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The formation generated by a finite group

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We prove here that the (saturated) formation generated by a finite soluble group has only finitely many (saturated) subformations. This answers a question asked by Professor W. Gaschütz. Some partial results are also given in the case of a formation generated by an arbitrary finite group.

At the Ninth Summer Research Institute of the Australian Mathematical Society, held in Canberra in 1969, Professor W. Gaschütz asked the following questions [3, 7.22]: "Does the (saturated) formation generated by a finite soluble group contain only finitely many (saturated) subformations?" We show in §1 that the answer to each of these questions is "Yes". The proof relies on that of the Oates-Powell theorem - 52.11 in Hanna Neumann [5] - which answers the corresponding question for the variety generated by a finite group. We have unsuccessfully considered the same question for the formation generated by a finite insoluble group. The best we can do in general is described in §2, where we show that among the subformations generated by quotients of direct powers of the generating group only finitely many are distinct. Incidentally, a corollary of Lemma 1.5 below - Corollary 1.6 - is the result of Peter M. Neumann [6] that a formation consisting of nilpotent groups is subgroup closed.

We refer the reader to the paper [1] of Carter, Fischer and Hawkes for notation and definitions relating to formations. In addition we remark that if $\underline{\underline{X}}$ is a class of groups then the formation generated by $\underline{\underline{X}}$

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is just $QR_0\underline{\underline{X}}$: this follows since $R_0Q\underline{\underline{X}} \subseteq QR_0\underline{\underline{X}}$ - if X is a subdirect product of groups $X_1/N_1, \ldots, X_r/N_r$, where each $X_i \in \underline{\underline{X}}$, then clearly X is a homomorphic image of a subdirect product of $X_1, \ldots, X_r$ under the natural homomorphism of $X_1 \times \ldots \times X_r$ onto $X_1/N_1 \times \ldots \times X_r/N_r$.

1. The soluble case

We denote by $\text{form}(G)$ the formation generated by the group G . A finite group G will be called *formation critical* if the formation generated by those proper factors of G which lie in $\text{form}(G)$ does not contain G (cf. 51.31 in [5]). Every formation is generated by its formation critical groups (cf. 51.41 in [5]). We remind the reader that a Cross variety is a variety generated by a single finite group. We shall prove

LEMMA 1.1. *A soluble Cross variety contains only finitely many formation critical groups.*

The remarks preceding Lemma 1.1 then yield

THEOREM 1.2. *The formation which consists of the finite groups of a soluble Cross variety has only finitely many subformations.*

Since the formation generated by a group is contained in the variety it generates this is enough to answer one of the questions of Gaschütz cited in the introduction. The other question can be answered easily using Theorem 1.2.

COROLLARY 1.3. *The saturated formation generated by a finite soluble group contains only finitely many saturated subformations.*

Proof. We appeal to the well-known theorems of Gaschütz and Lubeseder (B. Huppert [4, VI.7.5 and VI.7.25]) that a formation of finite soluble groups is saturated if and only if it can be locally defined.

Let F be the saturated formation generated by a finite soluble group G and let π be the set of primes dividing $|G|$. For $p \in \pi$, define $F(p) = \text{form}\left(G/O_{p'p}(G)\right)$, and define $F(p) = \emptyset$ otherwise: then it follows by the theorems just mentioned that F is the (saturated) formation defined locally by the $F(p)$. (Here, as usual, $O_{p'p}(G)$

denotes the largest p-nilpotent normal subgroup of G .)

Let $\mathfrak{G}$ be any saturated subformation of $\mathfrak{F}$, defined locally by formations $\mathfrak{G}(p)$. Then $\mathfrak{G}(p) = \emptyset$ for $p \notin \pi$. Furthermore, by replacing $\mathfrak{G}(p)$ by $\mathfrak{G}(p) \cap \mathfrak{F}(p)$ if necessary, we may suppose that $\mathfrak{G}(p) \subseteq \mathfrak{F}(p)$, $p \in \pi$. Now Theorem 1.2 ensures that each $\mathfrak{F}(p)$ has only finitely many subformations, and π is finite, so there are only finitely many possibilities for $\mathfrak{G}$.

The proof of Lemma 1.1 depends on the next two lemmas.

