smooth, connected and compact Riemannian manifold without boundary and $f : M \to M$ a C^1-diffeomorphism. f is said to be *partially hyperbolic* (cf. for example [16]; see also [7]) if there exists a non-trivial Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle into stable, center and unstable distributions, such that all unit vectors $v^\sigma \in E^\sigma_x$, $\sigma = c, s, u$, with $x \in M$ satisfy

$$\|D_x f v^s\| < \|D_x f v^c\| < \|D_x f v^u\|$$

and

$$\|D_x f|_{E^s_x}\| < 1 \quad \text{and} \quad \|D_x f^{-1}|_{E^u_x}\| < 1$$

for some suitable Riemannian metric on M. The stable distribution E^s and unstable distribution E^u are integrable to the stable and unstable foliations W^s and W^u respectively such that $TW^s = E^s$ and $TW^u = E^u$ (cf. [8]).

In this paper we always assume that f is a C^1-partially hyperbolic diffeomorphism of M and μ is an f-invariant probability measure. The notion of unstable metric entropy of μ with respect to f was introduced in [9], using a type of measurable partitions consisting of local unstable leaves that can be obtained by refining a finite partition into pieces of unstable leaves. We recall the construction of such measurable partitions and the definition of unstable metric entropy as follows. To begin with, we recall some standard notation and classical results on measurable partitions.

Let $(X, \mathcal{A}, \nu)$ be a standard probability space. For a partition α of X, let $\alpha(x)$ denote the element of α containing x. If α and β are two partitions such that $\alpha(x) \subset \beta(x)$ for all $x \in X$, we then write $\alpha \geq \beta$ or $\beta \leq \alpha$. For a measurable transformation $f : X \to X$ and a partition α of X, we denote $f^{-1}\alpha = \{f^{-1}A : A \in \alpha\}$. Clearly, $f^{-1}\alpha$ is a partition if α is. A partition ξ is *increasing* if $f^{-1}\xi \geq \xi$. $\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta\}$ is called the *join* of α and β. For a partition β, we denote $\beta_m^n = \vee_{i=m}^n f^{-i}\beta$. In particular, $\beta_0^{n-1} = \vee_{i=0}^{n-1} f^{-i}\beta$.

For a partition η, let $\mathcal{B}(\eta)$ denote the smallest sub-σ-algebra of $\mathcal{A}$ that contains all elements of η. A partition η of X is called *measurable* if there exists a countable set $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\eta)$ such that for almost every (a.e.) pair $C_1, C_2 \in \eta$, we can find some A_n which separates them in the sense that $C_1 \subset A_n$, $C_2 \subset X - A_n$. The *canonical system of conditional measures of* ν *relative to* η is a family of probability measures $\{\nu_x^\eta : x \in X\}$ with $\nu_x^\eta\big(\eta(x)\big) = 1$ such that for every measurable set $B \subset X$, $x \mapsto \nu_x^\eta(B)$ is $\mathcal{B}(\eta)$-measurable and

$$\nu(B) = \int_X \nu_x^\eta(B)\, d\nu(x).$$

The classical result of Rohlin (cf. [17]) says that if η is a measurable partition, then there exists a system of conditional measures relative to η. It is unique in the sense that two such systems coincide in a set of full ν-measure. For measurable partitions α and η, let

$$H_\nu(\alpha|\eta) := -\int_X \log \nu_x^\eta(\alpha(x))\, d\nu(x)$$

denote the conditional entropy of α given η with respect to ν.

Now consider a C^1-partially hyperbolic diffeomorphism $f : M \to M$. Take $\varepsilon_0 > 0$ small. Let $\mathcal{P} = \mathcal{P}_{\varepsilon_0}$ denote the set of finite Borel partitions of M whose elements have diameters smaller than or equal to ε_0, that is, $\operatorname{diam} \alpha := \sup\{\operatorname{diam} A : A \in \alpha\} \leq \varepsilon_0$. For each $\beta \in \mathcal{P}$, we can define a finer partition η such that $\eta(x) = \beta(x) \cap W^u_{\text{loc}}(x)$ for each $x \in M$, where $W^u_{\text{loc}}(x)$ denotes the local unstable manifold at x whose size is greater than the diameter ε_0 of β. Since W^u is a continuous foliation, η is a measurable partition with respect to any Borel probability measure on M. Let $\mathcal{P}^u$ denote the set of partitions η obtained in this way and *subordinate to unstable manifolds*. Here a partition η of M is said to be subordinate to unstable manifolds of f with respect to a measure μ if for μ-a.e. x, $\eta(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$. It is clear that if $\alpha \in \mathcal{P}$ is such that $\mu(\partial\alpha) = 0$, where $\partial\alpha := \cup_{A\in\alpha}\partial A$, then the corresponding η given by $\eta(x) = \alpha(x) \cap W^u_{\text{loc}}(x)$ is a partition subordinate to unstable manifolds of f.

Definition 2.1. The *conditional entropy of* f *with respect to a measurable partition* α *given* $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f, \alpha|\eta) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta).$$

The *conditional entropy of* f *given* $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f|\eta) = \sup_{\alpha\in\mathcal{P}} h_\mu(f, \alpha|\eta)$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

and the *unstable metric entropy of* f is defined as

$$h^u_\mu(f) = \sup_{\eta\in\mathcal{P}^u} h_\mu(f|\eta).$$

The following theorem is one of the main results in [9].

THEOREM. [9, Theorem A and Corollary A.2] *For any* $\alpha \in \mathcal{P}$ *and* $\eta \in \mathcal{P}^u$,

$$h^u_\mu(f) = h_\mu(f|\eta) = h_\mu(f,\alpha|\eta) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta).$$

Unstable metric entropy was independently studied by Yang [24], where it is defined as the entropy introduced by Ledrappier and Young [11]. Suppose that f is $C^{1+\epsilon}(\epsilon > 0)$ and μ is ergodic. Recall a hierarchy of metric entropies $h_\mu(f,\xi_i) := H_\mu(\xi_i|f\xi_i)$ introduced by Ledrappier and Young in [11], where $i = 1,\ldots,\tilde{u}$ and $\tilde{u}$ is the number of distinct positive Lyapunov exponents. For each i, ξ_i is an increasing partition subordinate to the ith level of the unstable leaves $W^{(i)}$ and is a generator. It is proved there that $h_\mu(f,\xi_{\tilde{u}}) = h_\mu(f)$, the metric entropy of μ. If there are u, $1 \le u \le \tilde{u}$, distinct Lyapunov exponents on an unstable subbundle, then the uth unstable foliation is exactly the unstable foliation of the partially hyperbolic system f. It is shown in [9] that the unstable metric entropy $h^u_\mu(f)$ is identical to $h_\mu(f,\xi_u)$ given by Ledrappier–Young. We remark that our definition of unstable metric entropy (and also Yang's in [24]) only requires f to be C^1, while the definition and results by Ledrappier–Young require the $C^{1+\epsilon}$-regularity of f.

Another notion introduced in [9] is the unstable topological entropy $h^u_{\mathrm{top}}(f)$. As a generalization, we define the unstable topological pressure associated with a potential $\varphi \in C(M,\mathbb{R})$ as follows. Denote by d^u the metric induced by the Riemannian structure on the unstable manifold and let $d^u_n(x,y) = \max_{0\le j\le n-1} d^u(f^j(x), f^j(y))$. Let $W^u(x,\delta)$ denote the open ball inside $W^u(x)$ with center x and radius δ with respect to d^u. Let E be a set of points in $\overline{W^u(x,\delta)}$ with pairwise d^u_n-distances at least ϵ. We call E an (n,ϵ) *u-separated subset* of $\overline{W^u(x,\delta)}$. Put

$$P^u(f,\varphi,\epsilon,n,x,\delta) := \sup\bigg\{\sum_{y\in E}\exp((S_n\varphi)(y)) : E \text{ is an } (n,\epsilon) \text{ u-separated subset of } \overline{W^u(x,\delta)}\bigg\},$$

where $(S_n\varphi)(y) = \sum_{i=0}^{n-1}\varphi^i(y)$.

Definition 2.2. We define *unstable topological pressure* of f with respect to *the potential* φ on M to be

$$P^u(f,\varphi) := \lim_{\delta\to 0}\sup_{x\in M} P^u(f,\varphi,\overline{W^u(x,\delta)}),$$

where

$$P^u(f,\varphi,\overline{W^u(x,\delta)}) := \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P^u(f,\varphi,\epsilon,n,x,\delta).$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

In fact, $P^u(f, \varphi, \overline{W^u(x, \delta)})$ is exactly the upper capacity topological pressure of the set $\overline{W^u(x, \delta)}$ defined by Pesin (see [13]). Naturally, one can define the unstable pressure via Caratheódory structure following [13]. For the unstable entropy case, this has been done in [15, 19].

Two alternative ways to define unstable topological pressure are by using (n, ϵ) u-spanning sets and by using open covers. We discuss it in detail in §2. Note that when $\varphi = 0$, the unstable topological pressure becomes the unstable topological entropy.

Let $\mathcal{M}_f(M)$ and $\mathcal{M}^e_f(M)$ denote the sets of all f-invariant and ergodic probability measures on M, respectively. Our first main result is the variational principle relating unstable topological pressure and unstable metric pressure, the sum of unstable metric entropy and integral of the potential.

THEOREM A. *Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism. Then, for any $\varphi \in C(M, \mathbb{R})$,*

$$P^u(f, \varphi) = \sup \left\{ h^u_\mu(f) + \int_M \varphi \, d\mu : \mu \in \mathcal{M}_f(M) \right\}.$$

Moreover,

$$P^u(f, \varphi) = \sup \left\{ h^u_\mu(f) + \int_M \varphi \, d\mu : \mu \in \mathcal{M}^e_f(M) \right\}.$$

As an immediate corollary, we recover the variational principle for unstable entropies obtained in [9].

COROLLARY A.1. *Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism. Then*

$$h^u_{\text{top}}(f) = \sup\{h^u_\mu(f) : \mu \in \mathcal{M}_f(M)\}.$$

Moreover,

$$h^u_{\text{top}}(f) = \sup\{h^u_\nu(f) : \nu \in \mathcal{M}^e_f(M)\}.$$

Let $P(f, \varphi)$ be the classical topological pressure of f associated to the potential φ (cf. [21, Ch. 9]). By the definition of $P^u(f, \varphi)$ and Theorem A, we have the following facts.

COROLLARY A.2. *We have $P^u(f, \varphi) \le P(f, \varphi)$.*

If f is $C^{1+\epsilon}$, the equality holds if there is no positive Lyapunov exponent in the center direction at ν-a.e. with respect to any ergodic measure ν.

The variational principle (Theorem A) gives a natural way of selecting members of $\mathcal{M}_f(M)$. The following concept of u-equilibrium state generalizes measure of maximal unstable entropy.

