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THE SPHERICAL GROWTH SERIES OF DYER GROUPS

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Abstract Graph products of cyclic groups and Coxeter groups are two families of groups that are defined by labelled graphs. The family of Dyer groups contains these both families and gives us a framework to study these groups in a unified way. This paper focuses on the spherical growth series of a Dyer group D with respect to the standard generating set. We give a recursive formula for the spherical growth series of D in terms of the spherical growth series of standard parabolic subgroups. As an application we obtain the rationality of the spherical growth series of a Dyer group. Furthermore, we show that the spherical growth series of D is closely related to the Euler characteristic of D.

Keywords: Dyer groups; Coxeter groups; right-angled Artin groups; the spherical growth series of groups; the rational Euler characteristic

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1. Introduction

Let (G, S) be a pair where G is a group and $S = \{s_1, \ldots, s_k\}$ is a generating set of G. One way to study the group G is by counting its elements algebraically/geometrically. Each element $g \in G$ can be written as a word $g = x_1 \ldots x_n$ where each letter x_i, $i = 1, \ldots, n$ lies in the alphabet $S \cup S^{-1} = \left\{s_1, \ldots, s_k, s_1^{-1}, \ldots, s_k^{-1}\right\}$. The *length of* g, denoted by $l(g) = l_S(g)$, is the minimal length of a word expression of g in the alphabet $S \cup S^{-1}$. We count the number of elements of length n in G and convert this sequence into a formal power series:

$$\mathcal{G}_{(G,S)}(t) := \sum_{n=0}^{\infty} |\{g \in G \mid l(g) = n\}| \cdot t^n.$$

Thus, $\mathcal{G}_{(G,S)}(t) = \sum_{n=0}^{\infty} a_n \cdot t^n$ where a_n is the number of vertices in a sphere of radius n in the Cayley-graph $Cay(G, S)$. This formal power series is called *the spherical growth* *series of G (with respect to S)* and tends to be an important measure of complexity for infinite groups.

Let us calculate the spherical growth series of the infinite cyclic group with the canonical generating set:

$$\mathcal{G}_{(\mathbb{Z},\{1\})}(t) = 1 + 2t + 2t^2 + 2t^3 + \ldots = 1 + 2(t + t^2 + t^3 \ldots) = 1 + \frac{2t}{1-t} = \frac{1+t}{1-t}.$$

Thus, $\mathcal{G}_{(\mathbb{Z},\{1\})}(t)$ is a rational function. By definition, a pair (G, S) has *rational growth series* if there exist polynomials $f(t)$ and $g(t)$ with integer coefficients such that $\mathcal{G}_{(G,S)}(t) = \frac{f(t)}{g(t)}$. Many groups that appear in geometric group theory have rational growth series, for example Coxeter groups [2, 17], surface groups [4, 5], virtually abelian groups [1] and hyperbolic groups [8, 9]. Note that the rationality of the spherical growth series depends on the generating set, e.g. there exist nilpotent groups and finite generating sets, such that the growth series are not rational [19]. The central object under our investigation is the growth series of Dyer groups. Throughout this paper (Γ, m, f) is a Dyer graph and (D, V) where $V = V(\Gamma)$ is the associated Dyer system. This means that the vertex set $V(\Gamma)$ is finite and is endowed with a map $f\colon V(\Gamma) \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ and the edge set $E(\Gamma)$ is endowed with a map $m\colon E(\Gamma) \to \mathbb{N}_{\geq 2}$. For two letters a, b and a natural number m we define $\pi(a, b, m) := ababab a\ldots$ where the length of the word is m. Further, we assume that for every edge $e = \{x, y\} \in E(\Gamma)$ if $m(e) \neq 2$, then $f(x) = f(y) = 2$. The associated *Dyer group* is defined as follows:

$$D := \langle V \mid x^{f(x)} \text{ if } f(x) \neq \infty,\ \pi(x, y, m(\{x, y\})) = \pi(y, x, m(\{x, y\})) \\ \text{if } \{x, y\} \in E(\Gamma)\rangle.$$

We note that, if $f(x) = 2$ for all $x \in V(\Gamma)$, then D is a Coxeter group, and if $f(x) = \infty$ for all $x \in V(\Gamma)$, then D is a right-angled Artin group. Further, if $m(e) = 2$ for all $e \in E(\Gamma)$, then D is called a numbered graph product.

For a subset $Y \subseteq V$ we denote by D_Y the subgroup in D which is generated by the set Y. This subgroup is called a *standard parabolic subgroup*. It is shown in [7] that (D_Y, Y) is itself a Dyer system which is associated to the Dyer graph (Γ_Y, m_Y, f_Y), where Γ_Y is the full subgraph of Γ spanned by Y, m_Y is the restriction of m to $E(\Gamma_Y)$, and f_Y is the restriction of f to $V(\Gamma_Y) = Y$.

Let (D, V) be a Dyer system. We define $V_2 := \{x \in V \mid f(x) = 2\}$, $V_\infty := \{x \in V \mid f(x) = \infty\}$, $V_p := \{x \in V \mid 2 < f(x) < \infty\}$. Let D_2 resp. D_∞ resp. D_p be the subgroup of D generated by V_2 resp. V_∞ resp. V_p. We have a decomposition $D = D_2 \times D_p \times D_\infty$ whenever Γ is complete. By definition, D is of *spherical type* if Γ is a complete graph and D_2 is finite. In particular, if D is of spherical type, then $D = D_2 \times D_p \times D_\infty$, D_p is a finite abelian group, and $D_\infty = \mathbb{Z}^l$ where $l = |V_\infty|$.

We state now our main result.

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Theorem 1.1. *Let (D, V) be a Dyer system.*

(1) If D is not of spherical type, then

$$\frac{(-1)^{|V|+1}}{\mathcal{G}_{(D,V)}(t)} = \sum_{Y \subsetneq V} \frac{(-1)^{|Y|}}{\mathcal{G}_{(D_Y,Y)}(t)}.$$

(2) If D is of spherical type, then we decompose $D_p = \prod_{x\in V_p} \mathbb{Z}/f(x)\mathbb{Z}$ *and let* $l = |V_\infty|$. *Then*

$$\mathcal{G}_{(D,V)}(t) = \mathcal{G}_{(D_2,V_2)}(t) \cdot \mathcal{G}_{(D_p,V_p)}(t) \cdot \mathcal{G}_{(D_\infty,V_\infty)}(t),$$

where

- D_2 *is a finite Coxeter group, hence* $\mathcal{G}_{(D_2,V_2)}(t)$ *can be calculated using the formula for finite Coxeter groups [17]:*

$$\mathcal{G}_{(D_2,V_2)}(t) = \prod_{i=1}^{k}(1 + t + \ldots + t^{m_i}),$$

where $m_1, \ldots, m_k$ *are the exponents of* (D_2, V_2).
- $\mathcal{G}_{(D_p,V_p)}(t) = \prod_{x\in V_p} \mathcal{G}_{(\mathbb{Z}/f(x)\mathbb{Z},\{1\})}(t)$. *If* $f(x) = 2r$, *then*

$$\mathcal{G}_{(\mathbb{Z}/f(x)\mathbb{Z},\{1\})}(t) = 1 + 2t + 2t^2 + \ldots + 2t^{r-1} + t^r.$$

If $f(x) = 2r + 1$, *then*

$$\mathcal{G}_{(\mathbb{Z}/f(x)\mathbb{Z},\{1\})}(t) = 1 + 2t + 2t^2 + \ldots + 2t^r.$$

- $\mathcal{G}_{(D_\infty,V_\infty)}(t) = \frac{(1+t)^l}{(1-t)^l}$.

