A Theorem on the Complete Integral

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The object of the present note is to show that a well-known theorem in the theory of non-linear partial differential equations, which is usually proved analytically,[1] admits of a geometrical proof which exhibits the relations concerned in a more intuitive manner.

Theorem: *Given a non-linear partial differential equation*

$$f(x, y, z, p, q) = 0. \tag{1}$$

Let $f_1(x, y, z, p, q)$, $f_2(x, y, z, p, q)$ *be two independent functions satisfying*

$$[f_1, f] = 0, \qquad [f_2, f] = 0.$$

Then if p, q *be eliminated from*

$$f = 0$$

$$f_1 = a_1 \tag{2}$$

$$f_2 = a_2 \tag{3}$$

where a_1 *and* a_2 *are arbitrary constants, the necessary and sufficient condition that the surface so obtained should be a complete integral of* (1) *is*

$$[f_1, f_2] = 0$$

where

$$[F, f] \equiv \frac{\partial F}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial F}{\partial y}\frac{\partial f}{\partial p} + \frac{\partial F}{\partial z}\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right) - \frac{\partial F}{\partial p}\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right) - \frac{\partial F}{\partial q}\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right).$$

Solving (1) and (2) for p and q, say

$$p = \phi(x, y, z, a_1)$$
$$q = \psi(x, y, z, a_1),$$

[1] See, for example, L. Bieberbach, *Differentialgleichungen* (Berlin, 1923), 220-221, or Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre* (Paris, 1891), 167 (§ 66).

and substituting in (3) we get a certain surface S. Now solutions of $[F, f] = 0$ are, by the theory of linear partial differential equations, the same as solutions of Charpit's equations for the characteristic strips of $f = 0$, *viz.*

$$\frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}} = \frac{dp}{-\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)} = \frac{dq}{-\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)}.$$

Hence if

$$f'_2 = a'_2 \tag{4}$$

were another independent integral of $[F, f] = 0$, then (1), (2), (4) would give another surface S' intersecting S along a characteristic *curve* of $f = 0$. S must then be generated by characteristic *curves* C of $f = 0$, whose associated strips satisfy $p = \phi$, $q = \psi$.

We have the identity

$$[F, f] \equiv -[f, F],$$

so that, since f_1 satisfies $[F, f] = 0$, f satisfies $[F, f_1] = 0$. Again, if we impose the relation

$$[f_1, f_2] = 0,$$

then f_2 satisfies $[F, f_1] = 0$. Also $[f_1, f_1] \equiv 0$, so $f = 0$, $f_1 = a_1$, $f_2 = a_2$ are three integrals of Charpit's equations corresponding to the partial differential equation

$$f_1 = a_1,$$

and S is, by similar reasoning, generated by characteristic curves C_1 of $f_1 = a_1$, whose strips again satisfy $p = \phi$, $q = \psi$.

It follows that at the intersection of a C and a C_1 the strips corresponding to C and to C_1 have a common surface element, which must therefore be the element of S at that point (since two intersecting lines determine an element); and as every point of S is an intersection of a C and a C_1, S must be an integral surface of $f = 0$ and $f_1 = a_1$ (and also of $f_2 = a_2$ since the argument is quite symmetrical with respect to f_1 and f_2).

Conversely, if S is an integral surface of $f = 0$, we can show that $[f_2, f_1] = 0$.

For then the normal at any point of S is given by $p = \phi$, $q = \psi$ (since the strips belonging to the curves C now lie on the surface S). Hence S is also an integral surface of $f_1 = a_1$ and so is generated by characteristic strips of $f_1 = a_1$. Let $f_3 = a_3$, $f_4 = a_4$ be the other two independent integrals of $[F, f_1] = 0$. Then

$$\left.\begin{aligned} f_3(x, y, z, \phi, \psi) = a_3 \\ f_4(x, y, z, \phi, \psi) = a_4 \end{aligned}\right\}$$

together give the congruence of characteristic curves of $[F, f_1] = 0$ whose associated strips satisfy $p = \phi$, $q = \psi$.

If $\lambda(a_3, a_4) = \text{const.}$, then these curves generate the surface $\lambda(f_3, f_4) = \text{const.}$, λ being an arbitrary function. By choosing λ suitably, they can be made to generate $f_2(x, y, z, \phi, \psi) = a_2$, since we know that this is generated by curves of the above congruence; f_2 must then be of the form $\lambda(f_3, f_4)$, and consequently must satisfy $[F, f_1] = 0$.

The condition $[f_1, f_2] = 0$ is thus both necessary and sufficient.

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