INTEGRAL REPRESENTATION THEOREMS IN PARTIALLY ORDERED VECTOR SPACES

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Abstract

Defining a Radon-type integration process we extend the Alexandroff, Fichtengolts-Kantorovich-Hildebrandt and Riesz integral representation theorems in partially ordered vector spaces.

We also identify some classes of operators with other classes of operator-valued set functions, the correspondence between operator and operator-valued set function being given by integration.

All these established results can be immediately applied in C^*-algebras (especially in W^*-algebras and AW^*-algebras of type I), in Jordan algebras, in partially ordered involutory $(O^*$-)algebras, in semifields, in quantum probability theory, as well as in the operator Feynman-Kac formula.

1. Introduction

The main object of this paper is to characterize the Alexandroff, Fichtengolts-Kantorovich-Hildebrandt and Riesz celebrated integral representation theorems, in partially ordered vector spaces.

The basic tool used in this approach is a good notion of a Radon-type integration process based on the order structure of the corresponding vector spaces.

For simplicity we confine our attention to finite partially ordered vector-valued σ-measures (respectively measures) and uniform order convergence,

but our process can be easily developed in the more general setting of locally finite partially ordered group-valued σ-meausres (respectively measures) and uniform order convergence almost everywhere, thus extending our results in [50].

Following this pattern our results generalize previous ones of S. Bochner [9], R. R. Christian [15], J. D. M. Wright [76–79] and S. Khurana [41], who established their results in integrable spaces of real or complex-valued functions with respect to partially ordered vector-valued σ-measures or measures, as well as of A. W. Wickstead [75] and R. Cristescu [17], who also defined types of integrability concerning vector lattice-valued functions with respect to vector lattice-valued σ-measures.

In the construction of our integral, in contrast to various concepts of vector integrals in topological vector spaces of R. G. Bartle, N. Dunford and J. Schwartz [5], R. G. Bartle [6], J. Batt and E. J. Berg [7], J. K. Brooks and P. W. Lewis [12], J. K. Brooks and N. Dinculeanu [13], N. Dinculeanu [22], I. Dobrakov [25, 26], N. Dunford [27], R. J. Easton and D. H. Tucker [28], J. R. Edwards and S. G. Wayment [31], R. K. Goodrich [35], A. Shuchat [64], K. Swong [68] and D. J. Uherka [72], there are no finiteness restrictions on variation or semivariation (respectively Gowurin, in terminology of [28], [31], [35], [72] of the corresponding vector σ-measures (respectively measures)).

On the other hand, J. J. Diestel and J. J. Uhl [21], N. Dinculeanu [22] and A. Shuchat [64], have shown the existence of Banach lattice valued positive σ-measures (respectively measures) with unbounded variation or semivariation.

As has been indicated in [21], even if the vector σ-measure (respectively measure) is real-valued, it need not to be of bounded variation or semivariation.

Thus covering another aspect of the problem of integration of vector-valued functions with respect to vector-valued σ-measures (respectively measures), we obtain integral representations analogous to those of the authors mentioned previously.

In particular the examined cases under considerations give extensions of the crucial concept of Radon measure in the sense of N. Bourbaki [11].

For other extensions of this very interesting concept of Radon measure (in the sense of N. Bourbaki) see also R. Cristescu [16, 17] and L. N. Tsitsas [71].

A σ-measure (respectively measure) on a σ-algebra $H_\sigma(T)$ (respectively algebra $H(T)$) of subsets of a space T with values in a partially ordered vector space (p.o.v.s) Y, is a positive σ-additive with respect to order convergence (respectively finitely additive) set function $m : H_\sigma(T) \to Y$ (respectively $m : H(T) \to Y$).

The defined integral has the form $\int_T f(t)\,dm(t)$, where the function f is defined on T and takes values in a second p.o.v.s. X. The integral has values in the Dedekind completion $\widehat{V}$ of a third p.o.v.s. V via a positive bilinear separately order continuous function from $X \times Y$ into a forth p.o.v.s. Z (denoted by " $\cdot$ ").