As a direct consequence we obtain the rationality of the spherical growth series of a Dyer system.

Corollary 1.2. *Let (D, V) be a Dyer system. The spherical growth series of D with respect to V is rational.*

Often there are interesting connections between special values of the spherical growth series of (G, S) with other properties of a group G, for example it was proven in [15] that for a Coxeter system (W, S), the value $\mathcal{G}_{(W,S)}(1)$ is closely related to the rational Euler characteristic of W which we denote by $\chi(W)$. More precisely:

$$\frac{1}{\mathcal{G}_{(W,S)}(1)} = \chi(W).$$

We prove that the same relation holds for all Dyer groups.

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Theorem 1.3. *(see Theorem 5.3) Let (D, V) be a Dyer system. Then*

$$\frac{1}{\mathcal{G}_{(D,V)}(1)} = \chi(D).$$

2. Preliminaries

We start this chapter by reviewing some standard facts of the word length and length functions.

Definition 2.1. *Let G be a finitely generated group and S be a finite generating set.*

(1) *For $g \in G$, $g \neq 1$ the word length of g is defined as:*

$$l(g) = l_S(g) = min\left\{n \mid g = s_1^{\epsilon_1} s_2^{\epsilon_2} \ldots s_n^{\epsilon_n}, s_i \in S, \epsilon_i \in \{-1, 1\}\right\}.$$

(2) *For $g \in G$, $g \neq 1$ the syllable length of g is defined as:*

$$l_{sy}(g) := min\left\{m \mid g = s_1^{a_1} s_2^{a_2} \ldots s_m^{a_m}, s_i \in S, a_i \in \mathbb{Z}\right\}.$$

And we set $l_{sy}(1) = l(1) = 0$.

We note that for a given group G and a finite generating set S consisting of elements of order two we have $l(g) = l_{sy}(g)$ for all $g \in G$.

The length of $g \in G$ is closely connected to the length of a special path in a geometric object which is associated to the group G, the Cayley-graph $Cay(G, S)$. Before we give a definition of this graph we recall the definition and some important facts about general graphs which we will need later on.

A graph Γ is a pair $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a set whose elements are called *vertices* and $E(\Gamma)$ is a subset of $\mathcal{P}_2(V) := \{X \mid X \subseteq V, |X| = 2\}$ whose elements are called *edges*. Usually, graphs are visualized graphically, where we draw for each vertex $x \in V(\Gamma)$ a point and label it with x and two points x, y are connected by a line if $\{x, y\} \in E(\Gamma)$.

Given a graph Γ and a vertex $x \in V(\Gamma)$ we define two subsets of $V(\Gamma)$ that are associated to x. The *link of* x, denoted by $lk(x)$ is defined as $lk(x) := \{y \in V(\Gamma) \mid \{x, y\} \in E(\Gamma)\}$ and the *star of* x, denoted by $st(x)$ is defined as $st(x) := lk(x) \cup \{x\}$. A graph Γ is called *complete* if $st(x) = V(\Gamma)$ for all $x \in V(\Gamma)$. A subgraph $\Omega \subseteq \Gamma$ is called *full* if for all pair of vertices $(v, w) \in V(\Omega) \times V(\Omega)$ we have $\{v, w\} \in E(\Omega)$ if and only if $\{v, w\} \in E(\Gamma)$.

Let G be a group and let S be a generating set for G. The *Cayley-graph for G with respect to S*, denoted by $Cay(G, S)$ is a graph with vertex set $V(Cay(G, S)) = G$ and edge set $E(Cay(G, S)) = \left\{\{g, gs\} \mid g \in G, s \in S \cup S^{-1}\right\}$. The distance between two vertices is defined as a number of edges in a shortest path connecting those vertices. Note that $l(g)$ is equal to the distance between the vertices 1_G and g. Hence the number of elements in G with word length n is equal to the number of vertices in the sphere with centre 1_G of radius n in $Cay(G, S)$. Let us consider the Cayley-graph of the free group F_2 with the generating set $\{x, y\}$ in Figure 1.

For example, the number of elements in F_2 with length 2 is equal to 12. One geometric way to count the elements in F_2 is by counting the vertices in the sphere with centre

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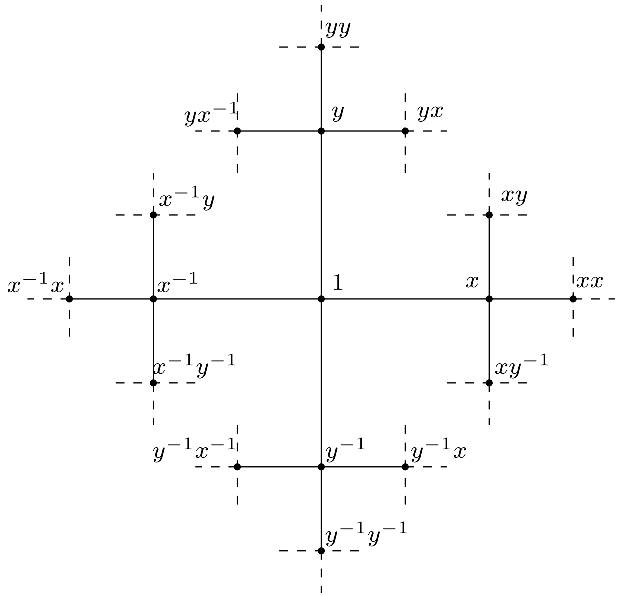

Figure 1. $Cay(F_2, \{x, y\})$.

1 of radius n in the Cayley-graph $Cay(F_2, \{x, y\})$. Let a_n be this number. We get the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_0 = 1$ and $a_n = 4 \cdot 3^{n-1}$ for $n \geq 1$. We convert this sequence into a formal power series $1 + a_1 t + a_2 t^2 + \ldots$ which leads us to the definition of the spherical growth series. The best general reference on growth series of groups is [11].

Definition 2.2. *Let G be a group and S be a finite generating set of G.*

(1) *The spherical growth series of G with respect to S is the formal series:*

$$\mathcal{G}_{(G,S)}(t) := \sum_{g\in G} t^{l(g)} = \sum_{n=0}^{\infty} |\{g \in G \mid l(g) = n\}| \cdot t^n.$$

(2) *The spherical growth series of a subset $A \subseteq G$ with respect to S is the formal series:*

$$\mathcal{G}_{(A,S)}(t) := \sum_{g\in A} t^{l(g)} = \sum_{n=0}^{\infty} |\{g \in A \mid l(g) = n\}| \cdot t^n.$$

We note that $\mathcal{G}_{(G,S)}(t)$ is an element in $\mathbb{Z}[[t]]$ the ring of formal power series in the variable t over $\mathbb{Z}$. We now give some examples.