LEMMA 1.4. *If Y is a subgroup of a nilpotent group X whose class is smaller than that of X then the class of the normal closure Y^X of Y in X is also smaller than the class of X .*

Proof. It will be enough to show (we do it by induction on d) that

$$(*) \qquad \gamma_d\left(Y^X\right) \leq \gamma_{d+1}(X)\gamma_d(Y) \ , \quad d \in \{1, 2, \ldots\} \ .$$

For $d = 1$ this is certainly the case since, for $x \in X$, $y \in Y$, $y^x = \left[x, y^{-1}\right]y$; hence

$$\gamma_1\left(Y^X\right) = Y^X \leq \gamma_2(X)Y = \gamma_2(X)\gamma_1(Y) \ .$$

Assume that (*) has been proved for some $d \geq 1$. Then

$$\begin{aligned} \gamma_{d+1}\left(Y^X\right) &= \left[\gamma_d\left(Y^X\right), Y^X\right] \leq \left[\gamma_{d+1}(X)\gamma_d(Y), Y^X\right] \\ &= \left[\gamma_{d+1}(X), Y^X\right]\left[\gamma_d(Y), Y^X\right] \leq \gamma_{d+2}(X)\left[\gamma_d(Y), \gamma_2(X)Y\right] \\ &= \gamma_{d+2}(X)\left[\gamma_d(Y), Y\right]\left[\gamma_d(Y), \gamma_2(X)\right] \leq \gamma_{d+2}(X)\gamma_{d+1}(Y) \end{aligned}$$

as required.

LEMMA 1.5. *Every supplement of the Fitting subgroup of a finite group G is in the formation generated by G .*

Proof. Let S be a subgroup of G and T a nilpotent normal subgroup of G such that $G = ST$. We show, by induction on the class of T , that $S \in \operatorname{form}(G)$.

Consider the following subgroups of $G \times G \times G$:

$$K = \{(s, s, s) : s \in S\} \ ,$$

$$D_1 = \{(t, t, 1) : t \in T\} \text{ and } D_2 = \{(1, t, t) : t \in T\} .$$

Let H be the subgroup they generate: then H, being subdirect, is in $\text{form}(G)$. Note that D_1 and D_2 are each normalized by K.

If T is abelian then D_1 and D_2 centralize each other. Also $D_1 D_2 \cap K = 1$ because an arbitrary element of $D_1 D_2$ may be written in the form (t, tu, u) for some $t, u \in T$ and it is in K only if $t = tu = u$, that is only if $t = u = 1$. Consequently, when T is abelian,

$$S \cong K \cong H/D_1 D_2 \in \text{form}(G) .$$

Suppose now that the class c of T is greater than 1. Put $M = Z(D_1)Z(D_2)$ and note that, as in the last paragraph, $M \cap K = 1$. Also M is normal in H. Since

$$[\gamma_{c-1}(D_1), D_2] = \{(1, w, 1) : w \in \gamma_c(T)\} \nleq M ,$$

the groups $D_1 M/M$ and $D_2 M/M$ generate a subgroup C of H/M of class c exactly. However $D_2 M/M$ has class $c - 1$ and so its normal closure B in C also has class $c - 1$, by Lemma 1.4. But B is normal in H/M and so, by induction, $KD_1 M/M \in \text{form}(H/M)$. Finally $D_1 M/M$ is normal in $KD_1 M/M$ and of class $c - 1$, so, again by induction, $KM/M \in \text{form}(KD_1 M/M)$. Therefore

$$S \cong K \cong KM/M \in \text{form}(G) .$$

The result of Peter M. Neumann referred to in the introduction follows immediately from Lemma 1.5.

COROLLARY 1.6. *A formation consisting of nilpotent groups is subgroup closed.*

We shall also require the following version of 51.37 of [5]: this consists simply of a rather precise statement of what emerges in the course of the proof of 51.37, and we refer the reader to [5] for details.

LEMMA 1.7. *Let A be a finite group which is generated by the subgroup L together with normal subgroups $M_1, M_2, \ldots, M_s$. Let $\underline{X}$ be the set of subgroups of A which can be generated by L together with some proper subset of $\{M_1, M_2, \ldots, M_s\}$.*

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Suppose that

$$[M_{\pi(1)}, M_{\pi(2)}, \ldots, M_{\pi(s)}] = 1$$

for every permutation π *of* $\{1, 2, \ldots, s\}$. *Then* A *is in* $QR_0\underline{\underline{X}}$.