In Section 2 we summarize some general definitions and preliminary results which will be needed for the later development.

In Section 3 we define the p.o.v.s. $M(T, X)$ (respectively $M_t(T, X)$) of $H_\sigma(T)$-measurable (respectively totally $H(T)$-measurable) functions from T into X and derive some approximation-type results in those spaces.

Next (Section 4) we develop an integration process for functions in $M(T, X)$ (respectively $M_t(T, X)$) with respect to a σ-measure (respectively measure) m and establish the continuity of this integral with respect to uniform order convergence on the p.o.v.s. $\mathcal{L}^1(T, X)$ (respectively $M_t(T, X)$) of $H_\sigma(T)$-integrable (respectively totally $H(T)$-measurable) functions, which contains the $H_\sigma(T)$-elementary integrable (respectively $H(T)$-simple) functions.

It is worth noting that S. Mazur and W. Orlicz [46] and D. Przeworska-Rolewicz and S. Rolewicz [53] have shown that the Riemann, Lebesgue or Bochner integrals do not behave well in some metric linear lattices. So those integrals are not continuous for the topology of uniform convergence on the space of simple functions.

N. Dunford has defined in [27] a spectral integral which also has the surprising and unexpected property of being discontinuous with respect to the topology of uniform convergence on the space of simple functions.

Section 5 treats Fichtengolts, Kantorovich and Hildebrand type theorems for some classes of operators defined on the space $\mathcal{L}^1(T, X)$ (respectively $M_t(T, X)$) and having values in Y.

Up to this point we have considered only order structure on the vector spaces X, Y, Z and no structure for the space T.

So in Sections 6, and 7, T (respectively X) is endowed with Hausdorff (respectively norm) topologies and Y, Z are p.o.v.s.

We compare notions of measurability with total boundness or continuity in $M(T, X)$ (respectively $M_t(T, X)$).

We also examine relations between $(\tau_u o)$- and (uo)-continuous linear operators with domain a subspace $Q(T, X)$ of $F(T, X)$ equipped with topology τ_u of uniform convergence and range Y.

We recall that classes of "topological-order" continuous linear operators have been extensively investigated in normed lattices by L. V. Kantorovich, B. Z. Vulikh and A. G. Pinsker in [37], [38], [74].

Further we examine two distinguished cases for T and establish general

forms of the following famous theorems in analysis.

The Alexandroff representation theorem (Section 8): If T is a normal topological space, U a positive $(\tau_u o)$-continuous linear operator with domain the partially ordered normed space $C_{tb}(T, X)$ of totally bounded continuous functions from T into X and with range Y then

$$U(f) = \int_T f(t)\,dm(t),$$

for a unique operator-valued Borel measure m.

The Riesz representation theorem (Section 9): If T is a locally compact topological space, U a positive $(\tau_u o)$-continuous linear operator with domain the partially ordered normed space $C_{oo}(T, X)$ of continuous functions with compact support from T into X and with range Y then $U(f) = \int_T f(t)\,dm(t)$, for a unique operator-valued Borel σ-measure m.

We use these representation theorems to identify some classes of $(\tau_u o)$-continuous linear operators from $C_{tb}(T, X)$ (respectively $C_{oo}(T, X)$) into Y with other classes of operator-valued Borel set functions, the correspondence between operator and set function being given by integration.

Next, if X is a partially ordered normed algebra and Y a partially ordered algebra, it makes sense to consider operators from $C_{tb}(T, X)$ (respectively $C_{oo}(T, X)$) into Y which are not only $(\tau_u o)$-continuous and linear but which also preserve multiplication.

Section 10 is devoted to answering the natural question: how the additional property of these operators being multiplicative is reflected in the representing σ-measures (respectively measures)?

We conclude the paper with significant and important applications in C^*-algebras, especially in W^*-algebras and in AW^*-algebras of type I [10, 18, 23, 24, 55, 80, 81], in Jordan algebras [2, 32, 65–67, 70, 82] in O^*-algebras [59, 61], in semifields [3, 44, 45, 56–58, 60, 62], in quantum probability theory [14, 18–20, 29, 30, 36, 51] as well as in the operator Feynman-Kac formula [34, 42, 43].