Example 2.3.

(1) $\mathcal{G}_{(\mathbb{Z}/4\mathbb{Z},\{1\})}(t) = 1 + 2t + t^2$.

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(2) $\mathcal{G}_{(\mathbb{Z}/5\mathbb{Z},\{1\})}(t) = 1 + 2t + 2t^2$.
(3) $\mathcal{G}_{(\mathbb{Z},\{1\})}(t) = 1 + 2t + 2t^2 + 2t^3 + \ldots = 1 + 2(t + t^2 + t^3 \ldots) = 1 + \frac{2t}{1-t} = \frac{1+t}{1-t}$.

Given two groups G and H with finite generating sets S_G resp. S_H, to construct a new group using given ones it is natural to use direct or free product construction. For direct and free products, there are formulas for the spherical growth series in terms of spherical growth series of the factors [11].

$$\mathcal{G}_{(G\times H, S_G\cup S_H)}(t) = \mathcal{G}_{(G,S_G)}(t)\cdot\mathcal{G}_{(H,S_H)}(t)\,,$$
$$\frac{1}{\mathcal{G}_{(G*H,S_G\cup S_H)}(t)} = \frac{1}{\mathcal{G}_{(G,S_G)}(t)} + \frac{1}{\mathcal{G}_{(H,S_H)}(t)} - 1.$$

A generalization of direct resp. free product construction is amalgamated products and graph products of groups. Let us recall a formula for the spherical growth series of an amalgamated product. First, we need a definition.

Definition 2.4. *Let (G, S) be a pair where G is a group generated by a finite set S. A pair (H, T) is admissible in (G, S), if H is a subgroup of G, $T \subseteq S$, and there exists a tranversal U for H in G such that if $g = hu$ with $g \in G$, $h \in H$, $u \in U$, then $l_S(g) = l_T(h) + l_S(u)$. We always assume that the transversal contains the identity as the representative of H.*

It was proven in [10] that if (L, R) is admissible in (H, S) and in (K, T), then the spherical growth series of $G = H *_L K$ can be computed using smaller pieces of G.

Proposition 2.5. *If (L, R) is admissible in (H, S) and in (K, T), then*

$$\frac{1}{\mathcal{G}_{(H*_L K, S\cup T)}(t)} = \frac{1}{\mathcal{G}_{(H,S)}(t)} + \frac{1}{\mathcal{G}_{(K,T)}(t)} - \frac{1}{\mathcal{G}_{(L,R)}(t)}.$$

Now we move on to graph products of groups. Given a finite graph Γ and a collection of finitely generated groups G_x for $x \in V(\Gamma)$, the *graph product of groups* is defined as:

$$G_\Gamma = \big(*_{x\in V(\Gamma)} G_x \big) / \langle\langle [g,h] \mid g \in G_x, h \in G_y, \{x,y\} \in E(\Gamma) \rangle\rangle.$$

We note that, if Γ is discrete, then the associated graph product of groups is the free product of the vertex groups and if Γ is complete, then the associated graph product of groups is the direct product of the vertex groups. If all vertex groups are infinite cyclic, then we call G_Γ a right-angled Artin group. For every vertex group G_x let S_x be a finite generating set and we set $S := \bigcup_{x\in V(\Gamma)} S_x$. A formula for the spherical growth series of G_Γ in terms of the spherical growth series of the vertex groups was proven in [10] for isomorphic vertex groups. Here we recall a special case of this formula where G_Γ is a

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right-angled Artin group and $S_x = \{1\}$ for every $x \in V$. Let c_i be the number of complete subgraphs in Γ on i vertices. Then

$$\frac{1}{\mathcal{G}_{(G_\Gamma,S)}(t)} = \sum_i (-1)^i c_i \frac{(\frac{1+t}{1-t}-1)^i}{(\frac{1+t}{1-t})^i}.$$

This formula was generalized for arbitrary vertex groups in [6]. Let G_Γ be a graph product of finitely generated vertex groups. We define for each complete subgraph $\Delta \subseteq \Gamma$, $P_\Delta(t) := \prod_{x\in V(\Delta)}(\frac{1}{\mathcal{G}_{(G_x,S_x)}(t)} - 1)$. Then

$$\frac{1}{\mathcal{G}_{(G_\Gamma,S)}(t)} = \sum P_\Delta(t),$$

where the summation is taken over all complete subgraphs of Γ including the empty one for which $P_\emptyset(t) = 1$.

We want also to point out that the geodesic growth series of graph products of cyclic groups was studied in [12].

Further groups for which it is possible to compute the spherical growth series using smaller building blocks of the group are Coxeter groups. Coxeter groups have special subgroups which can be considered as building blocks for the whole group. Given a finite graph Γ with an edge-labelling $m\colon E(\Gamma) \to \mathbb{N}_{\geq 2}$. The Coxeter group associated to Γ is given by the presentation:

$$W = \langle V(\Gamma) \mid x^2 \text{ for all } x \in V(\Gamma), (xy)^{m(\{x,y\})} \text{ for all } \{x,y\} \in E(\Gamma)\rangle.$$

For any subset $X \subseteq V(\Gamma)$ the subgroup generated by the set X is canonically isomorphic to the Coxeter group which is associated to the full subgraph of Γ with the vertex set X. This subgroup is called a *standard parabolic subgroup* and we denote it by W_X. A natural question is if it is possible to use the spherical growth series of special parabolic subgroups to obtain a formula for the spherical growth series of the whole group. It was proven in [17], [2] that it is indeed the case. Let (W,S) be a Coxeter system. If W is finite, then

$$\frac{t^m + (-1)^{|S|+1}}{\mathcal{G}_{(W,S)}(t)} = \sum_{X \subsetneq S} \frac{(-1)^{|X|}}{\mathcal{G}_{(W_X,X)}(t)},$$

where $m = max\,\{l(w) \mid w \in W\}$. The spherical growth series of a finite Coxeter group can also be calculated using the non-recursive formula:

$$\mathcal{G}_{(W,S)}(t) = \prod_{i=1}^{k}(1 + t + \ldots + t^{m_i}),$$

where $m_1,\ldots,m_k$ are the exponents of (W,S).

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If W is infinite, then

$$\frac{(-1)^{|S|+1}}{\mathcal{G}_{(W,S)}(t)} = \sum_{X \subsetneq S} \frac{(-1)^{|X|}}{\mathcal{G}_{(W_X,X)}(t)}.$$

In particular, the above formulas show that for a Coxeter group W there exists a polynomial $f(t)$ such that:

$$\frac{f(t)}{\mathcal{G}_{(W,S)}(t)} = \sum_{X \subsetneq S} \frac{(-1)^{|X|}}{\mathcal{G}_{(W_X,X)}(t)}.$$

3. Dyer groups

We begin this chapter with the definition of the main protagonist in this article, a Dyer group. For two letters a, b and a natural number m we define $\pi(a, b, m) := abababa\ldots$ where the length of the word is m. For example $\pi(a, b, 3) = aba$.