Proof of Lemma 1.1. We shall follow closely the proof of 52.23 of [5]. In particular Lemma 1.7 will be used in place of 51.37 of [5].

Let $\underline{\underline{V}}$ be a Cross variety. Then the results of Chapter 5 of [5] show that there are bounds on

(a) the class of a nilpotent group in $\underline{\underline{V}}$,

(b) the order of a chief factor of a group in $\underline{\underline{V}}$, and

(c) the order of a finite group in $\underline{\underline{V}}$ on a given number of generators.

Suppose that $\underline{\underline{V}}$ is soluble and that A is a formation critical group in $\underline{\underline{V}}$. Let F and Φ be, respectively, the Fitting and Frattini subgroups of A . By the results of W. Gaschütz [2] we know that F/Φ is the direct product of minimal normal subgroups $M_1/\Phi, M_2/\Phi, \ldots, M_s/\Phi$ of A/Φ and that F/Φ has a complement L/Φ in A/Φ . Also F/Φ is the Fitting subgroup of A/Φ and so $C_A(F/\Phi) = F$ by W.R. Scott [7, 7.4.7].

Suppose that s exceeds the bound given by (a) above. Then the groups L and $M_1, M_2, \ldots, M_s$ satisfy the conditions of Lemma 1.7; and furthermore the groups in the set $\underline{\underline{X}}$ of Lemma 1.7 all lie in form(A) by Lemma 1.5. Thus we deduce that form(A) is generated by $\underline{\underline{X}}$. But $A \notin \underline{\underline{X}}$, and so this contradicts the definition of A .

Consequently (a) and (b) give a bound on the order of F/Φ . Therefore we also have that $|A : F| = |A : C_A(F/\Phi)|$ is bounded. Thus we obtain a bound on $|A : \Phi|$ and hence a bound for the number of generators needed for A . Finally (c) gives a bound on the order of A and completes the proof of Lemma 1.1.

To conclude this section we record the following result which is presumably well-known (cf. VI.7.21 of [4]).

LEMMA 1.8. *Let* N *be a normal subgroup of a group* G *and* M *a* *normal subgroup of* G *which centralizes* N . *Then the group* G^* *obtained by split extending* N *by* G/M *with its natural action on* N *is in the formation generated by* G .

Proof. Let $G_1 = \{(g, g) : g \in G\}$ and $N_1 = \{(n, 1) : n \in N\}$ be subgroups of $G \times G$; and note that G_1 normalizes N_1 . Put $M_1 = \{(m, m) : m \in M\}$, so that M_1 is normalized by G_1 and centralized by N_1 . Then, clearly,

$$G^* \cong G_1N_1/M_1 \in \text{form}(G) .$$

2. Direct powers and formations

Throughout this section G will denote a given finite group and $g \to g^*$, $g \to g_i$ and $g \to g_i^*$ will denote fixed isomorphisms of G onto groups G^* , G_i and G_i^* , respectively. We shall employ the natural consequences of this notation - thus if K is a subgroup of G then K_i will denote the subgroup of G_i corresponding to K , and so on.

Let $U \leq V$ be normal subgroups of G with $[V, G] \leq U$ and let $D(U, V)$ be the subgroup of $G \times G^*$ consisting of all elements vw^* with $v, w \in V$ and $v \equiv w$ modulo U . Then $D(U, V) \triangleleft G \times G^*$. By a *central factor square* of G we shall mean a group isomorphic to a group $G \times G^*/D(U, V)$. Since any two isomorphisms $G \to G^*$ differ by an automorphism of G , such a group is determined up to isomorphism by U and V and is independent of the particular isomorphism $g \to g^*$ selected.

The main result of this section is the following.

THEOREM 2.1. *Let* H *be a homomorphic image of a direct power of* G . *Then the formation generated by* H *can be generated by a central factor square* $G \times G^*/D(U, V)$ *of* G *such that* U *and* V *are characteristic in* G *and the automorphism group of* G/U *acts trivially on* V/U .

Proof. The proof will be conducted by induction on $|G|$. If $|G| = 1$ the theorem clearly holds and so we may assume that $|G| > 1$ and that the theorem holds for all groups of order smaller than $|G|$.