Definition 2.3. Let $\varphi \in C(M, \mathbb{R})$. A member μ of $\mathcal{M}_f(M)$ is called a *u-equilibrium state for φ* if $P^u(f, \varphi) = h^u_\mu(f) + \int \varphi \, d\mu$.

Let $\mathcal{M}^u_\varphi(M, f)$ denote the set of all u-equilibrium states for φ.

A measure of maximal unstable entropy is a u-equilibrium state for the potential 0. A significant result in [9] is that the unstable metric entropy function is upper semicontinuous (cf. [9, Proposition 2.15], which is restated in Lemma 4.4 below). Therefore, a u-equilibrium state should always exist for partially hyperbolic diffeomorphisms. Furthermore, $\mathcal{M}^u_\varphi(M, f)$ has the following non-trivial properties.

THEOREM B.

(1) *$\mathcal{M}^u_\varphi(M, f)$ is convex.*
(2) *$\mathcal{M}^u_\varphi(M, f)$ is non-empty and compact.*
(3) *The extreme points of $\mathcal{M}^u_\varphi(M, f)$ are precisely the ergodic members of $\mathcal{M}^u_\varphi(M, f)$.*
(4) *If $\varphi, \psi \in C(M, \mathbb{R})$ and there exists $c \in \mathbb{R}$ such that $\varphi - \psi - c$ belongs to the closure of the set $\{h \circ f - h : h \in C(M, \mathbb{R})\}$ in $C(M, \mathbb{R})$, then $\mathcal{M}^u_\varphi(M, f) = \mathcal{M}^u_\psi(M, f)$.*

Gibbs u-states form a special class of invariant probability measures on M whose conditional measures along unstable leaves are absolutely continuous with respect to the Lebesgue measure on the leaves (cf. [1, 2], [14], see also [1, p. 221]). More precisely, we have the definition below.

Recall that a partition η is subordinate to unstable manifolds of f with respect to a measure μ if for μ-a.e. x, $\eta(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$. Also recall that for a measurable partition η of M, the conditional measures of a probability measure μ relative to a measurable partition η are denoted by $\{\mu^\eta_x\}$. Denote by m^u_x the Lebesgue measure on $W^u(x)$ induced by the intrinsic Riemannian structure on $W^u(x)$.

Definition 2.4. Let f be a $C^{1+\epsilon}$-partially hyperbolic diffeomorphism. $\mu \in \mathcal{M}_f(M)$ is called a *Gibbs u-state* of f if for every measurable partition η subordinate to unstable manifolds of f, μ -a.e. $x \in M$, μ^η_x is absolutely continuous with respect to m^u_x, that is, $\mu^\eta_x \ll m^u_x$ for μ -a.e. x.

For a C^1 partially hyperbolic diffeomorphism $f : M \to M$, denote $\varphi^u(x) = -\log|\det Df|_{E^u(x)}|$. Using some results in [24], we can relate the u-equilibrium states associated to φ^u to the Gibbs u-states of f.

THEOREM C. *Let f be $C^{1+\epsilon}$ and $\mu \in \mathcal{M}_f(M)$. Then μ is a Gibbs u-state of f if and only if μ is a u-equilibrium state of φ^u.*

COROLLARY C.1. *If f is $C^{1+\epsilon}$, then $P^u(f, \varphi^u) = 0$.*

COROLLARY C.2. *There always exists a Gibbs u-state for any $C^{1+\epsilon}$-partially hyperbolic diffeomorphism.*

Corollary C.2 can be easily extended to the partially hyperbolic attractor case, which recovers the existence result proved in [14].

It is well known that the (classical) topological pressure $P(f, \cdot)$ determines the set $\mathcal{M}_f(M)$ and the entropy $h_\mu(f)$ for all $\mu \in \mathcal{M}_f(M)$, in the sense of [21]. We recall the precise meaning as follows. A *finite signed measure* on M is a countably additive map

$\mu : \mathcal{B}(M) \to \mathbb{R}$, where $\mathcal{B}(M)$ is the σ-algebra of Borel subsets of M. Then $\mu \in \mathcal{M}_f(M)$ if and only if $\int_M \varphi \, d\mu \leq P(f, \varphi)$ for all $\varphi \in C(M, \mathbb{R})$. Moreover, $h_\nu(f) = \inf\{P(f, \varphi) - \int_M \varphi \, d\nu : \varphi \in C(M, \mathbb{R})\}$ holds if and only if the entropy map $\mu \mapsto h_\mu(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semicontinuous at ν. Rather surprisingly, the analogue holds for unstable pressure: the unstable topological pressure $P^u(f, \cdot)$, which might be considered as a partial pressure of the system, also determines the set $\mathcal{M}_f(M)$ and the entropy $h^u_\mu(f)$ for all $\mu \in \mathcal{M}_f(M)$. Moreover, we have a cleaner result since the unstable entropy map $\mu \mapsto h^u_\mu(f)$ is always upper semicontinuous.

THEOREM D.
(1) *Let $\mu : \mathcal{B}(M) \to \mathbb{R}$ be a finite signed measure. Then $\mu \in \mathcal{M}_f(M)$ if and only if $\int_M \varphi \, d\mu \leq P^u(f, \varphi)$, for all $\varphi \in C(M, \mathbb{R})$.*
(2) *Let $\nu \in \mathcal{M}_f(M)$. Then*

$$h^u_\nu(f) = \inf \left\{ P^u(f, \varphi) - \int_M \varphi \, d\nu : \varphi \in C(M, \mathbb{R}) \right\}.$$

As the existence of the u-equilibrium state has been guaranteed by the upper semicontinuity of the unstable entropy map, it is natural to ask when the u-equilibrium state is unique. This question is very subtle and already attracts a lot of interest in the case of the classical pressure. In this paper, we study the differentiability properties of the unstable pressure and their relations to the uniqueness of the u-equilibrium state. Such an approach is developed in [22] for the classical pressure.

To start with, we define a notion of tangent functional to the convex function $P^u(f, \cdot) : C(M, \mathbb{R}) \to \mathbb{R}$, which is closely related to the u-equilibrium state. The (classical) tangent functional can be found in [21, Definition 9.9].

Definition 2.5. Let $\varphi \in C(M, \mathbb{R})$. A *u-tangent functional to $P^u(f, \cdot)$ at φ* is a finite signed measure $\mu : \mathcal{B}(M) \to \mathbb{R}$ such that

$$P^u(f, \varphi + \psi) - P^u(f, \varphi) \geq \int_M \psi \, d\mu \quad \text{for all } \psi \in C(M, \mathbb{R}).$$

Let $t^u_\varphi(M, f)$ denote the set of all u-tangent functionals to $P^u(f, \cdot)$ at φ.

For the classical tangent functionals and equilibrium states, one has $\mathcal{M}_\varphi(M, f) \subset t_\varphi(M, f)$. The equality $\mathcal{M}_\varphi(M, f) = t_\varphi(M, f)$ holds under the assumption that $\mu \mapsto h_\mu(f)$ is upper semicontinuous at all the members of $t_\varphi(M, f)$ (cf. [21]). The assumption always holds for the u-tangent functionals.

THEOREM E. *Let $\varphi \in C(M, \mathbb{R})$, then $\mathcal{M}^u_\varphi(M, f) = t^u_\varphi(M, f)$.*

By a classical theorem in functional analysis (cf. [6]), the convex function $P^u(f, \cdot)$ on $C(M, \mathbb{R})$ has a unique tangent functional at a dense subset of $C(M, \mathbb{R})$. Combining with Theorem E, we have the following.

COROLLARY E.1. *The set $\{\varphi \in C(M, \mathbb{R}) : \mathcal{M}^u_\varphi(M, f)$ has a unique member$\}$ is dense in $C(M, \mathbb{R})$.*

To give sufficient conditions for a continuous function to have a unique u-equilibrium state, we consider two types of differentiability of unstable pressure.

Definition 2.6. The unstable topological pressure $P^u(f, \cdot) : C(M, \mathbb{R}) \to \mathbb{R}$ is said to be *Gateaux differentiable at* φ if

$$\lim_{t\to 0} \frac{1}{t}(P^u(f, \varphi + t\psi) - P^u(f, \varphi))$$

exists for any $\psi \in C(M, \mathbb{R})$.

THEOREM F. *$P^u(f, \cdot)$ is Gateaux differentiable at φ if and only if there is a unique unstable tangent functional to $P^u(f, \cdot)$ at φ.*

Combining Theorems E and F, we have the following.

COROLLARY F.1. *$P^u(f, \cdot)$ is Gateaux differentiable at φ if and only if there is a unique u-equilibrium state of φ.*

Now we consider the Fréchet differentiability of unstable topological pressure.

Definition 2.7. $P^u(f, \cdot) : C(M, \mathbb{R}) \to \mathbb{R}$ is said to be *Fréchet differentiable at* φ if there exists $\gamma \in C(M, \mathbb{R})^*$ such that

$$\lim_{\psi\to 0} \frac{|P^u(f, \varphi + \psi) - P^u(f, \varphi) - \gamma(\psi)|}{\|\psi\|} = 0.$$

Let $\mu_n \to \mu$ denote the convergence in weak* topology and $\|\mu_n - \mu\| \to 0$ the convergence in norm topology on $\mathcal{M}_f(M)$. The norm topology here is induced by the metric $\|\mu_1 - \mu_2\| := \sup\{|\int \varphi\, d\mu_1 - \int \varphi\, d\mu_2| : \varphi \in C(M, \mathbb{R}), \|\varphi\| \le 1\}$. We have the following equivalent ways to describe Fréchet differentiability of $P^u(f, \cdot)$.

THEOREM G. *The following statements are equivalent to each other.*

(1) *$P^u(f, \cdot)$ is Fréchet differentiable at φ.*

(2) *There exists a measure $\mu_\varphi \in \mathcal{M}_f(M)$ such that whenever $(\mu_n) \subset \mathcal{M}_f(M)$ with $h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_n \to P^u(f, \varphi)$, we have $\|\mu_n - \mu_\varphi\| \to 0$ as $n \to \infty$.*

(3) *$t^u_\varphi(M, f)$ has a unique member μ_φ and*

$$P^u(f, \varphi) > \sup\left\{h^u_\mu(f) + \int_M \varphi\, d\mu : \mu \text{ is ergodic and } \mu \ne \mu_\varphi\right\}.$$

(4) *$P^u(f, \cdot)$ is affine in a neighborhood of φ.*

(5) *$t^u_\varphi(M, f)$ has a unique member μ_φ and* $\sup\{\|\mu - \mu_\varphi\| : \mu \in t^u_{\varphi+\psi}(M, f)\} \to 0$ *as* $\psi \to 0$.

(6) *$t^u_\varphi(M, f)$ has a unique member μ_φ and* $\inf\{\|\mu - \mu_\varphi\| : \mu \in t^u_{\varphi+\psi}(M, f)\} \to 0$ *as* $\psi \to 0$.