Definition 3.1.

(1) *A Dyer graph is a triple* (Γ, m, f) *where* Γ *is a graph with finite vertex set* $V = V(\Gamma)$, $f\colon V \to \mathbb{N}_{\geq 2} \cup \{\infty\}$ *and* $m\colon E(\Gamma) \to \mathbb{N}_{\geq 2}$ *are maps. For every edge* $e = \{x, y\} \in E(\Gamma)$, *if* $m(e) \neq 2$, *then* $f(x) = f(y) = 2$.

(2) *The associated Dyer group is defined as follows:*

$$D := \langle V \mid x^{f(x)}, x \in V \text{ if } f(x) \neq \infty, \pi(x, y, m(\{x, y\})) = \pi(y, x, m(\{x, y\})) \text{ if } \{x, y\} \in E(\Gamma)\rangle.$$

(3) *The associated pair (D, V) where D is a Dyer group and* $V = V(\Gamma)$ *is called a Dyer system.*

3.1. Dyer tools

We start by recalling several results which were proven by Dyer in [7]. Let G be a group and $g \in G$. We denote the order of g by $o(g)$. If $o(g)$ is finite, then we write $\mathbb{Z}_{o(g)}$ for the cyclic group of cardinality $o(g)$ and if $o(g)$ is infinite, then we write $\mathbb{Z}_{o(g)}$ for the infinite cyclic group. More generally we use the notation $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ if n is a positive integer and $\mathbb{Z}_\infty = \mathbb{Z}$.

Let (D, V) be a Dyer system. By definition, a conjugate of a generator $x \in V$ is called a *reflection*. We define

$$R := \left\{gxg^{-1} \mid g \in D, x \in V\right\}.$$

R is the set of all reflections in D. For $\rho \in R$ we define a copy of $\mathbb{Z}_{o(\rho)}$ as $H_\rho = \left\{a[\rho] \mid a \in \mathbb{Z}_{o(\rho)}\right\}$. The set H_ρ is an abelian group whose group operation is defined by

$a[\rho] + b[\rho] := (a + b)[\rho]$. Hence H_ρ is isomorphic to $\mathbb{Z}_{o(\rho)}$. Further, we define

$$M = \bigoplus_{\rho \in R} H_\rho.$$

This set is an abelian group with canonical group operation $\sum a_\rho[\rho] + \sum b_\rho[\rho] = \sum(a_\rho + b_\rho)[\rho]$. Furthermore, this abelian group is a D-module where the structure of the D-module is defined for $g \in D$ by:

$$g \cdot \sum a_\rho[\rho] := \sum a_\rho[g\rho g^{-1}].$$

Let $g \in D$. We pick one syllabic representative $(x_1^{a_1}, x_2^{a_2}, \dots, x_l^{a_l})$ for g, that is, a tuple of syllables such that $g = x_1^{a_1} x_2^{a_2} \dots x_l^{a_l}$. For each $i \in \{1, \dots, l\}$ we define a reflection:

$$\rho_i := x_1^{a_1} x_2^{a_2} \dots x_{i-1}^{a_{i-1}} x_i x_{i-1}^{-a_{i-1}} \dots x_2^{-a_2} x_1^{-a_1}.$$

We set

$$N(g) = \sum_{i=1}^{l} a_i[\rho_i] \in M.$$

For $n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and $a \in \mathbb{Z}_n$ we denote by $\|a\|_n$ the word length of a with respect to the generating set $\{1\}$.

Theorem 3.2. ([7]) *Let (D, V) be a Dyer system. Let $g, h \in D$.*

(1) N(g) does not depend on the choice of the syllabic representative for g.
(2) Let $N(g) = \sum_{\rho \in R} a_\rho(g)[\rho]$. Then
  (a) $l_{sy}(g) = |\{\rho \in R \mid a_\rho(g) \neq 0\}|$.
  (b) $l(g) = \sum_{\rho \in R} \|a_\rho(g)\|_{o(\rho)}$.
(3) $N(gh) = N(g) + g \cdot N(h)$.

Let $g \in D$. A syllabic representative $(x_1^{a_1}, x_2^{a_2}, \dots, x_l^{a_l})$ for g is called *reduced* if $l = l_{sy}(g)$. The following is a direct consequence of part (2) of Theorem 3.2 and it will be often used hereafter.

Corollary 3.3. *Let $g \in D$ and $(x_1^{a_1}, x_2^{a_2}, \dots, x_l^{a_l})$ be a reduced syllabic representative for g. Then*

$$l(g) = \|a_1\|_{o(x_1)} + \|a_2\|_{o(x_2)} + \ldots + \|a_l\|_{o(x_l)}.$$

Proof. For each $i \in \{1, \dots, l\}$ we set

$$\rho_i = x_1^{a_1} x_2^{a_2} \dots x_{i-1}^{a_{i-1}} x_i x_{i-1}^{-a_{i-1}} \dots x_2^{-a_2} x_1^{-a_1}.$$

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Then

$$N(g) = \sum_{i=1}^{l} a_i[\rho_i].$$

Since $l = l_{sy}(g)$, by Theorem 3.2 (2–a) we have $\rho_i \neq \rho_j$ for $i \neq j$. By Theorem 3.2 (2–b) it follows that:

$$l(g) = \|a_1\|_{o(\rho_1)} + \|a_2\|_{o(\rho_2)} + \ldots + \|a_l\|_{o(\rho_l)} = \|a_1\|_{o(x_1)} + \|a_2\|_{o(x_2)} + \ldots + \|a_l\|_{o(x_l)}.$$

□

3.2. Standard parabolic subgroups

Let (D, V) be a Dyer system. For any subset $X \subseteq V$, we denote the subgroup generated by the set X by $D_X \subseteq D$. D_X is called the *standard parabolic subgroup* generated by X.

Let (Γ, m, f) be the Dyer graph associated with (D, V). We denote by Γ_X the full subgraph of Γ spanned by X, by m_X the restriction of m to $E(\Gamma_X)$, and by f_X the restriction of f to $V(\Gamma_X) = X$. Then (Γ_X, m_X, f_X) is a Dyer graph and we know by [7] that (D_X, X) is the Dyer system associated with (Γ_X, m_X, f_X) (see also [13, Proposition 2.7]).