Suppose then that

$$H = G_1^* \times G_2^* \times \ldots \times G_l^*/X$$

where $l^* \geq 1$ and $X \triangleleft D = G_1^* \times G_2^* \times \ldots \times G_l^*$. Let $\bar{X} = \cap X^\alpha$, the intersection being taken over all $\alpha \in \text{Aut}D$. Then clearly $D/\bar{X}$ and H generate the same formation and so we may assume that X is characteristic in D . We now distinguish two cases.

Case 1: X contains $Y_1^* \times \ldots \times Y_l^*$ for some $1 \neq Y \leq G$. Then $X \cap G_1^*$ is a non-trivial characteristic subgroup T_1^* of G_1^* and, since the symmetric group S_l acts naturally on D , X contains $T_1^* \times \ldots \times T_l^*$. But then H is a homomorphic image of a direct power of G/T and so, by our inductive hypothesis, $\text{form}(H)$ can be generated by a central factor square of G/T constructed with respect to characteristic subgroups U/T and V/T of G/T , with the properties described. The central factor square is isomorphic to $G \times G^*/D(U, V)$: furthermore U and V are characteristic in G and $\text{Aut}G/U$ acts trivially on V/U .

Case 2: X contains no subgroup of the form $Y_1^* \times \ldots \times Y_l^*$ with $1 \neq Y \leq G$. Let

$$E = G_1 \times G_2 \times \ldots \times G_{l+1}$$

and let L be the intersection of the kernels of the homomorphisms of E onto H . If ϕ is any such homomorphism and $\alpha \in \text{Aut}E$ then $\alpha\phi$ is another homomorphism of E onto H with kernel $(\ker \phi)^{\alpha^{-1}}$: thus L is characteristic in E .

We shall show that L contains no element of the form

(1) $$t = z_i w_j x$$

where $i \neq j$, z and w are distinct elements of G , and $x \in \prod_{k \neq i,j} G_k$. In fact suppose that L does contain such an element. Then, since the symmetric group S_{l+1} acts on E , we find that L also contains $u = w_i z_j x$. Hence L contains $tu^{-1} = y_i y_j^{-1}$ where

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$y = zw^{-1} \neq 1$, and by replacing y by a suitable power we may assume that y has prime order p . By applying further elements of S_{l+1} we find that L contains every element $y_l y_k y_m^{-1}$ with $k \neq m$ and so if $Y = \langle y \rangle$ and $Z = Y_1 \times \ldots \times Y_{l+1}$ we have

(2) $$|Z : Z \cap L| \leq p .$$

However let ψ be the homomorphism of E onto H determined by the conditions $g_i \to g_i^{*X}$ $(i \neq l + 1)$ and $g_{l+1} \to 1$. Then by the assumptions of Case 2 we have $y_j \notin \ker\psi$ for some j with $1 \leq j \leq l$. On the other hand there is a homomorphism χ of E onto H whose kernel contains y_j , and even the whole of G_j , but fails to contain some other y_k . Therefore $Z \cap \ker\psi$ and $Z \cap \ker\chi$ are distinct proper subgroups of Z and so

$$|Z : Z \cap \ker\psi \cap \ker\chi| \geq p^2 .$$

This contradicts (2) and shows that L contains no element of the form (1).

The set of all elements $g \in G$ such that $\prod_{i=1}^{l+1} g_i \in L$ is a subgroup V of G . Now if α is an automorphism of G and h an element of V then $(h^\alpha)_1 \prod_{i=2}^{l+1} h_i$ is in L , and consequently $h^\alpha = h$ by what we have just proved. Hence Aut$G$ fixes every element of V and in particular V is characteristic in G .

Now clearly H and E/L generate the same formation. However let

$$K_{ij} = \prod_{k \neq i,j} G_k \langle v_i v_j : v \in V \rangle \quad (1 \leq i,\ j \leq l + 1,\ i \neq j) .$$

Then $E/K_{ij} \cong G \times G^*/D(1, V)$ and $\bigcap K_{ij} = L$. Consequently E/L generates the same formation as $G \times G^*/D(1, V)$, and the proof is complete.

COROLLARY 2.2. *Only finitely many formations can be generated by subsets of* $QD_0(G)$.

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(Here $D_0(G)$ denotes the class of isomorphic copies of direct powers of G.)