(7) *$t^u_\varphi(M, f)$ has a unique member μ_φ and there is a weak* neighborhood V of μ_φ such that*

$$h^u_{\mu_\varphi}(f) > \sup\{h^u_\mu(f) : \mu \in V \text{ is ergodic and } \mu \ne \mu_\varphi\}.$$

It follows that Fréchet differentiability of $P^u(f, \cdot)$ implies the uniqueness of the u-equilibrium state. It is also clear that Fréchet differentiability of $P^u(f, \cdot)$ is stronger than Gateaux differentiability of $P^u(f, \cdot)$, either by the definitions or by Theorems F and G.

As a corollary of Theorem G(4), we have the following.

COROLLARY G.1. *The set of $\varphi \in C(M, \mathbb{R})$ such that $P^u(f, \cdot)$ is Fréchet differentiable at φ is open in $C(M, \mathbb{R})$.*

3. *Unstable topological pressure*

In this section, we redefine the unstable topological pressure via spanning sets and open covers, and discuss its basic properties.

3.1. *Definition using spanning sets.*

Recall that unstable topological pressure is defined in Definition 2.2 using (n, ϵ) u-separated sets. We can also define unstable topological pressure by using (n, ϵ) u-spanning sets as follows.

A set $F \subset W^u(x)$ is called an (n, ϵ) *u-spanning set* of $\overline{W^u(x, \delta)}$ if $\overline{W^u(x, \delta)} \subset \bigcup_{y \in F} B_n^u(y, \epsilon)$, where $B_n^u(y, \epsilon) = \{z \in W^u(x) : d_n^u(y, z) \leq \epsilon\}$ is the (n, ϵ) u-Bowen ball around y. Put

$$Q^u(f, \varphi, \epsilon, n, x, \delta) := \inf \left\{ \sum_{x \in F} \exp((S_n\varphi)(x)) : F \text{ is an } (n, \epsilon) \text{ u-spanning subset of } \overline{W^u(x, \delta)} \right\}.$$

Then, instead of Definition 2.2, we can also define

$$P^u(f, \varphi, \overline{W^u(x, \delta)}) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q^u(f, \varphi, \epsilon, n, x, \delta).$$

It is standard to verify that these two definitions for $P^u(f, \varphi, \overline{W^u(x, \delta)})$ coincide.

The following lemma is useful.

LEMMA 3.1. $P^u(f, \varphi) = \sup_{x \in M} P^u(f, \varphi, \overline{W^u(x, \delta)})$ *for any* $\delta > 0$.

Proof. It is easy to see that $P^u(f, \varphi) \leq \sup_{x \in M} P^u(f, \varphi, \overline{W^u(x, \delta)})$ for any $\delta > 0$ since $\delta \mapsto \sup_{x \in M} P^u(f, \varphi, \overline{W^u(x, \delta)})$ is increasing.

Let us prove the other direction for some fixed $\delta > 0$. For any $\rho > 0$, there exists $y \in M$ such that

$$\sup_{x \in M} P^u(f, \varphi, \overline{W^u(x, \delta)}) \leq P^u(f, \varphi, \overline{W^u(y, \delta)}) + \frac{\rho}{3}. \tag{3.1}$$

Pick $\epsilon_0 > 0$ such that

$$\begin{aligned} P^u(f, \varphi, \overline{W^u(y, \delta)}) &= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q^u(f, \varphi, \epsilon, n, y, \delta) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log Q^u(f, \varphi, \epsilon_0, n, y, \delta) + \frac{\rho}{3}. \end{aligned} \tag{3.2}$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

We can also choose $\delta_1 > 0$ small enough such that $\delta_1 < \delta$ and

$$P^u(f,\varphi) \geq \sup_{x\in M} P^u(f,\varphi,\overline{W^u(x,\delta_1)}) - \frac{\rho}{3}. \tag{3.3}$$

Then there exist $y_i \in \overline{W^u(y,\delta)}$, $1 \leq i \leq N$, where N only depends on δ, δ_1 and the Riemannian structure on $\overline{W^u(y,\delta)}$, such that

$$\overline{W^u(y,\delta)} \subset \bigcup_{i=1}^{N} \overline{W^u(y_i,\delta_1)}. \tag{3.4}$$

Then we have

$$\begin{aligned}
&\sup_{x\in M} P^u(f,\varphi,\overline{W^u(x,\delta)})\\
&\quad\leq P^u(f,\varphi,\overline{W^u(y,\delta)}) + \frac{\rho}{3} \quad \text{by (3.1)}\\
&\quad\leq \limsup_{n\to\infty} \frac{1}{n}\log Q^u(f,\varphi,\epsilon_0,n,y,\delta) + \frac{2\rho}{3} \quad \text{by (3.2)}\\
&\quad\leq \limsup_{n\to\infty} \frac{1}{n}\log\bigg(\sum_{i=1}^{N} Q^u(f,\varphi,\epsilon_0,n,y_i,\delta_1)\bigg) + \frac{2\rho}{3} \quad \text{by (3.4)}\\
&\quad\leq \limsup_{n\to\infty} \frac{1}{n}\log NQ^u(f,\varphi,\epsilon_0,n,y_j,\delta_1) + \frac{2\rho}{3} \quad \text{for some } 1\leq j\leq N\\
&\quad= \limsup_{n\to\infty} \frac{1}{n}\log Q^u(f,\varphi,\epsilon_0,n,y_j,\delta_1) + \frac{2\rho}{3}\\
&\quad\leq \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log Q^u(f,\varphi,\epsilon,n,y_j,\delta_1) + \frac{2\rho}{3}\\
&\quad= P^u(f,\varphi,\overline{W^u(y_j,\delta_1)}) + \frac{2\rho}{3}\\
&\quad\leq \sup_{x\in M} P^u(f,\varphi,\overline{W^u(x,\delta_1)}) + \frac{2\rho}{3}\\
&\quad\leq P^u(f,\varphi) + \rho \quad \text{by (3.3)}.
\end{aligned}$$

Since $\rho > 0$ is arbitrary, we have $\sup_{x\in M} P^u(f,\varphi,\overline{W^u(x,\delta)}) \leq P^u(f,\varphi)$. □

3.2. *Definition using open covers.* We proceed to define the unstable topological pressure by using open covers. Let $\mathcal{C}_M$ denote the set of Borel covers of M and $\mathcal{C}_M^o \subset \mathcal{C}_M$ the set of open covers of M. Given $\mathcal{U} \in \mathcal{C}_M$, denote $\mathcal{U}_m^n := \bigvee_{i=m}^n f^{-i}\mathcal{U}$. Put

$$p^u(f,\varphi,\mathcal{U},n,x,\delta) := \inf\bigg\{\sum_{B\in\mathcal{V}} \sup_{y\in B\cap\overline{W^u(x,\delta)}} \exp((S_n\varphi)(y)) : \mathcal{V}\in\mathcal{C}_M, \mathcal{V}\succeq \mathcal{U}_0^{n-1}\bigg\}.$$

If $B \cap \overline{W^u(x,\delta)} = \emptyset$, we set $\sup_{y\in B\cap\overline{W^u(x,\delta)}} \exp((S_n\varphi)(y)) = 0$.

Definition 3.2. We define

$$\widetilde{P}^u(f,\varphi) := \lim_{\delta\to 0}\sup_{x\in M} \widetilde{P}^u(f,\varphi,\overline{W^u(x,\delta)}),$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

where

$$\widetilde{P}^u(f, \varphi, \overline{W^u(x, \delta)}) := \sup_{\mathcal{U} \in \mathcal{C}_M^o} \limsup_{n\to\infty} \frac{1}{n} \log p^u(f, \varphi, \mathcal{U}, n, x, \delta).$$

Remark 3.3. It is not clear whether the sequence log $p^u(f, \varphi, \epsilon, n, x, \delta)$ is subadditive or not, so we have used lim sup in the definition above. This is one of the main differences from the case for classical topological pressure.

Observe that for $\delta > 0$ small enough, there exists $C > 1$ such that for any $x \in M$,

$$d(y, z) \le d^u(y, z) \le Cd(y, z) \quad \text{for any } y, z \in \overline{W^u(x, \delta)} \tag{3.5}$$

since M is compact and W^u is a continuous foliation. By some similar arguments to those in the proofs of [**21**], we can verify that Definitions 2.2 and 3.2 for unstable topological pressure coincide.

PROPOSITION 3.4. *We have* $\widetilde{P}^u(f, \varphi, \overline{W^u(x, \delta)}) = P^u(f, \varphi, \overline{W^u(x, \delta)})$. *As a consequence,*

$$\widetilde{P}^u(f, \varphi) = P^u(f, \varphi).$$

3.3. *Basic properties of unstable topological pressure.* Here we list some properties of unstable topological pressure. The proof is straightforward by definition and hence is omitted.

PROPOSITION 3.5. *If* $\varphi, \psi \in C(M, \mathbb{R})$ *with norm* $\|\cdot\|$ *and* $c \in \mathbb{R}$, *then the following statements are true.*

(1) $P^u(f, 0) = h^u_{\text{top}}(f)$.
(2) $P^u(f, \varphi + c) = P^u(f, \varphi) + c$.
(3) $\varphi \le \psi$ *implies that* $P^u(f, \varphi) \le P^u(f, \psi)$. *In particular,* $h^u_{\text{top}}(f) + \inf \varphi \le P^u(f, \varphi) \le h^u_{\text{top}}(f) + \sup \varphi$.
(4) $|P^u(f, \varphi) - P^u(f, \psi)| \le \|\varphi - \psi\|$.
(5) $P^u(f, \cdot)$ *is convex.*
(6) $P^u(f, \varphi + h \circ f - h) = P^u(f, \varphi)$. *Moreover, if* $\varphi - \psi$ *belongs to the closure of the set* $\{h \circ f - h : h \in C(M, \mathbb{R})\}$ *in* $C(M, \mathbb{R})$, *then* $P^u(f, \psi) = P^u(f, \varphi)$.
(7) $P^u(f, \varphi + \psi) \le P^u(f, \varphi) + P^u(f, \psi)$.
(8) $P^u(f, c\varphi) \le cP^u(f, \varphi)$ *if* $c \ge 1$ *and* $P^u(f, c\varphi) \ge cP^u(f, \varphi)$ *if* $c \le 1$.
(9) $|P^u(f, \varphi)| \le P^u(f, |\varphi|)$.

4. *The variational principle*

4.1. *Some properties of unstable metric entropy.* In this subsection, we collect some important properties of unstable metric entropy proved in [**9**]. In particular, they will be used in the proof of the variational principle (Theorem A) and in describing the set $\mathcal{M}^u_\varphi(M, f)$ in Theorem B.

LEMMA 4.1. [**9**, Corollary A.2] $h^u_\mu(f) = h_\mu(f, \alpha|\eta) = \lim_{n\to\infty}(1/n)H_\mu(\alpha_0^{n-1}|\eta)$ *for any* $\alpha \in \mathcal{P}$ *and* $\eta \in \mathcal{P}^u$.