The subject of σ-measures (respectively measures) and integrals taking values in partially ordered vector spaces of operators acting on a Hilbert space H has received much attention in the recent mathematical and physics literature [8, 14, 18–20, 29, 30, 36, 51, 79].

Thus our results can be immediately applied to integrating operator-valued measurable functions with respect to positive operator-valued σ-measures (respectively measures) acting on H.

In particular the results of Section 10 answer the question when the representing σ-measures (respectively measures) are projection-valued.

On the other hand our results have important applications to semifield-valued σ-measures (respectively measures) and integrals.

This very interesting concept has been defined and studied by the Russian school of Sarymsakov, Benderskiĭ, Čilin, Rubšteĭn, Halikulov, Kučkarov and others in a series of papers [44, 45, 56–58, 60, 62].

The class of semifields is much wider than the class of topological semi-fields [3], and it is contained in the class of K_σ-spaces.

As is well known, many attempts have been made to use the setting of p.o.v.s. in the theory of the mathematical foundation of quantum mechanics (see, for example, [36]).

To this end we describe in Section 11 how the presented results can be directly applied to a state space of quantum mechanics, integrating vector-valued random variables in a sense similar to that of E. Rowecka [54] or H. Umegaki and A. T. Bharucha-Reid [73] with respect to an operator-valued instrument (respectively observable).

The fruitful concept of instruments has been introduced by E. B. Davies and J. T. Lewis in [20], generalizing that of observables in a statistical system of quantum probability theory. Accordingly there is a canonical procedure for associating an observable to an instrument. It is easily verified that an instrument (respectively observable) is a σ-measure (respectively measure) taking values in a certain partially ordered Banach space of linear operators.

On the other hand, the main problems in mathematical physics concerning a perturbation formula of the Feynman-Kac type for the solution semigroups of some initial-value problems lead to the development of an integration theory which does not presuppose finite variation or semivariation of the corresponding measures (respectively σ-measures) (cf. [42, 43]).

The resulting notions and techniques (based on the partial ordering) can be immediately applied to express the superpositions of some semigroups of operators, including those which describe the motion of quantum mechanical particles in a potential force field.

Now another available application of our methods employed here seems to be in neutron diffusion theory. In that direction the reader may consult the excellent treatise of P. M. Morse and H. Feschbach [48].

We refer to [1, 10, 22–24, 34, 37, 52, 55, 63, 69, 74, 83] for general references about p.o.v.s., measure theory, normed algebras and semigroups of linear operators.

2. Notation and terminology

We now give the basic properties of a p.o.v.s. that will be needed in the subsequent development.

Throughout this paper X will denote a p.o.v.s. over the real field $\mathbb{R}$ with

positive cone X^+. We call X positively generated if $X = X^+ - X^+$.

Let $(x_\delta)_{\delta \in \Delta}$ be a net in X and $x \in X$. We say that $(x_\delta)_{\delta \in \Delta}$ is increasing (respectively decreasing) if $x_\delta \geq x_{\delta'}$ (respectively $x_\delta \leq x_{\delta'}$) whenever $\delta \geq \delta'$. The notation $x_\delta \uparrow x$ means that the net $(x_\delta)_{\delta \in \Delta}$ is increasing and $x = \sup\{x_\delta : \delta \in \Delta\}$, and similarly for $x_\delta \downarrow x$. We say that $(x_\delta)_{\delta \in \Delta}$ order converges to x $(x_\delta \xrightarrow{o} x$ or $o\text{-}\lim x_\delta = x)$ if there exists a decreasing net $(y_\delta)_{\delta \in \Delta}$ such that $y_\delta \downarrow 0$ and $-y_\delta \leq x_\delta - x \leq y_\delta$, for all $\delta \in \Delta$. Recall that this kind of order convergence does not in general coincide with that defined in [50], even if X is a vector lattice [49]. For details about various other concepts of order convergence see [47] and [49].