Proof. Let F be a formation generated by groups $H_1, H_2, \ldots$ belonging to $QD_0(G)$. By Theorem 2.1 the formation generated by H_i can be generated by one of the groups $G \times G^*/D(U, V)$ and so each H_i can be replaced by one of these groups. Thus there are only finitely many possibilities for F.

It may be worth remarking that the formations generated by the groups $G \times G^*/D(U, V)$ are not necessarily distinct as can be seen by taking G to be cyclic of order two.

3. The formation generated by SL(2, 5)

As an application of Theorem 2.1 we shall briefly indicate how the lattice of subformations of the formation F generated by $G = SL(2, 5)$ may be determined. It will be clear that the method is of more general application, but in the interests of brevity we shall refrain from stating a cumbersome general result. Much of the detail will be left to the reader.

In the sequel $G_1, G_2, \ldots$ will denote isomorphic copies of G and Z_i will be the centre of G_i, with $Z_i = \langle z_i \rangle$.

Firstly we remark that G has the following property.

(3) *If M is a finite group with a central subgroup W such that $M/W \cong G$ then M splits over W.*

Proof. By [4, V.25.4] the Schur multiplicator of G is trivial. Let ϕ be an epimorphism of a free group F of finite rank onto M and let $R = \phi^{-1}(W)$. Then $F/R \cong G$, and, since the Schur multiplicator of G is isomorphic to $R \cap F'/[F, R]$ (V.23.5 of [4]), we have $R \cap F' \leq [F, R]$. Therefore $M' \cap W = 1$ and $M = M' \times W$.

It is clear that any group $L \in F$ has a central subgroup W of exponent 2 such that L/W belongs to the formation generated by the alternating group A_5. Furthermore the latter formation consists of all finite direct powers of A_5; and so L/W is perfect. Therefore

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$L = L'W$. It follows that $L' = L''$ and that if U is a complement for $L' \cap W$ in W then $L = L' \times U$. Notice that, by Lemma 1.8, every central subgroup of G belongs to F , and so F contains the cyclic group C_2 of order 2 .

Now we have already pointed out that $\text{form}(G) = QR_0(G)$. We shall show that any group $H \in R_0(G)$ is a direct product of copies of G and of C_2 . By the remarks made in the preceding paragraph it is enough to consider the case where H is perfect and is subdirectly contained in $G_1 \times \ldots \times G_n$ for some n . Our assertion follows by induction on n unless $H \cap G_1 = Z_1$, and we shall now exclude this possibility.

Suppose then that $H \cap G_1 = Z_1$. Induction shows that $H = A_1A_2 \ldots A_k$ where the A_i are normal subgroups of H containing Z_1 such that $A_i/Z_1 \cong G$ and $A_i \cap A_j = Z_1$ if $i \neq j$. By (3) we have $A_i = Z_1 \times B_i$ with $B_i \cong G$. B_i is characteristic in A_i and so normal in H , and it follows easily that $H = Z_1 \times B_1 \times \ldots \times B_k$. As $H = H'$ we have a contradiction.

It now follows that any group $L \in F$ has the form

$$(G_1 \times \ldots \times G_l/X) \times C$$

where $l \geq 0$, $X \leq Z_1 \times \ldots \times Z_l$ and C is elementary abelian of exponent 2 . If $X = Z_1 \times \ldots \times Z_l$ it is clear that the formation generated by L can be generated by one of the groups 1 , C_2 , A_5 and $A_5 \times C_2$. Otherwise, by Theorem 2.1, the formation generated by L can be generated either by G or by the central square S of G .

Finally we show that the formations F and F^* generated by G and S are distinct. Otherwise $G \in F^*$ and G would be a homomorphic image of a group $G_1 \times \ldots \times G_l/X$ with $X \leq Z_1 \times \ldots \times Z_l$ and X an intersection of kernels of homomorphisms of $G_1 \times \ldots \times G_l$ onto S . It is easy to see that any such kernel contains the element $\bar{z} = z_1 \ldots z_l$. Therefore G would be a homomorphic image of the group $G_1 \times \ldots \times G_l/\langle\bar{z}\rangle$. However this is not the case and so $G \notin F^*$.

Thus F *has precisely six subformations*. They are generated respectively by the groups 1 , C_2 , A_5 , $A_5 \times C_2$, S , and G itself.

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