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

LEMMA 4.2. [9, Corollary 3.2] *For any $\eta \in \mathcal{P}^u$ subordinate to unstable manifolds and any $\epsilon > 0$,*

$$h_\mu(f|\eta) = \lim_{n\to\infty} -\frac{1}{n}\log \mu_x^\eta(B_n^u(x,\epsilon)), \quad \mu\text{-a.e. } x.$$

LEMMA 4.3. [9, Proposition 2.14] *For any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^u$, the map $\mu \mapsto H_\mu(\alpha|\eta)$ from $\mathcal{M}(M)$ to $\mathbb{R}^+ \cup \{0\}$ is concave.*

Furthermore, the map $\mu \mapsto h_\mu^u(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is affine.

Recall that for each partition $\alpha \in \mathcal{P}$, the partition ζ given by $\zeta(x) = \alpha(x) \cap W^u_{\mathrm{loc}}(x)$ for any $x \in M$ is denoted by α^u. Conversely, for each partition $\eta \in \mathcal{P}^u$, there is a partition $\beta \in \mathcal{P}$ such that $\eta(x) = \beta(x) \cap W^u_{\mathrm{loc}}(x)$ for any $x \in M$. Denote such β by $\eta^{\not u}$.

LEMMA 4.4. [9, Proposition 2.15]

(a) *Let $\nu \in \mathcal{M}(M)$. For any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^u$ with $\mu(\partial\alpha) = 0$ and $\mu(\partial\eta^{\not u}) = 0$, the map $\mu \mapsto H_\mu(\alpha|\eta)$ from $\mathcal{M}(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semicontinuous at μ, i.e.*

$$\limsup_{\nu\to\mu} H_\nu(\alpha|\eta) \le H_\mu(\alpha|\eta).$$

(b) *The unstable entropy map $\mu \mapsto h_\mu^u(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semicontinuous at μ, i.e.*

$$\limsup_{\nu\to\mu} h_\nu^u(f) \le h_\mu^u(f).$$

The second part of the above lemma also follows from [24, Theorem D].

4.2. *Proof of the variational principle.* At first, we prove Proposition 4.6 stated below, which is one inequality of the variational principle (Theorem A). The following lemma is well known.

LEMMA 4.5. *Suppose that $0 \le p_1, \ldots, p_m \le 1$, $s = p_1 + \cdots + p_m$ and $a_1, \ldots, a_m \in \mathbb{R}$. Then*

$$\sum_{i=1}^m p_i(a_i - \log p_i) \le s\bigg(\log \sum_{i=1}^m e^{a_i} - \log s\bigg).$$

The above lemma is almost identical to [3], except that we have removed the condition $s \le 1$.

Proof. If $s = 0$, then the inequality holds trivially. Suppose that $s > 0$. Let $p_i' = p_i/s$. Then $\sum_{i=1}^m p_i' = 1$ and we can apply [21, Lemma 9.9] to get

$$\sum_{i=1}^m \frac{p_i}{s}\bigg(a_i - \log\frac{p_i}{s}\bigg) \le \log\bigg(\sum_{i=1}^m e^{a_i}\bigg).$$

Simplifying it, we have the inequality in the lemma. □

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

PROPOSITION 4.6. *Let μ be any f-invariant probability measure. Then*

$$h^u_\mu(f) + \int_M \varphi \, d\mu \le P^u(f, \varphi).$$

Proof. Let $\mu = \int_{\mathcal{M}^e_f(M)} \nu \, d\tau(\nu)$ be the unique ergodic decomposition where τ is a probability measure on the Borel subsets of $\mathcal{M}_f(M)$ and $\tau(\mathcal{M}^e_f(M)) = 1$. Since $\mu \mapsto h^u_\mu(f)$ is affine and upper semicontinuous by Lemmas 4.3 and 4.4, so is $h^u_\mu(f) + \int_M \varphi \, d\mu$ and hence

$$h^u_\mu(f) + \int_M \varphi \, d\mu = \int_{\mathcal{M}^e_f(M)} \left(h^u_\nu(f) + \int_M \varphi \, d\nu \right) d\tau(\nu) \tag{4.6}$$

by a classical result in convex analysis (cf. [5, Fact A.2.10 on p. 356]). So, we only need to prove the proposition for ergodic measures.

Suppose that μ is ergodic. Let $\rho > 0$ be arbitrary. Take $\eta \in \mathcal{P}^u$ subordinate to unstable manifolds and then take $\varepsilon > 0$. By Lemma 4.2, we have

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu^\eta_y(B^u_n(y, \varepsilon)) = h^u_\mu(f|\eta), \quad \mu\text{-a.e. } y.$$

Hence, for μ-a.e. y, there exists $N(y) = N(y, \varepsilon) > 0$ such that if $n \ge N(y)$, then

$$\mu^\eta_y(B^u_n(y, \varepsilon)) \le e^{-n(h^u_\mu(f|\eta)-\rho)}$$

and

$$\frac{1}{n}(S_n\varphi)(y) \ge \int_M \varphi \, d\mu - \rho. \tag{4.7}$$

Denote $E_n = E_n(\varepsilon) = \{y \in M : N(y) = N(y, \varepsilon) \le n\}$. Then $\mu\big(\cup_{n=1}^\infty E_n\big) = 1$. So, there exists $n > 0$ large enough such that $\mu(E_n) > 1 - \rho$. Hence, there exists $x \in M$ such that $\mu^\eta_x(E_n) = \mu^\eta_x(E_n \cap \eta(x)) > 1 - \rho$. Fix such n and x. If $y \in \eta(x)$, $\mu^\eta_y = \mu^\eta_x$. We have

$$\mu^\eta_x(B^u_n(y, \varepsilon)) \le e^{-n(h_\mu(f|\eta)-\rho)} \quad \text{for all } y \in E_n \cap \eta(x). \tag{4.8}$$

Now we take $\delta > 0$ such that $W^u(x, \delta) \supset \eta(x)$. Let F be an $(n, \varepsilon/2)$ u-spanning set of $\overline{W^u(x, \delta)} \cap E_n$ satisfying

$$\overline{W^u(x, \delta)} \cap E_n \subset \bigcup_{z\in F} B^u_n(z, \varepsilon/2)$$

and $B^u_n(z, \varepsilon/2) \cap E_n \ne \emptyset$ for any $z \in F$. Let $y(z)$ be an arbitrary point in $B^u_n(z, \varepsilon/2) \cap E_n$. We have

$$\begin{aligned} 1 - \rho < \mu^\eta_x(\overline{W^u(x, \delta)} \cap E_n) &\le \mu^\eta_x\Big(\bigcup_{z\in F} B^u_n(z, \varepsilon/2) \Big) \\ &\le \sum_{z\in F} \mu^\eta_x(B^u_n(z, \varepsilon/2)) \le \sum_{z\in F} \mu^\eta_x(B^u_n(y(z), \varepsilon)). \end{aligned} \tag{4.9}$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

Using (4.7), (4.8) and then applying Lemma 4.5 with $p_i = \mu_x^\eta(B_n^u(y(z),\varepsilon))$, $a_i = (S_n\varphi)(y(z))$, we have

$$\sum_{z\in F}\mu_x^\eta(B_n^u(y(z),\varepsilon))\bigg(n\bigg(\int_M \varphi\,d\mu-\rho\bigg)+n(h_\mu^u(f|\eta)-\rho)\bigg)$$
$$\le \sum_{z\in F}\mu_x^\eta(B_n^u(y(z),\varepsilon))((S_n\varphi)(y(z))-\log\mu_x^\eta(B_n^u(y(z),\varepsilon)))$$
$$\le \bigg(\sum_{z\in F}\mu_x^\eta(B_n^u(y(z),\varepsilon))\bigg)\bigg(\log\sum_{z\in F}\exp((S_n\varphi)(y(z)))-\log\sum_{z\in F}\mu_x^\eta(B_n^u(y(z),\varepsilon))\bigg).$$

Then, combining (4.9), we get

$$n\bigg(\int_M\varphi\,d\mu-\rho\bigg)+n(h_\mu^u(f|\eta)-\rho)$$
$$\le \log\sum_{z\in F}\exp((S_n\varphi)(y(z)))-\log\sum_{z\in F}\mu_x^\eta(B_n^u(y(z),\varepsilon))$$
$$\le \log\sum_{z\in F}\exp((S_n\varphi)(y(z)))-\log(1-\rho). \tag{4.10}$$

Let $\tau_\varepsilon := \{|\varphi(x)-\varphi(y)| : d(x,y)\le\varepsilon\}$. Then, for any $z\in F$, $\exp((S_n\varphi)(y(z)))\le \exp((S_n\varphi)(z)+n\tau_\varepsilon)$. Dividing by n and taking the lim sup on both sides of (4.10), we have

$$\int_M\varphi\,d\mu+h_\mu^u(f|\eta)-2\rho\le \limsup_{n\to\infty}\frac{1}{n}\log\sum_{z\in F}\exp((S_n\varphi)(z))+\tau_\varepsilon.$$

Moreover, we can choose a sequence of F such that

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{z\in F}\exp((S_n\varphi)(z))\le P^u(f,\varphi).$$

Since $\rho>0$ is arbitrary and $\tau_\varepsilon\to 0$ as $\varepsilon\to 0$, one has $\int_M\varphi\,d\mu+h_\mu^u(f|\eta)\le P^u(f,\varphi)$. □

Proof of Theorem A. We first prove that for any $\rho>0$, there exists $\mu\in\mathcal{M}_f(M)$ such that $h_\mu^u(f)+\int_M\varphi\,d\mu\ge P^u(f,\varphi)-\rho$. Combining with Proposition 4.6, we obtain the first equality in Theorem A.

For some $\delta>0$ small enough, we can find a point $x\in M$ such that

$$P^u(f,\varphi,\overline{W^u(x,\delta)})\ge P^u(f,\varphi)-\rho.$$

Take $\varepsilon>0$ small enough. Let E_n be an (n,ε) u-separated set of $\overline{W^u(x,\delta)}$ with cardinality $N^u(f,\varepsilon,n,x,\delta)$ such that

$$\log\sum_{y\in E_n}\exp((S_n\varphi)(y))\ge \log P^u(f,\varphi,\varepsilon,n,x,\delta)-1.$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

Define

$$\nu_n := \frac{\sum_{y\in E_n} \exp((S_n\varphi)(y))\delta_y}{\sum_{z\in E_n} \exp((S_n\varphi)(z))}$$

and

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} f^i \nu_n.$$

Since the set $\mathcal{M}(M)$ of all probability measures on M is a compact space with weak* topology, there exists a subsequence $\{n_k\}$ of natural numbers such that $\lim_{k\to\infty} \mu_{n_k} = \mu$. Obviously $\mu \in \mathcal{M}_f(M)$.