We call X almost Archimedean (respectively Archimedean) if $-\varepsilon y \leq x \leq \varepsilon y$ (respectively $\varepsilon x \leq y$) for some $y \in X^+$ and all real numbers $\varepsilon > 0$ implies $x = 0$ (respectively $x \leq 0$).

We call X monotone complete (respectively σ-complete) if every majorised increasing net (respectively sequence) in X has a supremum in X. We say that X has the diagonal property if whenever $\{x_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\} \subseteq X$ with $o\text{-}\lim_n x_{m,n} = x_m \in X$, for each $m \in \mathbb{N}$ and if $o\text{-}\lim x_m = x$ then there exists a strictly increasing sequence $\{n_m : m \in \mathbb{N}\} \subseteq \mathbb{N}$ such that $o\text{-}\lim x_{m,n_m} = x$.

Let Y be another p.o.v.s. A linear operator $F : X \to Y$ is order (respectively σ-order) continuous if $F(x_\delta) \xrightarrow{o} 0$ (respectively $F(x_n) \xrightarrow{o} 0$) whenever $(x_\delta)_{\delta \in \Delta}$ (respectively $(x_n)_{n \in \mathbb{N}}$) is a net (respectively sequence) in X such that $x_\delta \xrightarrow{o} 0$ (respectively $x_n \xrightarrow{o} 0$).

We denote by $L_r(X, Y)$ the p.o.v.s. of regular operators from X into Y. By definition,

$$L_r(X, Y) := \{F : X \to Y : \text{ there exist positive linear operators } F_i : X \to Y,$$
$$i = 1, 2 \text{ with } F = F_1 - F_2\}.$$

Also we put

$$L_y(X, Y) \text{ (respectively } L_\sigma(X, Y))$$
$$:= \{F : X \to Y : \text{ there exist positive order (respectively } \sigma\text{-order)}$$
$$\text{continuous linear operators } F_i : X \to Y,$$
$$i = 1, 2 \text{ with } F = F_1 = F_2\}.$$

In particular we write $X^\sim$, $X_\gamma^\sim$, $X_\sigma^\sim$ for $L_r(X, \mathbb{R})$, $L_y(X, \mathbb{R})$, $L_\sigma(X, \mathbb{R})$.

Let Z be a p.o.v. subspace of $X^\sim$ separating points of the positively generated p.o.v.s. X. Then the canonical map $i : X \to i(X) \subseteq Z^\sim$, $x \mapsto i(x) := F_x$ $(F_x(f) = f(x)$, for every $f \in Z)$ is an algebraic and order homomorphism of X into the p.o.v. subspace $i(X)$ of $Z_\gamma^\sim \subseteq Z^\sim$. When

$Z = X_\gamma^\sim$ (respectively $Z = X_\sigma^\sim$) i also satisfies the following property: $0 \leq i(x_\delta) \uparrow i(x)$ (respectively $0 \leq i(x_n) \uparrow i(x)$), whenever $(x_\delta)_{\delta \in \Delta}$ (respectively $(x_n)_{n \in \mathbb{N}}$) is a net (respectively sequence) in X with $0 \leq x_\delta \uparrow x$ (respectively $0 \leq x_n \uparrow x$), $x \in X$. In this case we identify X with its image $i(X)$ and so X is considered as a p.o.v. subspace of $Z_\gamma^\sim$. We call X order reflexive (or perfect) if $Z = X_\gamma^\sim$ and the embedding i is onto, that is, when $X = (X_\gamma^\sim)_\gamma^\sim$.

As is well known, even if X is a Banach lattice the concepts of topological reflexivity and order reflexivity are not equivalent.

If $U \in L_r(X, Y)$ the restriction of the algebraic dual $U^\sharp$ of U to $Y^\sim$ will be denoted by $U^\sim$ and is called the order adjoint of U. Clearly $U^\sim \in L_\gamma(Y^\sim, X^\sim)$.

