ISOMORPHISMS BETWEEN ENDOMORPHISM RINGS OF PROJECTIVE MODULES

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Let R and S be arbitrary rings, ${}_RM$ and ${}_SN$ countably generated free modules, and let $\phi:\text{End}({}_RM)\to\text{End}({}_SN)$ be an isomorphism between the endomorphism rings of M and N. Camillo [**3**] showed in 1984 that these assumptions imply that R and S are Morita equivalent rings. Indeed, as Bolla pointed out in [**2**], in this case the isomorphism ϕ must be induced by some Morita equivalence between R and S. The same holds true if one assumes that ${}_RM$ and ${}_SN$ are, more generally, non-finitely generated free modules.

In this note, we make the observation that the above results of Camillo and Bolla cannot be extended to a class of modules broader than that of non-finitely generated free modules in any natural way. More precisely, let $\mathcal{M}$ be now the class of all the countably generated locally free projective modules (over arbitrary rings); we give examples to show that: (1) there exist modules ${}_RM$ and ${}_SN$ in the class $\mathcal{M}$ such that $\text{End}({}_RM)\cong\text{End}({}_SN)$, while R and S are not Morita equivalent; (2) there exist ${}_RM$ in the class $\mathcal{M}$ and an automorphism δ of the endomorphism ring $\text{End}({}_RM)$ such that δ cannot be induced by any Morita auto-equivalence of the ring R.

All the rings in this paper are supposed to be associative and with identity element. A module ${}_RM$ is called *locally free* [**4**] if each finite set of elements of M is contained in a finitely generated free direct summand. If ${}_RM$ is a left R-module, then $\text{End}({}_RM)$ denotes the endomorphism ring of ${}_RM$ (and endomorphisms act opposite scalars) and $f\,\text{End}({}_RM)$ will denote the subring (not necessarily with identity) of $\text{End}({}_RM)$, given by

$$f\,\text{End}({}_RM)=\{f\in\text{End}({}_RM)\mid f=g\circ h,\, h:{}_RM\to R^n,\, g:{}_RR^n\to{}_RM,\text{ for some integer }n\}.$$

In particular, when ${}_RM$ is free and countably generated, then $\text{End}({}_RM)$ is isomorphic to the ring of row-finite matrices $\mathbb{RFM}(R)$ and $f\,\text{End}({}_RM)$ is then isomorphic to the subring of the matrices with a finite number of non-zero columns, $\mathbb{FC}(R)$.

We start with the following lemma, which will be needed for the construction of the announced examples. Notice that this lemma could also be obtained from [**9**, Corollary 1], but we give a different proof of it.

LEMMA. *Let D be a division ring. Then, the rings $\mathbb{RFM}(D)$ and $\mathbb{RFM}(\mathbb{RFM}(D))$ are not Morita equivalent rings.*

Proof. To simplify the notation, let us put $E=\mathbb{RFM}(D)$ and $S=\mathbb{RFM}(E)$. By [**1**, Exercise 7, p. 23], E has only one non-trivial ideal which is just the left socle of E, E_0. S has the non-trivial ideal $S_0=\mathbb{FC}(E)$, and S_0-mod is a category equivalent to E-mod [**5**, Theorem 2.4]. By using this equivalence and [**6**, Proposition 3.5], we see that S_0 has exactly one non-trivial ideal I satisfying that $S_0IS_0=I$. However, such an I is also an ideal of S, so that S has at least two non-trivial ideals. Thus, E and S cannot be Morita equivalent rings [**1**, Proposition 21.11].

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EXAMPLE 1. *There exist rings R, S and modules ${}_RP$, ${}_SQ$ such that ${}_RP$, ${}_SQ$ are (non-finitely) countably generated locally free and projective modules with* $\mathrm{End}({}_RP) \cong \mathrm{End}({}_SQ)$, *but R, S are not Morita equivalent rings.*

Proof. Let D be a division ring and V, W, left D-vector spaces with $\dim(V) = \aleph_0$ and $\dim(W) = \aleph_1$. We put $A = \mathrm{End}({}_DV)$, $B = \mathrm{End}({}_DW)$, and then $T = \mathbb{RFM}(A)$, $U = \mathbb{RFM}(B)$. Note that, by the Lemma, A and T cannot be Morita-equivalent rings. The rings R and S are now constructed by taking $R = A \times U$, $S = B \times T$. Finally, we choose the modules ${}_RP$ and ${}_SQ$ by putting ${}_RP = A^{(\mathbb{N})} \oplus U$, ${}_SQ = B^{(\mathbb{N})} \oplus T$. Since $\mathrm{End}({}_RP) \cong \mathrm{End}({}_AA^{(\mathbb{N})}) \times U$-and similarly for ${}_SQ$ – we have that $\mathrm{End}({}_RP) \cong T \times U \cong U \times T \cong \mathrm{End}({}_SQ)$.