We can choose a partition $\eta \in \mathcal{P}^u$ such that $W^u(x, \delta) \subset \eta(x)$ (by shrinking δ if necessary). That is, $W^u(x, \delta)$ is contained in a single element of η. Then choose $\alpha \in \mathcal{P}$ such that $\mu(\partial\alpha) = 0$ and $\operatorname{diam}(\alpha) < \varepsilon/C$, where $C > 1$ is as in (3.5). Hence, we have

$$\begin{aligned} H_{\nu_n}(\alpha_0^{n-1}|\eta) + \int_M (S_n\varphi)\, d\nu_n &= \sum_{y\in E_n} \nu_n(\{y\})(-\log \nu_n(\{y\}) + (S_n\varphi)(y)) \\ &= \log \sum_{y\in E_n} \exp((S_n\varphi)(y)). \end{aligned}$$

Fix a natural number $q > 1$. For any natural number $n > q$, $j = 0, 1, \ldots, q-1$, put $a(j) = [n - j/q]$, where $[a]$ denotes the integer part of $a > 0$. Then

$$\bigvee_{i=0}^{n-1} f^{-i}\alpha = \bigvee_{r=0}^{a(j)-1} f^{-(rq+j)}\alpha_0^{q-1} \vee \bigvee_{t\in S_j} f^{-t}\alpha,$$

where $S_j = \{0, 1, \ldots, j-1\} \cup \{j + qa(j), \ldots, n-1\}$.

For a partition $\alpha \in \mathcal{P}$, denote by α^u the partition in $\mathcal{P}^u$ whose elements are given by $\alpha^u(x) = \alpha(x) \cap W^u_{\mathrm{loc}}(x)$. Note that

$$f^{rq}\bigg(\bigvee_{i=0}^{r-1} f^{-iq}\alpha_0^{q-1} \vee f^j\Big(\eta \vee \bigvee_{t\in S_j} f^{-t}\alpha\Big)\bigg) = f\alpha \vee \cdots \vee f^{rq+j}\alpha \vee f^{rq+j}\eta \geq f\alpha^u.$$

We get that

$$\begin{aligned} &H_\nu\bigg(\bigvee_{r=0}^{a(j)-1} f^{-rq}\alpha_0^{q-1} \Big| f^j\Big(\eta \vee \bigvee_{t\in S_j} f^{-t}\alpha\Big)\bigg) \\ &\quad\leq H_\nu(\alpha_0^{q-1}|f^j\eta) + \sum_{r=1}^{a(j)-1} H_{f^{rq}\nu}\bigg(\alpha_0^{q-1}\Big| f^{rq}\bigg(\bigvee_{i=0}^{r-1} f^{-iq}\alpha_0^{q-1} \vee f^j\Big(\eta \vee \bigvee_{t\in S_j} f^{-t}\alpha\Big)\bigg)\bigg) \\ &\quad\leq H_\nu(\alpha_0^{q-1}|f^j\eta) + \sum_{r=1}^{a(j)-1} H_{f^{rq}\nu}(\alpha_0^{q-1}|f\alpha^u). \end{aligned} \tag{4.11}$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

Also,

$$H_\nu\Big(\bigvee_{r=0}^{a(j)-1} f^{-(rq+j)}\alpha_0^{q-1}|\eta\vee\bigvee_{t\in S_j} f^{-t}\alpha\Big)$$
$$= H_{f^j\nu}\Big(\bigvee_{r=0}^{a(j)-1} f^{-rq}\alpha_0^{q-1}|f^j(\eta\vee\bigvee_{t\in S_j} f^{-t}\alpha)\Big). \tag{4.12}$$

Replacing ν by ν_n and $f^j\nu_n$ in (4.12) and (4.11), respectively, we get

$$\begin{aligned}
&\log\sum_{y\in E_n}\exp((S_n\varphi)(y))\\
&=\sum_{y\in E_n}\nu_n(\{y\})\big(-\log\nu_n(\{y\})+(S_n\varphi)(y)\big)\\
&=H_{\nu_n}(\alpha_0^{n-1}|\eta)+\int_M(S_n\varphi)\,d\nu_n\\
&=H_{\nu_n}\Big(\bigvee_{r=0}^{a(j)-1} f^{-(rq+j)}\alpha_0^{q-1}\vee\bigvee_{t\in S_j} f^{-t}\alpha|\eta\Big)+\int_M(S_n\varphi)\,d\nu_n\\
&\le\sum_{t\in S_j}H_{\nu_n}(f^{-t}\alpha|\eta)+H_{\nu_n}\Big(\bigvee_{r=0}^{a(j)-1} f^{-rq-j}\alpha_0^{q-1}\Big|\eta\vee\bigvee_{t\in S_j} f^{-t}\alpha\Big)+\int_M(S_n\varphi)\,d\nu_n\\
&\le\sum_{t\in S_j}H_{\nu_n}(f^{-t}\alpha|\eta)+H_{f^j\nu_n}\Big(\bigvee_{r=0}^{a(j)-1} f^{-rq}\alpha_0^{q-1}\Big|f^j(\eta\vee\bigvee_{t\in S_j} f^{-t}\alpha)\Big)+\int_M(S_n\varphi)\,d\nu_n\\
&\le\sum_{t\in S_j}H_{\nu_n}(f^{-t}\alpha|\eta)+H_{f^j\nu_n}(\alpha_0^{q-1}|f^j\eta)+\sum_{r=1}^{a(j)-1}H_{f^{rq+j}\nu_n}(\alpha_0^{q-1}|f\alpha^u)+\int_M(S_n\varphi)\,d\nu_n.
\end{aligned}$$

It is clear that $\mathrm{card}S_j\le 2q$. Denote by d the number of elements of α. Summing the inequalities over j from 0 to $q-1$ and dividing by n, by Lemma 4.3 we get

$$\begin{aligned}
&\frac{q}{n}\log\sum_{y\in E_n}\exp((S_n\varphi)(y))\\
&\le\frac{1}{n}\sum_{j=0}^{q-1}\sum_{t\in S_j}H_{\nu_n}(f^{-t}\alpha|\eta)+\frac{1}{n}\sum_{j=0}^{q-1}H_{f^j\nu_n}(\alpha_0^{q-1}|f^j\eta)\\
&\quad+\frac{1}{n}\sum_{i=0}^{n-1}H_{f^i\nu_n}(\alpha_0^{q-1}|f\alpha^u)+\frac{q}{n}\int_M(S_n\varphi)\,d\nu_n\\
&\le\frac{2q^2}{n}\log d+\frac{1}{n}\sum_{j=0}^{q-1}H_{f^j\nu_n}(\alpha_0^{q-1}|f^j\eta)+H_{\mu_n}(\alpha_0^{q-1}|f\alpha^u)+q\int_M\varphi\,d\mu_n.
\end{aligned}\tag{4.13}$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

Let $\{n_k\}$ be a sequence of natural numbers such that:

(1) $\mu_{n_k} \to \mu$ as $k \to \infty$;

(2) $\lim_{k\to\infty}(1/n_k) \log P^u(f, \varphi, \varepsilon, n_k, x, \delta) = \limsup_{n\to\infty}(1/n) \log P^u(f, \varphi, \varepsilon, n, x, \delta)$.

Since $\mu(\partial\alpha) = 0$ and μ is invariant, $\mu(\partial\alpha_0^{q-1}) = 0$ for any $q \in \mathbb{N}$. By Lemma 4.4,

$$\limsup_{k\to\infty} H_{\mu_{n_k}}(\alpha_0^{q-1} | f\alpha^u) \leq H_\mu(\alpha_0^{q-1} | f\alpha^u).$$

Thus, replacing n by n_k in (4.13) and letting $k \to \infty$, we get

$$q \limsup_{n\to\infty} \frac{1}{n} \log P^u(f, \varphi, \varepsilon, n, x, \delta) \leq H_\mu(\alpha_0^{q-1} | f\alpha^u) + q \int_M \varphi \, d\mu.$$

Then, by Lemma 4.1,

$$P^u(f, \varphi, \overline{W^u(x, \delta)}) \leq \lim_{q\to\infty} \frac{1}{q} H_\mu(\alpha_0^{q-1} | f\alpha^u) + \int_M \varphi \, d\mu = h_\mu^u(f) + \int_M \varphi \, d\mu.$$

Thus, $h_\mu^u(f) + \int_M \varphi \, d\mu \geq P^u(f, \varphi) - \rho$. Since ρ is arbitrary, we get by combining with Proposition 4.6,

$$P^u(f, \varphi) = \sup \left\{ h_\mu^u(f) + \int_M \varphi \, d\mu : \mu \in \mathcal{M}_f(M) \right\}.$$

We prove the second equation in Theorem A.

Let $\rho > 0$ be sufficiently small. Then there exists an invariant measure μ such that $h_\mu^u(f) + \int_M \varphi \, d\mu > P^u(f, \varphi) - \rho/2$. By (4.6), there exists an ergodic measure ν such that

$$h_\nu^u(f) + \int_M \varphi \, d\nu > h_\mu^u(f) + \int_M \varphi \, d\mu - \rho/2 > P^u(f, \varphi) - \rho.$$

Since ρ is arbitrary, we have

$$P^u(f, \varphi) = \sup \left\{ h_\mu^u(f) + \int_M \varphi \, d\mu : \mu \in \mathcal{M}_f^e(M) \right\}. \qquad \square$$

The proof of Corollary A.1 is straightforward and hence is omitted.

Proof of Corollary A.2. The inequality $P^u(f, \varphi) \leq P(f, \varphi)$ follows from the definition directly.

If f is $C^{1+\epsilon}$ and there is no positive Lyapunov exponent in the center direction, then, by the Ledrappier–Young formula [11] and [9, Theorem A], $h_\mu(f) = h_\mu^u(f)$ for any $\mu \in \mathcal{M}_f^e(M)$. Then, by Theorem A and the classical variational principle for pressure (cf. [21, Theorem 9.10]),

$$\begin{aligned} P^u(f, \varphi) &= \sup \left\{ h_\mu^u(f) + \int_M \varphi \, d\mu : \mu \in \mathcal{M}_f^e(M) \right\} \\ &= \sup \left\{ h_\mu(f) + \int_M \varphi \, d\mu : \mu \in \mathcal{M}_f^e(M) \right\} \\ &= P(f, \varphi). \end{aligned} \qquad \square$$

5. *U-equilibrium states*

In this section, we shall first give some fundamental properties for the set of u-equilibrium states, proving Theorem B. Then for the particular potential $\varphi^u = -\log|\det Df|_{E^u}|$ we relate the u-equilibrium states at φ^u to the Gibbs u-states of f.

5.1. *Properties of $\mathcal{M}^u_\varphi(M, f)$.*

Proof of Theorem B. It follows from Lemma 4.3 that $\mu \mapsto h^u_\mu(f) + \int_M \varphi\, d\mu$ is affine. Hence, (1) holds.