Also it is easily verified that if X, Y are positively generated and $U \in L_\gamma(X, Y)$ (respectively $U \in L_\sigma(X, Y)$) then the restriction U' of $U^\sim$ to $Y_\gamma^\sim$ (respectively $Y_\sigma^\sim$) maps $Y_\gamma^\sim$ (respectively $Y_\sigma^\sim$) into $X_\gamma^\sim$ (respectively $X_\sigma^\sim$). In particular $U \in L_r(X, Y)$ implies $(U')' := U'' \in L_\gamma((X^\sim)_\gamma^\sim, (Y_\gamma)_\gamma^\sim)$ and if $Y_\gamma^\sim$ separates points of Y then $U''(x'') = (U(x))''$ with x'' the image of x under the canonical map of X into $(X^\sim)_\gamma^\sim$.

The following simple facts concerning "order duals" of p.o.v.s. will prove to be useful later.

If X is positively generated and Y monotone complete (respectively σ-complete) then $L_r(X, Y)$, $L_\gamma(X, Y)$ (respectively $L_\sigma(X, Y)$) are also monotone complete (respectively σ-complete).

A Dedekind completion for X is a pair $(j, \widehat{X})$, where $\widehat{X}$ is a Dedekind complete vector lattice and $j: X \to \widehat{X}$ is a linear injection with the properties

(a) for each $x \in X$, $x \geq 0$ if and only if $j(x) \geq 0$ (j is bipositive);

(b) $\{j(x): x \in X$ with $j(x) \leq \hat{y}\} \neq \varnothing$ and $\hat{y} = \sup\{j(x): x \in X$ with $j(x) \leq \hat{y}\}$ for every $\hat{y} \in \widehat{X}$;

(c) $j(x) = \sup\{j(x_q): q \in Q\}$ in $\widehat{X}$, whenever $\{x_q : q \in Q\} \subseteq X$ with $x = \sup\{x_q : q \in Q\}$ in X.

Moreover, a positively generated p.o.v.s. X possesses a Dedekind completion if and only if X is Archimedean. For a proof of this last result see [52, Proposition 1.19].

Next, let X be a partially ordered algebra that is a positively generated p.o.v.s. which is at the same time an algebra such that $x \cdot y \geq 0$, whenever $x, y \in X$ with $x, y \geq 0$ and let Z be a partially ordered subalgebra of $X^\sim$ separating points of X.

Suppose also that the multiplication in X is separately order continuous. Then the canonical embedding $i: X \to i(X) \subseteq Z_\gamma^\sim \subseteq Z^\sim$ is an algebraic and order isomorphism of X onto the partially ordered subalgebra $i(X)$ of $Z^\sim$ if we endow $Z^\sim$ with the following product, under which $Z^\sim$ becomes

a partially ordered algebra. Given F, $G \in Z^{\sim}$, let $F \cdot G$ be defined by $(F \cdot G)(u) = F \cdot (G \cdot u)$, $u \in Z$, where $(G \cdot u)(a) := G \cdot u_a$ and $u_a(x) := u(a \cdot x)$, for all a, $x \in X$.

3. Partially ordered vector-valued measurable functions

In this section $H_\sigma(T)$ (respectively $H(T)$) will denote a σ-algebra (respectively algebra) of subsets of T and X a positively generated p.o.v.s.

We also denote by $F(T, X)$ the p.o.v.s. of functions from T into X, where the vector and ordering operation are defined pointwise. We identify X with the subspace of $F(T, X)$ of all constant functions.

A net $(f_\delta)_{\delta \in \Delta}$ pointwise o-converges to $f \in F(T, X)$ on T (p-$\lim f_\delta = f$ on T) if o-$\lim f_\delta(t) = f(t)$, for every $t \in T$. It is well known that there exist p.o.v.s. $F(T, X)$ in which o-convergence does not imply p-convergence and conversely. However the following theorem can be easily verified.