Lemma 3.4. *Let (D, V) be a Dyer system. Let D_X be a standard parabolic subgroup of D. Then for any $g \in D_X$ we have $l_X(g) = l_V(g)$.*

Proof. Let $g \in D_X$. Let $(x_1^{a_1}, x_2^{a_2}, \ldots, x_l^{a_l})$ be a reduced syllabic representative for g. We know by [13, Lemma 2.5] that $x_1, x_2, \ldots, x_l \in X$, hence, by Corollary 3.3,

$$l_V(g) = \|a_1\|_{o(x_1)} + \|a_2\|_{o(x_2)} + \ldots + \|a_l\|_{o(x_l)} \geq l_X(g).$$

It is clear that we also have $l_X(g) \geq l_V(g)$, thus $l_X(g) = l_V(g)$. □

Proposition 3.5. *Let (D, V) be a Dyer system and D_X be a standard parabolic subgroup. Then for every $g \in D$*

(1) there exist a unique $g_0 \in gD_X$ of minimal syllabic length in gD_X and g_0 satisfies:

$$l_{sy}(g_0h) = l_{sy}(g_0) + l_{sy}(h) \textit{and } l(g_0h) = l(g_0) + l(h)$$

for all $h \in D_X$.

(2) there exists a unique $g_0' \in D_Xg$ of minimal syllabic length in D_Xg and g_0 satisfies

$$l_{sy}(hg_0') = l_{sy}(h) + l_{sy}(g_0') \textit{and } l(hg_0') = l(h) + l(g_0')$$

for all $h \in D_X$.

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Proof. The statements regarding syllabic length were proved in [13, Proposition 2.8]. Hence we know that there exists a unique $g_0 \in gD_X$ such that for all $h \in D_X$ we have:

$$l_{sy}(g_0 h) = l_{sy}(g_0) + l_{sy}(h).$$

Let $h \in D_X$. Let $(x_1^{a_1}, \ldots, x_p^{a_p})$ be a reduced syllabic representative for g_0 and $(y_1^{b_1}, \ldots, y_q^{b_q})$ be a reduced syllabic representative for h. We know from the above that $(x_1^{a_1}, \ldots, x_p^{a_p}, y_1^{b_1}, \ldots, y_q^{b_q})$ is a reduced syllabic representative for $g_0 h$, hence, by Corollary 3.3,

$$l(g_0 h) = \|a_1\|_{o(x_1)} + \ldots + \|a_p\|_{o(x_p)} + \|b_1\|_{o(y_1)} + \ldots + \|b_q\|_{o(y_q)} = l(g_0) + l(h).$$

The proof of part (2) is the same as for part (1). □

Corollary 3.6. *Let* (Γ, m, f) *be a Dyer graph and (D, V) be the associated Dyer system. Every pair* (D_X, X) *where* D_X *is a standard parabolic subgroup is admissible.*

Proof. For a standard parabolic subgroup D_X and an element $g \in D$ there exists a unique $g_0 \in gD_X$ such that $l(g_0 h) = l(g_0) + l(h)$ for all $h \in D_X$. We take these minimal elements as a transversal. Lemma 3.4 and Proposition 3.5 show that this transversal is admissible. □

Corollary 3.7. *Let* (Γ, m, f) *be a Dyer graph. Let* $v \in V(\Gamma)$*. If* $st(v) \neq V(\Gamma)$*, then*

$$D = D_{V-\{v\}} *_{D_{lk(v)}} D_{st(v)},$$

and

$$\frac{1}{\mathcal{G}_{(D,V)}(t)} = \frac{1}{\mathcal{G}_{(D_{V-\{v\}}, V-\{v\})}(t)} + \frac{1}{\mathcal{G}_{(D_{st(v)}, st(v))}(t)} - \frac{1}{\mathcal{G}_{(D_{lk(v)}, lk(v))}(t)}.$$

Proof. The proof of the equality $D = D_{V-\{v\}} *_{D_{lk(v)}} D_{st(v)}$ follows by analysing the presentation of D and the canonical presentation of the amalgam. By Corollary 3.6 the groups in this amalgamated product are admissible. Thus by Proposition 2.5 we get the above equality of the spherical growth series. □

4. *X*-minimality

Definition 4.1. *Let (D, V) be a Dyer system and* $g \in D$*. For* $X \subseteq V$ *the element* g *is called X-minimal if* $l_{sy}(g) \leq l_{sy}(gh)$ *for all* $h \in D_X$*.*

Note that, by Proposition 3.5, if g is X-minimal, then $l_{sy}(gh) = l_{sy}(g) + l_{sy}(h)$ and $l(gh) = l(g) + l(h)$ for all $h \in D_X$. Note also that, if $X \subseteq Y \subseteq V$ and g is Y-minimal, then g is also X-minimal, since $gD_X \subseteq gD_Y$.

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Definition 4.2. *Let (D, V) be a Dyer system and $X \subseteq V$. We define two subsets of D as follows:*

$$A_X = A_X(D) := \{g \in D \mid g \text{is } X\text{-minimal}\} \text{ and } B_X = B_X(D) := A_X - (\cup_{X \subsetneq Y} A_Y).$$

An important feature of these sets is that, for $X \subseteq V$, the set A_X is the disjoint union of those B_Y with $X \subseteq Y$. This property is less obvious than it seems and follows from the following lemma.

Lemma 4.3. *Let (D, V) be a Dyer system, let $g \in D$, and let $X, Y \subseteq V$. If g is both X-minimal and Y-minimal, then g is $(X \cup Y)$-minimal.*

Proof. Let g_0 be the unique $(X \cup Y)$-minimal element in $gD_{X\cup Y}$. By Proposition 3.5 there exists $h \in D_{X\cup Y}$ such that $g = g_0 h$ and $l_{sy}(g) = l_{sy}(g_0) + l_{sy}(h)$. Suppose $g_0 \neq g$, that is, $l_{sy}(h) \geq 1$. Let $(x_1^{a_1}, \ldots, x_p^{a_p})$ be a reduced syllabic representative for h. We know from [13, Lemma 2.5] that $x_1, \ldots, x_p \in X \cup Y$. But, if $x_p \in X$, then g is not X-minimal, and if $x_p \in Y$, then g is not Y-minimal. This is a contradiction, hence $h = 1$ and $g = g_0$. □

Corollary 4.4. *Let (D, V) be a Dyer system and let $X \subseteq V$. Then A_X is the disjoint union of those B_Y with $X \subseteq Y$.*

The following lemma is a particular case of the well-known general Möbius inversion formula (see [18, Section 3.7] or [14] for example).