It follows from Lemma 4.4 that $\mu \mapsto h^u_\mu(f) + \int_M \varphi\, d\mu$ is upper semicontinuous. The set $\mathcal{M}^u_\varphi(M, f)$ is non-empty because an upper semicontinuous function on a compact space attains its supremum. If $\mu_n \in \mathcal{M}^u_\varphi(M, f)$ and $\mu_n \to \mu$ in $\mathcal{M}_f(M)$, then $h^u_\mu(f) + \int_M \varphi\, d\mu \geq \limsup_{n\to\infty} h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_n = P^u(f, \varphi)$. This together with Theorem A proves that $\mathcal{M}^u_\varphi(M, f)$ is compact. Thus (2) is proved.

If $\mu \in \mathcal{M}^u_\varphi(M, f)$ is ergodic, then it is an extreme point of $\mathcal{M}_f(M)$ and hence of $\mathcal{M}^u_\varphi(M, f)$. Now let $\mu \in \mathcal{M}^u_\varphi(M, f)$ be an extreme point of $\mathcal{M}^u_\varphi(M, f)$ and suppose that $\mu = p\mu_1 + (1-p)\mu_2$ for some $\mu_1, \mu_2 \in \mathcal{M}_f(M)$ and $p \in [0, 1]$. By Lemma 4.3, $P^u(f, \varphi) = h^u_\mu(f) + \int_M \varphi\, d\mu = p(h^u_{\mu_1}(f) + \int_M \varphi\, d\mu_1) + (1 - p)(h^u_{\mu_2}(f) + \int_M \varphi\, d\mu_2)$. By the variational principle, Theorem A, we must have $\mu_1, \mu_2 \in \mathcal{M}^u_\varphi(M, f)$. Hence, $\mu_1 = \mu_2 = \mu$ since μ is an extreme point of $\mathcal{M}^u_\varphi(M, f)$. It means that μ is an extreme point of $\mathcal{M}_f(M)$ as well. Thus, μ is ergodic. This proves (3).

Now we prove (4). By Proposition 3.5(2), (4) and (6), $P^u(f, \varphi) = P^u(f, \psi) + c$. On the other hand, it is easy to see that $\int_M \varphi\, d\mu = \int_M \psi\, d\mu + c$ and hence $h^u_\mu(f) + \int_M \varphi\, d\mu = h^u_\mu(f) + \int_M \psi\, d\mu + c$. Thus, $\mathcal{M}^u_\varphi(M, f) = \mathcal{M}^u_\psi(M, f)$. □

5.2. *Gibbs u-states.*

There are two leading cases for a potential φ. First, φ is the constant function 0. In this case, the unstable topological pressure is just the unstable topological entropy (Proposition 3.5(1)) and Theorem B(2) gives existence of measure of maximal unstable metric entropy.

Second, $\varphi^u = -\log|\det Df|_{E^u}|$. Theorem B(2) gives existence of u-equilibrium states with respect to φ^u. We start to prove Theorem C, which claims that when f is $C^{1+\epsilon}$ such u-equilibrium states coincide with Gibbs u-states first studied in [14].

LEMMA 5.1. (cf. [24, Proposition 5.2]) *If f is $C^{1+\epsilon}$ and $\mu \in \mathcal{M}_f(M)$, then*

$$h^u_\mu(f) \leq \int_M -\varphi^u\, d\mu.$$

The equality holds if and only if μ is a Gibbs u-state of f.

Remark 5.2. We have replaced $h_\mu(f, \mathcal{F}^u)$ in [24] by $h^u_\mu(f)$, where $\mathcal{F}^u$ is the unstable foliation of f. Indeed, see [9], we have $h^u_\mu(f) = h_\mu(f, \mathcal{F}^u)$. See also the discussion after Definition 2.1.

We characterize Gibbs u-states of f by u-equilibrium states of f with respect to φ^u.

Proof of Theorem C. By Theorem A and Lemma 5.1,

$$P^u(f,\varphi^u)=\sup\left\{h^u_\mu(f)+\int_M \varphi^u\,d\mu\right\}=0.$$

By Lemma 5.1 again, μ is a Gibbs u-state of f if and only if μ is a u-equilibrium state of φ^u. □

Corollary C.1 was already obtained in the proof of Theorem C.

Proof of Corollary C.2. Since a u-equilibrium state for any continuous function φ always exists by Theorem B(2), we know from Theorem C that a Gibbs u-state always exists. □

6. *Unstable topological pressure determines $\mathcal{M}_f(M)$*

Proof of Theorem D. We modify the proof [21, Theorem 9.11] for the unstable pressure case. If $\mu\in\mathcal{M}_f(M)$, then it is easy to see that $\int_M\varphi\,d\mu\le P^u(f,\varphi)$ by Theorem A. Now let $\mu:\mathcal{B}(M)\to\mathbb{R}$ be a finite signed measure such that $\int_M\varphi\,d\mu\le P^u(f,\varphi)$ for all $\varphi\in C(M,\mathbb{R})$. Firstly, we show that μ is a measure. Let $\varphi\ge 0$. If $\epsilon>0$ and $n>0$ is large enough, then, by Proposition 3.5(3),

$$\begin{aligned}\int_M n(\varphi+\epsilon)\,d\mu&=-\int_M -n(\varphi+\epsilon)\,d\mu\\&\ge -P^u(f,-n(\varphi+\epsilon))\\&\ge -(h^u_{\mathrm{top}}(f)+\sup(-n(\varphi+\epsilon)))\\&=-h^u_{\mathrm{top}}(f)+n\inf(\varphi+\epsilon)>0.\end{aligned}$$

Hence, $\int_M(\varphi+\epsilon)\,d\mu>0$. Since $\epsilon>0$ is arbitrary, $\int_M\varphi\,d\mu\ge 0$ and μ is a measure.

Next we show that μ is a probability measure. For $n\in\mathbb{Z}$, $\int_M n\,d\mu\le P^u(f,n)=h^u_{\mathrm{top}}(f)+n$. If $n>0$, then $\mu(M)\le(1/n)h^u_{\mathrm{top}}(f)+1$. Hence, $\mu(M)\le 1$ by letting $n\to\infty$. If $n<0$, then $\mu(M)\ge(1/n)h^u_{\mathrm{top}}(f)+1$. Letting $n\to-\infty$, $\mu(M)\ge 1$. It follows that $\mu(M)=1$.

At last we show that $\mu\in\mathcal{M}_f(M)$. For $n\in\mathbb{Z}$, $n\int_M(\varphi\circ f-\varphi)\,d\mu\le P^u(f,n(\varphi\circ f-\varphi))=h^u_{\mathrm{top}}(f)$ by Proposition 3.5(6) and (1). If $n>0$, then $\int_M(\varphi\circ f-\varphi)\,d\mu\le(1/n)h^u_{\mathrm{top}}(f)$. Hence, $\int_M(\varphi\circ f-\varphi)\,d\mu\le 0$ by letting $n\to\infty$. If $n<0$, then $\int_M(\varphi\circ f-\varphi)\,d\mu\ge(1/n)h^u_{\mathrm{top}}(f)$. Hence, $\int_M(\varphi\circ f-\varphi)\,d\mu\ge 0$ by letting $n\to-\infty$. Therefore, $\int_M\varphi\circ f\,d\mu=\int_M\varphi\,d\mu$. So, $\mu\in\mathcal{M}_f(M)$.

The proof is an adaption of that of [21, Theorem 9.12]. Recall that we already know that the unstable entropy map $\mu\mapsto h^u_\mu(f)$ is upper semicontinuous. The fact that $h^u_\nu(f)\le\inf\left\{P^u(f,\varphi)-\int_M\varphi\,d\nu:\varphi\in C(M,\mathbb{R})\right\}$ immediately follows from Theorem A. To prove the other direction, let $b>h^u_\nu(f)$. Put

$$C=\{(\mu,t)\in\mathcal{M}_f(M)\times\mathbb{R}:0\le t\le h^u_\mu(f)\}.$$

By Lemma 4.3, C is a convex subset of $C(M,\mathbb{R})^*\times\mathbb{R}$, where $C(M,\mathbb{R})^*$ is endowed with the weak* topology. Then $(\nu,b)\notin\bar{C}$ as $\mu\mapsto h^u_\mu(f)$ is upper semicontinuous at ν. By a classical result [6, p. 417], there is a continuous linear functional $F:C(M,\mathbb{R})^*\times\mathbb{R}\to\mathbb{R}$ such that $F(\mu,t)\le F(\nu,b)$ for any $(\mu,t)\in\bar{C}$. We can suppose that F has the form

$F(\mu, t) = \int_M \psi \, d\mu + td$ for some $\psi \in C(M, \mathbb{R})$ and $d \in \mathbb{R}$. Then $\int_M \psi \, d\mu + td < \int_M \psi \, d\nu + bd$ for any $(\mu, t) \in \bar{C}$. In particular, $\int_M \psi \, d\mu + h^u_\mu(f)d < \int_M \psi \, d\nu + bd$ for any $\mu \in \mathcal{M}_f(M)$. Setting $\mu = \nu$, we have $h^u_\nu(f)d < bd$. Hence, $d > 0$. We have

$$\int_M \frac{\psi}{d} \, d\mu + h^u_\mu(f) < \int_M \frac{\psi}{d} \, d\nu + b$$

for any $\mu \in \mathcal{M}_f(M)$. By the variational principle (Theorem A),

$$P^u\Big(f, \frac{\psi}{d}\Big) \leq \int_M \frac{\psi}{d} \, d\nu + b.$$

Then

$$b \geq P^u\Big(f, \frac{\psi}{d}\Big) - \int_M \frac{\psi}{d} \, d\nu \geq \inf\{P^u(f, \varphi) - \int_M \varphi \, d\nu : \varphi \in C(M, \mathbb{R})\}.$$

Therefore, $h^u_\nu(f) \geq \inf\{P^u(f, \varphi) - \int_M \varphi \, d\nu : \varphi \in C(M, \mathbb{R})\}$. □

7. *Differentiability properties of the unstable topological pressure*

In this section, we consider the differentiability properties of the unstable topological pressure. We shall first give the relation between the u-tangent functionals and the u-equilibrium states and then consider the Gateaux differentiability and Fréchet differentiability of the unstable topological pressure. The equivalence of the Gateaux differentiability of $P^u(f, \cdot)$ and the existence of the unique unstable tangent functional $P^u(f, \cdot)$ at a given φ is obtained and several necessary and sufficient conditions for $P^u(f, \cdot)$ to be Fréchet differentiable at a given φ are given.

7.1. *U-tangent functionals.*

Proof of Theorem E. First, we show that $t^u_\varphi(M, f) \subset \mathcal{M}^u_\varphi(M, f)$. Let $\mu \in t^u_\varphi(M, f)$. By Proposition 3.5(7), for all $\psi \in C(M, \mathbb{R})$,

$$\int_M \psi \, d\mu \leq P^u(f, \varphi + \psi) - P^u(f, \varphi) \leq P^u(f, \psi).$$

By Theorem D(1), $\mu \in \mathcal{M}_f(M)$.