${}_RP$ and ${}_SQ$ are projective modules, because ${}_AA^{(\mathbb{N})}$, ${}_UU$, ${}_BB^{(\mathbb{N})}$ and ${}_TT$ are projective; they are (non-finitely) countably generated, since so are ${}_AA^{(\mathbb{N})}$ and ${}_BB^{(\mathbb{N})}$, while ${}_RU$ and ${}_ST$ are cyclic. Moreover, for each $n \geq 1$, we have ${}_AA \cong {}_AA^n$ and ${}_BB \cong {}_BB^n$, because A and B are endomorphism rings of non-finitely generated vector spaces (see, for instance, [**7**, Example 1.3.33]). As a consequence, any finite family of elements of ${}_RP$ being included in a direct summand of the form $A^n \oplus U$, we deduce that ${}_RP$–and ${}_SQ$–are locally free modules.

It remains to show that R and S are not Morita equivalent. By [**3**, Theorem], it is enough to prove that the rings $\mathbb{RFM}(R)$ and $\mathbb{RFM}(S)$ are not isomorphic. Suppose we had such an isomorphism, so that $\mathbb{RFM}(A) \times \mathbb{RFM}(U) \cong \mathbb{RFM}(B) \times \mathbb{RFM}(T)$. But the endomorphism ring of a vector space is always indecomposable as a ring, and so we can infer that each of the four factors above is indecomposable as a ring. It follows from [**1**, Proposition 7.8] that we should have either $\mathbb{RFM}(A) \cong \mathbb{RFM}(B)$ or $\mathbb{RFM}(A) \cong \mathbb{RFM}(T)$. By applying again [**3**, Theorem], this would imply that A is Morita equivalent to one of the rings B or T. But A is not equivalent to T by the Lemma, as we saw before. Finally, A and B are not equivalent rings because A has exactly one non-trivial ideal, while B has two [**8**, p. 360].

We now turn to the above-mentioned result of Bolla [**2**]: if δ is an isomorphism between endomorphism rings of the non-finitely generated free modules ${}_RP$ and ${}_SQ$, then δ is induced by a Morita equivalence. By Example 1, this is no longer the case if ${}_RP$ and ${}_SQ$ are supposed to belong to the class $\mathcal{M}$ of non-finitely generated locally free projective modules. We show next that δ need not be induced by an equivalence, even if we assume that the rings R and S are already Morita equivalent.

EXAMPLE 2. *There exist a non-finitely generated projective and locally free left R-module ${}_RP$ and an isomorphism δ of the endomorphism ring* $\mathrm{End}({}_RP)$ *such that δ is not induced by any Morita auto-equivalence of R.*

Proof. Let D be a division ring, ${}_DV$ a non-finitely generated left D-vector space and $A = \mathrm{End}({}_DV)$. Then we put $R = D \times A$, and ${}_RP = V \times A$, so that $\mathrm{End}({}_RP) \cong A \times A$. ${}_RP$ is obviously a non-finitely generated projective module which is locally free because $A^k \cong A$ for any $k \geq 1$. Let δ be the automorphism of the ring $A \times A$ given by $\delta(a, a') = (a', a)$. Now, if $F: R\text{-mod} \to R\text{-mod}$ is any Morita equivalence satisfying that $F(P) = Q \cong P$, then F induces a ring isomorphism $\mathrm{End}({}_RP) \xrightarrow{\theta(F)} \mathrm{End}({}_RP)$ such that $\theta(F)(f\,\mathrm{End}({}_RP)) = f\,\mathrm{End}({}_RP)$, because the morphisms in $f\,\mathrm{End}({}_RP)$ are characterised in $\mathrm{End}({}_RP)$ by the property of factoring through some finitely generated projective. But $\theta(F) \neq \delta$ because if $\alpha: V \to V$ is the projection of V onto the first coordinate, then $(\alpha, 1) \in f\ \mathrm{End}(_R P)$ and $\delta(\alpha, 1) = (1, \alpha) \notin f\ \mathrm{End}(_R P)$. This shows that δ is not induced by an equivalence.

REFERENCES

1. F. W. Anderson and K. R. Fuller, *Rings and categories of modules.* (Springer-Verlag, 1974).

2. M. Bolla, Isomorphisms between infinite matrix rings, *Linear Algebra and Appl.* **69** (1985), 239–247.

3. V. Camillo, Morita equivalence and infinite matrix rings, *Proc. Amer. Math. Soc.* **90** (1984), 186–188.

4. W. N. Franzsen and P. Schultz, The endomorphism ring of a locally free module, *J. Austral. Math. Soc. Ser. A* **35** (1983), 308–326.

5. J. L. García, The characterization problem for endomorphism rings, *J. Austral. Math. Soc. Ser. A* **50** (1991), 116–137.

6. J. L. García and J. J. Simón, Morita equivalence for idempotent rings, *J. Pure Appl. Algebra* **76** (1991), 39–56.

7. L. H. Rowen, *Ring theory*, *Vol.* 1 (Academic Press, 1988).

8. K. G. Wolfson, An ideal-theoretic characterization of the ring of all linear transformations, *Amer. J. Math.* **75** (1953), 358–386.

9. L. Bonami, Morita-equivalent rings are isomorphic, *Arch. Math.* **34** (1980), 108–110.

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