Lemma 4.5. *Let V be a set and $\mathcal{P}(V)$ be the set of all subsets of V. Further, let G be an abelian group. If the functions $f\colon \mathcal{P}(V) \to G$ and $g\colon \mathcal{P}(V) \to G$ satisfy*

$$f(X) = \sum_{X \subseteq Y} g(Y) \text{ for all } X \in \mathcal{P}(V),$$

then they satisfy

$$g(X) = \sum_{X \subseteq Y} (-1)^{|Y-X|} f(Y) \text{ for all } X \in \mathcal{P}(V).$$

Proposition 4.6. *Let (D, V) be a Dyer system. For $X \subseteq V$ we have*

$$\mathcal{G}_{(B_X,V)}(t) = \sum_{X \subseteq Y} (-1)^{|Y-X|} \frac{\mathcal{G}_{(D,V)}(t)}{\mathcal{G}_{(D_Y,Y)}(t)}.$$

In particular, for $X = \emptyset$ we obtain:

$$\mathcal{G}_{(B_\emptyset,V)}(t) = \sum_{Y} (-1)^{|Y|} \frac{\mathcal{G}_{(D,V)}(t)}{\mathcal{G}_{(D_Y,Y)}(t)},$$

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which is equivalent to:

$$\frac{\mathcal{G}_{(B_\emptyset,V)}(t)+(-1)^{|V|+1}}{\mathcal{G}_{(D,V)}(t)}=\sum_{Y\subsetneq V}\frac{(-1)^{|Y|}}{\mathcal{G}_{(D_Y,Y)}(t)}.$$

Proof. Let (D, V) be a Dyer system. We define two functions $f\colon \mathcal{P}(V)\to\mathbb{Z}[[t]]$ and $g\colon \mathcal{P}(V)\to\mathbb{Z}[[t]]$ where $\mathbb{Z}[[t]]$ is the formal power series ring with coefficients in the group $\mathbb{Z}$ as follows:

$$f(X)(t)=\sum_{g\in A_X}t^{l(g)}\text{and } g(X)(t)=\sum_{g\in B_X}t^{l(g)}.$$

By Corollary 4.4 the set A_X is a disjoint union of those B_Y, where $X\subseteq Y$. Hence we have

$$f(X)(t)=\sum_{X\subseteq Y}g(Y)(t),$$

and by Lemma 4.5 we obtain

$$g(X)(t)=\sum_{X\subseteq Y}(-1)^{|Y-X|}f(Y)(t).$$

By definition we have $g(X)(t)=\mathcal{G}_{(B_X,V)}(t)$. Thus we obtain

$$\mathcal{G}_{(B_X,V)}(t)=\sum_{X\subseteq Y}(-1)^{|Y-X|}f(Y)(t).$$

Further, $D=\cup_{g\in A_Y}gD_Y$ and this union is disjoint. More precisely, two cosets are equal or disjoint. Assume that there exist $g_1,g_2\in A_Y$, $g_1\neq g_2$ such that $g_1D_Y=g_2D_Y$. Since g_1 and g_2 are both Y-minimal it follows by Proposition 3.5 that $g_1=g_2$, contradiction. We obtain

$$\mathcal{G}_{(D,V)}(t)=\sum_{g\in A_Y}\sum_{u\in D_Y}t^{l(g)+l(u)}=f(Y)(t)\cdot\mathcal{G}_{(D_Y,V)}(t).$$

Finally, by Lemma 3.4 we have $\mathcal{G}_{(D_Y,V)}(t)=\mathcal{G}_{(D_Y,Y)}(t)$, hence

$$\mathcal{G}_{(B_X,V)}(t)=\sum_{X\subseteq Y}(-1)^{|Y-X|}f(Y)(t)=\sum_{X\subseteq Y}(-1)^{|Y-X|}\frac{\mathcal{G}_{(D,V)}(t)}{\mathcal{G}_{(D_Y,Y)}(t)}.$$

□

Our next task is to give a good description of the set $B_\emptyset$. We are particularly interested in properties of (D, V) that ensure the set $B_\emptyset$ to be empty.

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Definition 4.7. *Let* (Γ, m, f) *be a Dyer graph and* (D, V) *be the associated Dyer system. We define* $V_2 := \{x \in V | f(x) = 2\}$, $V_\infty := \{x \in V | f(x) = \infty\}$, $V_p := \{x \in V | 2 < f(x) < \infty\}$. *Let* D_2 *resp.* D_∞ *resp.* D_p *be the standard parabolic subgroup of* D *generated by* V_2 *resp.* V_∞ *resp.* V_p.

The Dyer group D *is called of spherical type if* Γ *is a complete graph and* D_2 *is a finite Coxeter group.*

Note that, if D is of spherical type, then $D = D_2 \times D_p \times D_\infty$, $D_p = \prod_{x \in V_p} \mathbb{Z}/f(x)\mathbb{Z}$, and $D_\infty = \mathbb{Z}^l$ where $l = |V_\infty|$.

The description of $B_\emptyset$ when $D = D_2$ is a Coxeter group is well-known and it is a direct consequence of the following.

Proposition 4.8. ([2]) *Let* (W, S) *be a Coxeter system. The following conditions on an element* $w_0 \in W$ *are equivalent.*

(a) *For each* $u \in W$, $l(w_0) = l(w_0 u^{-1}) + l(u)$.
(b) *For each* $s \in S$, $l(w_0 s) < l(w_0)$.

Moreover, w_0 *exists if and only if* W *is finite. If* w_0 *satisfies (a) and/or (b), then* w_0 *is unique,* w_0 *is an involution, and* $w_0 S w_0 = S$.

The element w_0 of Proposition 4.8 is called the *longest element* of W, if it exists. The following is a straightforward consequence of Proposition 4.8.

Corollary 4.9. *Let* (W, S) *be a Coxeter system. We have* $B_\emptyset(W) \neq \emptyset$ *if and only if* W *is finite. If* W *is finite, then* $B_\emptyset(W) = \{w_0\}$, *where* w_0 *is the longest element of* W.

In the general case we have the following.

Lemma 4.10. *Let* (D, V) *be a Dyer system.*

(1) We have $B_\emptyset \neq \emptyset$ *if and only if* D *is of spherical type.*
(2) Suppose D *is of spherical type. Set* $V_p = \{x_1, \ldots, x_k\}$ *and* $V_\infty = \{y_1, \ldots, y_l\}$. *Let* $g \in D$. *Then* $g \in B_\emptyset$ *if and only if* g *can be written in the form:*

$$g = w_0 x_1^{a_1} \ldots x_k^{a_k} y_1^{b_1} \ldots y_l^{b_l},$$

where w_0 *is the longest element of* D_2, $a_i \in (\mathbb{Z}/f(x_i)\mathbb{Z}) - \{0\}$ *for all* $i \in \{1, \ldots, k\}$, *and* $b_j \in \mathbb{Z} - \{0\}$ *for all* $j \in \{1, \ldots, l\}$.

Proof. We first prove that, if $B_\emptyset \neq \emptyset$, then D is of spherical type. We will then show that, if D is of spherical type, then $B_\emptyset \neq \emptyset$ and the elements of $B_\emptyset$ are as described in part (2).