Then, for all $\psi \in C(M, \mathbb{R})$,

$$P^u(f, \varphi + \psi) - P^u(f, \varphi) \geq \int_M \psi \, d\mu = \int_M (\varphi + \psi) \, d\mu - \int_M \varphi \, d\mu,$$

which implies that

$$P^u(f, \varphi + \psi) - \int_M (\varphi + \psi) \, d\mu \geq P^u(f, \varphi) - \int_M \varphi \, d\mu.$$

Since $\psi \in C(M, \mathbb{R})$ is arbitrary, one has

$$\inf\left\{P^u(f, h) - \int_M h \, d\mu : h \in C(M, \mathbb{R})\right\} \geq P^u(f, \varphi) - \int_M \varphi \, d\mu.$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

By Theorem D(2), we have $h^u_\mu(f) \ge P^u(f,\varphi) - \int_M \varphi\, d\mu$. Combining with the variational principle (Theorem A), we have $P^u(f,\varphi) = h^u_\mu(f) + \int_M \varphi\, d\mu$. Thus, $\mu \in \mathcal{M}^u_\varphi(M, f)$.

Conversely, if $\mu \in \mathcal{M}^u_\varphi(M, f)$, then $P^u(f,\varphi) = h^u_\mu(f) + \int_M \varphi\, d\mu$. We have

$$P^u(f,\varphi+\psi) - P^u(f,\varphi) \ge h^u_\mu(f) + \int_M (\varphi+\psi)\, d\mu - h^u_\mu(f) - \int_M \varphi\, d\mu$$
$$= \int_M \psi\, d\mu \quad \text{for all } \psi \in C(M,\mathbb{R}),$$

where the variational principle (Theorem A) is used in the first inequality. Therefore, $\mu \in t^u_\varphi(M, f)$. □

Remark 7.1. The proof of Theorem E is based on Theorem D. Our proof of $t^u_\varphi(M, f) \subset \mathcal{M}_f(M)$ using Theorem D(1) is much shorter than the one in [**21**, Theorem 9.14].

7.2. *Gateaux differentiability.* Since $P^u(f,\cdot)$ is convex (Proposition 3.5(5)), for any $\varphi, \psi \in C(M,\mathbb{R})$ the map $t \mapsto 1/t(P^u(f,\varphi+t\psi) - P^u(f,\varphi))$ is increasing and hence the following two limits exist.

Definition 7.2. We define

$$d^+P^u(f,\varphi)(\psi) := \lim_{t\to 0^+} \frac{1}{t}(P^u(f,\varphi+t\psi) - P^u(f,\varphi))$$

and

$$d^-P^u(f,\varphi)(\psi) := \lim_{t\to 0^-} \frac{1}{t}(P^u(f,\varphi+t\psi) - P^u(f,\varphi)).$$

The following proposition is immediate.

PROPOSITION 7.3. *We have*
(1) $d^-P^u(f,\varphi)(\psi) = -d^+P^u(f,\varphi)(-\psi)$;
(2) $d^-P^u(f,\varphi)(\psi) \le d^+P^u(f,\varphi)(\psi)$.

Recall that the unstable topological pressure $P^u(f,\cdot)$ is said to be Gateaux differentiable at φ if

$$\lim_{t\to 0} \frac{1}{t}(P^u(f,\varphi+t\psi) - P^u(f,\varphi))$$

exists for any $\psi \in C(M,\mathbb{R})$ (see Definition 2.6). By Proposition 7.3, $P^u(f,\cdot)$ is Gateaux differentiable at φ if and only if for any $\psi \in C(M,\mathbb{R})$,

$$d^+P^u(f,\varphi)(\psi) = -d^+P^u(f,\varphi)(-\psi).$$

LEMMA 7.4. $d^+P^u(f,\varphi)(\psi) = \sup\{\int_M \psi\, d\mu : \mu \in t^u_\varphi(M, f)\}$ *for any* $\varphi, \psi \in C(M,\mathbb{R})$.

Proof. If $\mu \in t^u_\varphi(M, f)$, then by definition $\int_M \psi\, d\mu \le 1/t(P^u(f,\varphi+t\psi) - P^u(f,\varphi))$ for all $t > 0$. Taking the limit, one has $\int_M \psi\, d\mu \le d^+P^u(f,\varphi)(\psi)$.

Conversely, let us denote $c = d^+P^u(f, \varphi)(\psi)$. Consider a linear functional γ on the linear subspace $\{t\psi : t \in \mathbb{R}\}$ given by $\gamma(t\psi) = tc$. Due to the convexity of $P^u(f, \cdot)$, $\gamma(t\psi) = tc \leq P^u(f, \varphi + t\psi) - P^u(f, \varphi)$, i.e., γ is bounded above by the continuous convex function $h \mapsto P^u(f, \varphi + h) - P^u(f, \varphi)$ on the subspace $\{t\psi : t \in \mathbb{R}\}$. Then the Hahn–Banach theorem asserts that γ can be extended to a linear functional on $C(M, \mathbb{R})$ such that

$$\gamma(h) \leq P^u(f, \varphi + h) - P^u(f, \varphi)$$

for any $h \in C(M, \mathbb{R})$. Then, using the Riesz representation theorem, there exists $\mu \in t^u_\varphi(M, f)$ such that

$$\int_M \psi \, d\mu = \gamma(\psi) = c = d^+P^u(f, \varphi)(\psi). \qquad \square$$

Proof of Theorem F. If $P^u(f, \cdot)$ is Gateaux differentiable at φ, then, for any $\psi \in C(M, \mathbb{R})$,

$$d^+P^u(f, \varphi)(\psi) = -d^+P^u(f, \varphi)(-\psi).$$

By Lemma 7.4, we have for any $\psi \in C(M, \mathbb{R})$,

$$\begin{aligned} \sup\left\{\int_M \psi \, d\mu : \mu \in t^u_\varphi(M, f)\right\} &= -\sup\left\{\int_M (-\psi) \, d\mu : \mu \in t^u_\varphi(M, f)\right\} \\ &= \inf\left\{\int_M \psi \, d\mu : \mu \in t^u_\varphi(M, f)\right\}. \end{aligned}$$

It follows that $t^u_\varphi(M, f)$ consists of a single element μ_φ.

Conversely, suppose that $t^u_\varphi(M, f)$ consists of a single element μ_φ. Then, by Lemma 7.4,

$$d^+P^u(f, \varphi)(\psi) = \int_M \psi \, d\mu_\varphi = -\int_M (-\psi) \, d\mu_\varphi = -d^+P^u(f, \varphi)(-\psi).$$

So, $P^u(f, \cdot)$ is Gateaux differentiable at φ. $\square$

Corollary F.1 follows directly from Theorems E and F.

7.3. *Fréchet differentiability.*

LEMMA 7.5. *The following two statements are mutually equivalent:*

(1) *$P^u(f, \cdot)$ has a unique u-tangent functional at φ;*

(1) *there exists a unique measure μ_φ such that every $(\mu_n) \subset \mathcal{M}_f(M)$ with $h^u_{\mu_n}(f) + \int_M \varphi \, d\mu_n \to P^u(f, \varphi)$ satisfies $\mu_n \to \mu_\varphi$ as $n \to \infty$.*

Proof. (1) $\Rightarrow$ (2): If $h^u_{\mu_n}(f) + \int_M \varphi \, d\mu_n \to P^u(f, \varphi)$ and $\mu_n \to \mu$ for some $\mu \in \mathcal{M}_f(M)$ as $n \to \infty$, by the upper semicontinuity of the unstable entropy map, we have

$$h^u_\mu(f) + \int_M \varphi \, d\mu = P^u(f, \varphi).$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

Namely, $\mu \in \mathcal{M}^u_\varphi(M, f) = t^u_\varphi(M, f)$. Hence, by (1) there is only one such measure μ, which is denoted by μ_φ. Thus, (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Let μ be a u-tangent functional to $P^u(f, \cdot)$ at φ. Note that such μ exists by Theorems E and B(2). Put $\mu_n = \mu$. Then $h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_n \to P^u(f, \varphi)$. Thus, $\mu = \mu_\varphi$. It follows that μ_φ is the unique u-tangent functional of $P^u(f, \cdot)$ at φ. □

Recall that $P^u(f, \cdot)$ is called Fréchet differentiable at φ if there exists $\gamma \in C(M, \mathbb{R})^*$ such that

$$\lim_{\psi\to 0} \frac{|P^u(f, \varphi+\psi) - P^u(f, \varphi) - \gamma(\psi)|}{\|\psi\|} = 0$$

(see Definition 2.7). If $P^u(f, \cdot)$ is Fréchet differentiable at φ, then

$$\lim_{t\to 0} \frac{|P^u(f, \varphi+t\psi) - P^u(f, \varphi) - t\gamma(\psi)|}{t\|\psi\|} = 0.$$

Thus, $P^u(f, \cdot)$ is Gateaux differentiable at φ. Moreover, $\gamma(\psi) = \int_M \psi\, d\mu_\varphi$, where μ_φ is the unique u-tangent functional to $P^u(f, \cdot)$ at φ by Theorem F and Lemma 7.4.

Proof of Theorem G. (1) $\Rightarrow$ (2): Suppose that $P^u(f, \cdot)$ is Fréchet differentiable at φ. By the discussion above, $P^u(f, \cdot)$ is Gateaux differentiable at φ. Hence, $\gamma(\psi) = \int_M \psi\, d\mu_\varphi$, where μ_φ is the unique u-tangent functional to $P^u(f, \cdot)$ at φ. Let $(\mu_n) \subset \mathcal{M}_f(M)$ with $h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_n \to P^u(f, \varphi)$ as $n \to \infty$. Put $\epsilon_n := P^u(f, \varphi) - h^u_{\mu_n}(f) - \int_M \varphi\, d\mu_n$. For any $\epsilon \in (0, \frac{1}{2})$, there exists $\delta > 0$ such that whenever $\|\psi\| < \delta$, we have

$$0 \le P^u(f, \varphi+\psi) - P^u(f, \varphi) - \int_M \psi\, d\mu_\varphi \le \epsilon\|\psi\|.$$

Then

$$\begin{aligned}
\int_M \psi\, d\mu_n - \int_M \psi\, d\mu_\varphi &= P^u(f, \varphi) + \int_M \psi\, d\mu_n - P^u(f, \varphi) - \int_M \psi\, d\mu_\varphi \\
&= h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_n + \epsilon_n + \int_M \psi\, d\mu_n - P^u(f, \varphi) - \int_M \psi\, d\mu_\varphi \\
&\le P^u(f, \varphi+\psi) - P^u(f, \varphi) - \int_M \psi\, d\mu_\varphi + \epsilon_n \\
&\le \epsilon\delta + \epsilon_n.
\end{aligned}$$

This is also true for $-\psi$ and hence we have $|\int_M \psi\, d\mu_n - \int_M \psi\, d\mu_\varphi| \le \epsilon\delta + \epsilon_n$ whenever $\|\psi\| \le \delta$. Thus, if n is large enough, then for any $\tilde{\psi} \in C(M, \mathbb{R})$ with $\|\tilde{\psi}\| \le 1$,

$$\begin{aligned}
\left| \int_M \tilde{\psi}\, d\mu_n - \int_M \tilde{\psi}\, d\mu_\varphi \right| &= \frac{1}{\delta}\left| \int_M \delta\tilde{\psi}\, d\mu_n - \int_M \delta\tilde{\psi}\, d\mu_\varphi \right| \\
&\le \frac{1}{\delta}(\epsilon\delta + \epsilon_n) \\
&= \epsilon + \frac{\epsilon_n}{\delta} < 2\epsilon,
\end{aligned}$$

which means that $\|\mu_n - \mu_\varphi\| \to 0$.