Suppose $B_\emptyset \neq \emptyset$. This means that there exists $g \in D$ such that $g \notin A_{\{x\}}$ for all $x \in V$. So, we can pick $g \in D$ such that, for all $x \in V$, there exists $a \in \mathbb{Z}_{f(x)} - \{0\}$ such that $l_{sy}(gx^a) \leq l_{sy}(g)$. We start by showing that Γ is complete. Let $x, y \in V$, $x \neq y$. Set $X = \{x, y\}$. We know that there exist $a \in \mathbb{Z}_{f(x)} - \{0\}$ and $b \in \mathbb{Z}_{f(y)} - \{0\}$ such as

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$l_{sy}(gx^a) \le l_{sy}(g)$ and $l_{sy}(gy^b) \le l_{sy}(g)$. On the other hand by Proposition 3.5, there exist $g_0 \in A_X$ and $h \in D_X$ such that $g = g_0 h$. By Proposition 3.5 we have

$$l_{sy}(g) = l_{sy}(g_0) + l_{sy}(h)\,,\ l_{sy}(gx^a) = l_{sy}(g_0) + l_{sy}(hx^a)\,,\ l_{sy}(gy^b) = l_{sy}(g_0) + l_{sy}(hy^b).$$

Thus, $l_{sy}(hx^a) \le l_{sy}(h)$ and $l_{sy}(hy^b) \le l_{sy}(h)$. If x and y are not connected by an edge, then $D_X = \mathbb{Z}_{f(x)} * \mathbb{Z}_{f(y)}$ and there is no h in $\mathbb{Z}_{f(x)} * \mathbb{Z}_{f(y)}$ such that $l_{sy}(hx^a) \le l_{sy}(h)$ and $l_{sy}(hy^b) \le l_{sy}(h)$. So, x and y are connected by an edge.

Since Γ is a complete graph, we have $D = D_2 \times D_p \times D_\infty$, $D_p = \prod_{x \in V_p} \mathbb{Z}/f(x)\mathbb{Z}$, and $D_\infty = \mathbb{Z}^l$, where $l = |V_\infty|$. Assume that $B_\emptyset \ne \emptyset$. Let $g \in B_\emptyset$. We write $g = g_2 g_p g_\infty$ with $g_2 \in D_2$, $g_p \in D_p$, and $g_\infty \in D_\infty$. For each $x \in V_2$ we have

$$l(g_2x) + l_{sy}(g_p) + l_{sy}(g_\infty) = l_{sy}(gx) < l_{sy}(g) = l(g_2) + l_{sy}(g_p) + l_{sy}(g_\infty),$$

hence $l(g_2x) < l(g_2)$.

By Proposition 4.8 this implies that D_2 is a finite Coxeter group and g_2 is the longest element of D_2. So, if $B_\emptyset \ne \emptyset$, then D is of spherical type.

Suppose now that D is of spherical type. Then $D = D_2 \times D_p \times D_\infty$, D_2 is a finite Coxeter group, $D_p = \prod_{x \in V_p} \mathbb{Z}/f(x)\mathbb{Z}$, and $D_\infty = \mathbb{Z}^l$, where $l = |V_\infty|$. Set $V_p = \{x_1, \dots, x_k\}$ and $V_\infty = \{y_1, \dots, y_l\}$. Let $g \in B_\emptyset$. Write g in the form $g = w x_1^{a_1} \dots x_k^{a_k} y_1^{b_1} \dots y_l^{b_l}$ with $w \in D_2$, $a_i \in \mathbb{Z}/f(x_i)\mathbb{Z}$ for all $i \in \{1, \dots, k\}$, and $b_j \in \mathbb{Z}$ for all $j \in \{1, \dots, l\}$. Let $i \in \{1, \dots, k\}$. If we had $a_i = 0$, then we would have $l_{sy}(gx_i^c) > l_{sy}(g)$ for all $c \in (\mathbb{Z}/f(x_i)\mathbb{Z}) - \{0\}$, hence we would have $g \notin B_\emptyset$. So $a_i \ne 0$ for all $i \in \{1, \dots, k\}$. Similarly, $b_j \ne 0$ for all $j \in \{1, \dots, l\}$. If w were not the longest element of D_2, then there would exist $x \in V_2$ such that $l(wx) > l(w)$, hence there would exist $x \in V_2$ such that $l_{sy}(gx) > l_{sy}(g)$. So, w is the longest element of D_2.

Let $g \in D$ which can be written in the form $g = w_0 x_1^{a_1} \dots x_k^{a_k} y_1^{b_1} \dots y_l^{b_l}$, where w_0 is the longest element of D_2, $a_i \in (\mathbb{Z}/f(x_i)\mathbb{Z}) - \{0\}$ for all $i \in \{1, \dots, k\}$, and $b_j \in \mathbb{Z} - \{0\}$ for all $j \in \{1, \dots, l\}$. Notice that such an element always exists. It is easily seen that $l_{sy}(gx_i^{-a_i}) < l_{sy}(g)$ for all $i \in \{1, \dots, k\}$ and $l_{sy}(gy_j^{-b_j}) < l_{sy}(g)$ for all $j \in \{1, \dots, l\}$. On the other hand, if $x \in V_2$, then $l(w_0x) < l(w_0)$, hence $l_{sy}(gx) < l_{sy}(g)$. So, $g \in B_\emptyset$. □

Let (D, V) be a Dyer system of spherical type. So, $D = D_2 \times D_p \times D_\infty$, D_2 is a finite Coxeter group, $D_p = \prod_{x \in V_p} \mathbb{Z}/f(x)\mathbb{Z}$, and $D_\infty = \mathbb{Z}^l$, where $l = |V_\infty|$. Let $x \in V_p$. If $f(x) = 2r$ is even we set $P_x(t) = 2t + 2t^2 + \dots + 2t^{r-1} + t^r$, and if $f(x) = 2r + 1$ is odd we set $P_x(t) = 2t + 2t^2 + \dots + 2t^r$. Then we set

$$P_D(t) = t^m \left(\prod_{x \in V_p} P_x(t) \right) \frac{2^l t^l}{(1-t)^l},$$

where m is the maximal length in D_2.

As an immediate corollary we obtain the following whose first part finishes the proof of Theorem 1.1.

Corollary 4.11. *Let (D, V) be a Dyer system.*

(1) If D is not of spherical type, then

$$\frac{(-1)^{|V|+1}}{\mathcal{G}_{(D,V)}(t)} = \sum_{Y \subsetneq V} \frac{(-1)^{|Y|}}{\mathcal{G}_{(D_Y,Y)}(t)}.$$

(2) If D is of spherical type, then

$$\frac{P_D(t) + (-1)^{|V|+1}}{\mathcal{G}_{(D,V)}(t)} = \sum_{Y \subsetneq V} \frac{(-1)^{|Y|}}{\mathcal{G}_{(D_Y,Y)}(t)}.$$

Proof. By Proposition 4.6 it suffices to show that $\mathcal{G}_{(B_\emptyset,V)}(t) = 0$ if D is not of spherical type and that $\mathcal{G}_{(B_\emptyset,V)}(t) = P_D(t)$ if D is of spherical type. If D is not of spherical type then, by Lemma 4.10, $B_\emptyset = \emptyset$, hence $\mathcal{G}_{(B_\emptyset,V)}(t) = 0$. Suppose D is of spherical type. Let $V_p = \{x_1, \ldots, x_k\}$ and $V_\infty = \{y_1, \ldots, y_l\}$, and let m be the maximal length in D_2. Then, by Lemma 4.10,

$$\mathcal{G}_{(B_\emptyset,V)}(t) = t^m \left(\prod_{i=1}^{k} \mathcal{G}_{(\mathbb{Z}/f(x_i)\mathbb{Z}-\{0\},\{1\})}(t)\right) \left(\prod_{i=1}^{l} \mathcal{G}_{(\mathbb{Z}-\{0\},\{1\})}(t)\right) =$$
$$t^m \left(\prod_{i=1}^{k} P_{x_i}(t)\right) \left(\frac{2t}{1-t}\right)^l = P_D(t).$$

□

We end this chapter with the proof of Theorem 1.1.