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

(2) $\Rightarrow$ (3): Assume that (2) holds. By Lemma 7.5, μ_φ is the unique member of $t^u_\varphi(M, f)$. Moreover, μ_φ is ergodic by Theorem B(3). By Theorem A, there exist ergodic measures (μ_n) with $h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_n \to P^u(f, \varphi)$. It follows from (2) that $\|\mu_n - \mu_\varphi\| \to 0$. On the other hand, distinct ergodic measures are mutually singular and hence have norm distance two. Thus, there exists $N \in \mathbb{N}$ such that $\mu_n = \mu_\varphi$ for any $n \geq N$. This implies that $P^u(f, \varphi) > \sup\{h^u_\mu(f) + \int_M \varphi\, d\mu : \mu \text{ is ergodic and } \mu \neq \mu_\varphi\}$.

(3) $\Rightarrow$ (4): Assume that (3) holds. Put

$$c = P^u(f, \varphi) - \sup\left\{h^u_\mu(f) + \int_M \varphi\, d\mu : \mu \text{ is ergodic and } \mu \neq \mu_\varphi\right\}.$$

Suppose that $\|\varphi - \psi\| < c/2$. Then, by Proposition 3.5(4),

$$\begin{aligned}&\sup\left\{h^u_\mu(f) + \int_M \psi\, d\mu : \mu \text{ is ergodic and } \mu \neq \mu_\varphi\right\}\\ &\quad\leq P^u(f, \varphi) + \|\varphi - \psi\| - c \leq P^u(f, \psi) + 2\|\varphi - \psi\| - c < P^u(f, \psi).\end{aligned}$$

It implies that every $\psi \in C(M, \mathbb{R})$ with $\|\varphi - \psi\| < c/2$ has μ_φ as the unique u-equilibrium state by Theorem A. Thus, $P^u(f, \psi) = h^u_{\mu_\varphi}(f) + \int_M \psi\, d\mu_\varphi$ for all ψ in the ball centered at φ of radius $c/2$. In other words, $P^u(f, \cdot)$ is affine in a neighborhood of φ.

(4) $\Rightarrow$ (5): Suppose that $P^u(f, \cdot)$ is affine in a neighborhood V of φ, i.e., there exist a linear functional γ and $c \in \mathbb{R}$ such that $P^u(f, \psi) = c + \gamma(\psi)$ for any $\psi \in V$. It is easy to check that $P^u(f, \cdot)$ is Fréchet differentiable at any $\psi \in V$. Thus, $P^u(f, \cdot)$ is Gateaux differentiable at ψ and $\gamma(\psi) = \int_M \psi\, d\mu_0$, where μ_0 is the unique u-tangent functional to $P^u(f, \cdot)$ at every $\psi \in V$. (5) follows immediately.

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (1): Assume that (6) holds. Let $\mu \in t^u_{\varphi+\psi}(M, f) = \mathcal{M}^u_{\varphi+\psi}(M, f)$ by Theorem E. Then $P^u(f, \varphi + \psi) = h^u_\mu(f) + \int_M(\psi + \varphi)\, d\mu$. We have

$$\begin{aligned}0 &\leq P^u(f, \varphi + \psi) - P^u(f, \varphi) - \int_M \psi\, d\mu_\varphi\\ &\leq \left(h^u_\mu(f) + \int_M (\psi + \varphi)\, d\mu\right) - \left(h^u_\mu(f) + \int_M \varphi\, d\mu\right) - \int_M \psi\, d\mu_\varphi\\ &= \int_M \psi\, d\mu - \int_M \psi\, d\mu_\varphi\\ &\leq \|\psi\| \cdot \|\mu - \mu_\varphi\|.\end{aligned}$$

Hence, $0 \leq P^u(f, \varphi + \psi) - P^u(f, \varphi) - \int_M \psi\, d\mu_\varphi \leq \|\psi\| \inf\{\|\mu - \mu_\varphi\| : \mu \in t^u_{\varphi+\psi}(M, f)\}$. So, $P^u(f, \cdot)$ is Fréchet differentiable at φ.

(3) $\Rightarrow$ (7): Assume that (3) holds. By Theorem E, $\mu_\varphi \in \mathcal{M}^u_\varphi(M, f)$. Denote $c = P^u(f, \varphi) - \sup\{h^u_\mu(f) + \int_M \varphi\, d\mu : \mu \text{ is ergodic and } \mu \neq \mu_\varphi\}$. Define

$$V := \{\mu \in \mathcal{M}_f(M) : \left|\int_M \varphi\, d\mu - \int_M \varphi\, d\mu_\varphi\right| < c/2\}.$$

https://doi.org/10.1017/etds.2020.105 Published online by Cambridge University Press

Then V is a weak* neighborhood of μ_φ. If $\mu \in V$ is ergodic and $\mu \neq \mu_\varphi$, then

$$\begin{aligned} h^u_\mu(f) &\leq h^u_\mu(f) + \int_M \varphi\, d\mu - \int_M \varphi\, d\mu_\varphi + c/2 \\ &\leq P^u(f,\varphi) - \int_M \varphi\, d\mu_\varphi - c/2 \\ &= h^u_{\mu_\varphi}(f) - c/2. \end{aligned}$$

Thus, (7) is proved.

(7) $\Rightarrow$ (3): Assume that (7) holds. Assume that $P^u(f,\varphi) = \sup\{h^u_\mu(f) + \int_M \varphi\, d\mu :$ μ is ergodic and $\mu \neq \mu_\varphi\}$. Then there exists a sequence of ergodic measures $\{\mu_n\}$ with $\mu_n \neq \mu_\varphi$ ergodic such that $h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_n \to P^u(f,\varphi)$. By Lemma 7.5, $\mu_n \to \mu_\varphi$ and hence $\mu_n \in V$ whenever n is sufficiently large. Thus,

$$\limsup_{n\to\infty} h^u_{\mu_n}(f) + \int_M \varphi\, d\mu_\varphi = P^u(f,\varphi).$$

This implies that $\limsup_{n\to\infty} h^u_{\mu_n}(f) = h^u_{\mu_\varphi}(f)$, contradicting (7). So, (3) holds. □

Acknowledgements. The authors would like to thank the referees for many helpful suggestions. W. Wu is supported by NSFC No. 11701559. Y. Zhu is supported by NSFC Nos. 11771118 and 11926103. Weisheng Wu is the corresponding author.

References

[1] C. Bonatti, L. J. Díaz, and M. Viana. *Dynamics Beyond Uniform Hyperbolicity. A Global Geometric and Probabilitistic Perspective (Encyclopaedia of Mathematical Sciences, 102)*. Springer, Berlin, 2005.

[2] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.* **115**(2000), 157–193.

[3] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*. Springer, Berlin, 1975.

[4] J. Buzzi. C^r surface diffeomorphisms with no maximal entropy measure. *Ergod. Th. & Dynam. Sys.* **34**(6) (2014), 1770–1793.

[5] T. Downarowicz. *Entropy in Dynamical Systems (New Mathematical Monographs, 18)*. Cambridge University Press, Cambridge 2011.

[6] N. Dunford and J. T. Schwartz. *Linear Operators Part I: General Theory*. Interscience, New York, NY, 1958.

[7] M. Grayson, C. Pugh, and M. Shub. Stably ergodic diffeomorphisms. *Ann. of Math. (2)* **140**(1994), 295–329.

[8] M. W. Hirsch, C. C. Pugh and M. Shub. Invariant manifolds. *Bull. Amer. Math. Soc.* **76**(5) (1970), 1015–1019.

[9] H. Hu, Y. Hua, and W. Wu. Unstable entropies and variational principle for partially hyperbolic diffeomorphisms. *Adv. Math.* **321**(2017), 31–68.

[10] Y. Hua, R. Saghin, and Z. Xia. Topological entropy and partially hyperbolic diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **28**(3) (2008), 843–862.

[11] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms: part II: relations between entropy, exponents and dimension. *Ann. Math.* **122**(3) (1985), 540–574.

[12] M. Misiurewicz. Diffeomorphism without any measure of maximal entropy. *Bull. Acad. Pol. Sci.* **21**(1973), 903–910.

[13] Ya. B. Pesin. *Dimension Theory in Dynamical Systems: Contemporary Views and Applications (Chicago Lectures in Mathematics)*. University of Chicago Press, Chicago, IL, 2008.

[14] Ya. B. Pesin, and Ya. G. Sinai. Gibbs measures for partially hyperbolic attractors. *Ergod. Th. & Dynam. Sys.* **2**(3–4) (1982), 417–438.

[15] G. Ponce. Unstable entropy of partially hyperbolic diffeomorphisms along non-compact subsets. *Nonlinearity* **32**(7) (2019), 2337–2351.
[16] F. Rodriguez Hertz, M. A. Rodriguez Hertz, and R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1d-center bundle. *Invent. Math.* **172**(2) (2008), 353–381.
[17] V. A. Rohlin. On the fundamental ideas of measure theory. *Amer. Math. Soc. Translations* **171** (1952), 55 pp. Engl. Transl. of *Mat. Sbornik* **25** (1949), 107–150.
[18] D. Ruelle. Statistical mechanics on a compact set with $\mathbb{Z}^{\nu}$ action satisfying expansiveness and specification. *Trans. Amer. Math. Soc.* **187**(1973), 237–251.
[19] X. Tian and W. Wu. Unstable entropies and dimension theory of partially hyperbolic systems. *Preprint*, 2018, arXiv:1811.03797.
[20] P. Walters. A variational principle for the pressure of continuous transformations. *Amer. J. Math.* **97**(4) (1975), 937–971.
[21] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, New York, NY, 1982.
[22] P. Walters. Differentiability properties of the pressure of a continuous transformation on a compact metric space. *J. Lond. Math. Soc.* **46**(3) (1992), 471–481.
[23] L. Wang and Y. Zhu. Center specification property and entropy for partially hyperbolic diffeomorphisms. *Discrete Contin. Dyn. Syst.* **36**(1) (2016), 469–479.
[24] J. Yang. Entropy along expanding foliations. *Preprint*, 2016, arXiv:1601.05504.