Proof of Theorem 1.1. The first part follows from Corollary 4.11 which we already mentioned above. We assume now that D is of spherical type, then $D = D_2 \times D_p \times D_\infty$. Hence we can use the formula for direct products on page 5. We get

$$\mathcal{G}_{(D,V)}(t) = \mathcal{G}_{(D_2,V_2)}(t) \cdot \mathcal{G}_{(D_p,V_p)}(t) \cdot \mathcal{G}_{(D_\infty,V_\infty)}(t).$$

Further, since $\mathcal{G}_{(\mathbb{Z},\{1\})}(t) = \frac{1+t}{1-t}$ we get $\mathcal{G}_{(D_\infty,V_\infty)}(t) = \frac{(1+t)^l}{(1-t)^l}$ where l is the cardinality of V_∞. A direct calculation shows the formulas for the spherical growth series of finite cyclic groups with the standard generating sets which ends the proof of the second part of the theorem. □

5. Euler characteristic

We start this chapter by recalling the definition and useful formulas of the Euler characteristic of groups. Following [3] a group G is said to be of *finite homological type* if the virtual cohomological dimension of G is finite and for every G-module M which is

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finitely generated as an abelian group, $H_i(G, M)$ is finitely generated for all i. If G is torsion-free and of finite homological type, then its Euler characteristic is defined by:

$$\chi(G) := \sum(-1)^i rk_{\mathbb{Z}}(H_i(G)).$$

If G is of finite homological type and has a torsion free subgroup H of finite index, then the Euler characteristic of G is defined by:

$$\chi(G) := \frac{\chi(H)}{[G : H]}.$$

We list some useful properties of the Euler characteristic.

Proposition 5.1. ([3])*[Proposition 7.3]*

(1) Let $1 \to A \to B \to C \to 1$ be a short exact sequence where A and C are of finite homological type. If B is virtually torsion-free, then B is of finite homolocial type and

$$\chi(B) = \chi(A) \cdot \chi(C).$$

(2) Let $G = A *_B C$ be an amalgamated product where A, B, C are of finite homological type. If G is virtually torsion free, then G is of finite homological type and

$$\chi(G) = \chi(A) + \chi(C) - \chi(B).$$

As a corollary we obtain

Corollary 5.2. *Let (D, V) and (D', V') be Dyer systems.*

(1) $\chi(D \times D') = \chi(D) \cdot \chi(D')$.
(2) If $D = D_{V-\{x\}} *_{D_{lk(x)}} D_{st(x)}$, then

$$\chi(D) = \chi(D_{V-\{x\}}) + \chi(D_{st(x)}) - \chi(D_{lk(x)}).$$

Proof. It was proven in [16, Corollary 1.2] that every Dyer group is a subgroup of finite index in a Coxeter group. Further, it was proven in [15] that Coxeter groups are of finite homological type. Since the property of being of finite homological type is preserved by taking finite index subgroups [3, Lemma 6.1], we know that every Dyer group is of finite homological type and is therefore virtually torsion free. Proposition 5.1 shows the results of the corollary. □

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Theorem 5.3 *Let* (Γ, m, f) *be a Dyer graph and (D, V) be the associated Dyer system. Then*

$$\frac{1}{\mathcal{G}_{(D,V)}(1)} = \chi(D).$$

Proof. Assume first that Γ is complete. In this case $D = D_2 \times D_p \times D_\infty$, where D_p is finite and $D_\infty \cong \mathbb{Z}^l$. Hence, by Corollary 5.2

$$\chi(D) = \chi(D_2) \cdot \chi(D_p) \cdot \chi(\mathbb{Z}^l).$$

Since D_2 is a Coxeter group we know by [15] that $\chi(D_2) = \frac{1}{\mathcal{G}_{(D_2,V_2)}(1)}$. We get

$$\chi(D) = \frac{1}{\mathcal{G}_{(D_2,V_2)}(1)} \cdot \chi(D_p) \cdot \chi(\mathbb{Z}^l).$$

Further, since D_p is finite we have $\chi(D_p) = \frac{1}{|D_p|}$ and $\mathcal{G}_{(D_p,V_p)}(1) = |D_p|$. Thus

$$\chi(D) = \frac{1}{\mathcal{G}_{(D_2,V_2)}(1)} \cdot \frac{1}{\mathcal{G}_{(D_p,V_p)}(1)} \cdot \chi(\mathbb{Z}^l).$$

We know that $\chi(\mathbb{Z}^l) = 0$ and $\frac{1}{\mathcal{G}_{(D_\infty,V_\infty)}(1)} = \frac{(1-1)^l}{(1+1)^l} = 0$ if $l > 0$, and $\chi(\mathbb{Z}^l) = 1$ and $\frac{1}{\mathcal{G}_{(D_\infty,V_\infty)}(1)} = 1$ if $l = 0$. Hence

$$\chi(D) = \frac{1}{\mathcal{G}_{(D_2,V_2)}(1)} \cdot \frac{1}{\mathcal{G}_{(D_p,V_p)}(1)} \cdot \frac{1}{\mathcal{G}_{(D_\infty,V_\infty)}(1)} = \frac{1}{\mathcal{G}_{(D,V)}(1)}.$$

Now assume that Γ is not complete, then there exists $x \in V(\Gamma)$ such that $st(x) \neq V(\Gamma)$. Then we have

$$D = D_{V-\{x\}} *_{D_{lk(x)}} D_{st(x)}.$$

By Corollary 3.7 we obtain

$$\frac{1}{\mathcal{G}_{(D,V)}(1)} = \frac{1}{\mathcal{G}_{(D_{V-\{x\}},V-\{x\})}(1)} + \frac{1}{\mathcal{G}_{(D_{st(x)},st(x))}(1)} - \frac{1}{\mathcal{G}_{(D_{lk(x)},lk(x))}(1)}.$$

By Corollary 5.2 we get

$$\chi(D) = \chi(D_{V-\{x\}} *_{D_{lk(x)}} D_{st(x)}) = \chi(D_{V-\{x\}}) + \chi(D_{st(x)}) - \chi(D_{lk(x)}).$$

We decompose $D_{V-\{x\}}$, $D_{st(x)}$ and $D_{lk(x)}$ again in amalgamated products. Using this strategy we will get a linear combination of Euler characteristics resp. spherical growth

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series of standard parabolic subgroups of D where all defining graphs are complete. Applying the above formulas we get

$$\frac{1}{\mathcal{G}_{(D,V)}(1)} = \chi(D).$$

□

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