THEOREM 3.1. *Let X be monotone complete (respectively σ-complete), $(f_\delta)_{\delta \in \Delta}$ (respectively $(f_n)_{n \in N}$) a net (respectively sequence) in $F(T, X)$ and $f \in F(T, X)$. The following assertions are equivalent*:
  (i) *o-$\lim f_\delta = f$ (respectively o-$\lim f_n = f$)*;
  (ii) *p-$\lim f_\delta = f$ (respectively p-$\lim f_n = f$) on T.*

On the other hand, a net $(f_\delta)_{\delta \in \Delta}$ in $F(T, X)$ uniformly o-converges to $f \in F(T, X)$ on T(u-$\lim f_\delta = f$ on T) if there exists a decreasing net $(u_\delta)_{\delta \in \Delta}$ in X such that

$$-u_\delta \leq f_\delta(t) - f(t) \leq u_\delta, \quad \text{for all } (\delta, t) \in \Delta \times T \text{ and } u_\delta \downarrow 0.$$

Now let $E(T, X)$ (respectively $S(T, X)$) be the set of $H_\sigma(T)$-elementary (respectively $H(T)$-simple) measurable functions from T into X. By definition,

$$E(T, X) := \{f \in F(T, X) : \text{there exists a countable partition } (A_n)_{n \in N} \text{ of } T$$
$$\text{in } H_\sigma(T) \text{ such that } f(t) = x_n, \text{ for every } t \in A_n, x_n \in X, n \in N\}$$

(respectively $S(T, X) := \{f \in F(T, X)$: there exists a finite partition of T in $H(T)$ such that $f(t) = x_i$, for every $t \in A_i$, $x_i \in X$, $i = 1, 2, \ldots, n\}$).

Clearly, given $f \in E(T, X)$ there exists a sequence $(g_n)_{n \in N}$ in $S(T, X)$ such that p-$\lim g_n = f$ on T.

Let $f \in F(T, X)$; f is $H_\sigma(T)$-measurable (respectively totally $H(T)$-measurable) if there exists a net $(f_\delta)_{\delta \in \Delta}$ in $E(T, X)$ (respectively $S(T, X)$)

such that u-lim $f_\delta = f$ on T. This net will be called an approximating net for f.

We put

$$M(T, X) := \{f \in F(T, X) : f \text{ is } H_\sigma(T)\text{-measurable}\}$$

(respectively $M_t(T, X) := \{f \in F(T, X) : f \text{ is totally } H(T)\text{-measurable}\}$). Clearly $M(T, X)$ (respectively $M_t(T, X)$), $E(T, X)$ and $S(T, X)$ are p.o.v. subspaces of $F(T, X)$ with $S(T, X) \subseteq M_t(T, X)$, $S(T, X) \subseteq E(T, X) \subseteq M(T, X)$.

The following theorems concern the space $M(T, X)$, but similar results are also valid for the space $M_t(T, X)$.

The proof of the next theorem is similar to that of [50, Theorem 2.2] (recall that X is of countable type if for every decreasing net $(x_\delta)_{\delta \in \Delta}$ in X with $x_\delta \downarrow 0$ there exists an increasing sequence $\{\delta_n : n \in \mathbb{N}\} \subseteq \Delta$ such that $x_{\delta_n} \downarrow 0$).

THEOREM 3.2. *Let X be a vector lattice of countable type and $f \in M(T, X)$. Then there exists an increasing (respectively decreasing) sequence $(f_n)_{n \in \mathbb{N}}$ (respectively $(g_n)_{n \in \mathbb{N}}$) in $E(T, X)$ with u-lim $f_n = f$ (respectively u-lim $g_n = f$) on T.*

THEOREM 3.3. *Let X be a p.o.v.s. with diagonal property and of countable type.*

(i) *If $(f_\delta)_{\delta \in \Delta}$ is a net in $M(T, X)$ such that u-lim $f_\delta = f$ on T then $f \in M(T, X)$.*

(ii) *For every $f \in M(T, X)$ there exists a sequence $(h_n)_{n \in \mathbb{N}}$ in $S(T, X)$ such that p-lim $h_n = f$ on T.*

PROOF. (i) This can be proved similarly to [50, Theorem 2.3(i)].

(ii) Let $f \in M(T, X)$. By definition there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $E(T, X)$ and $(x_n)_{n \in \mathbb{N}}$ in X such that

$$-x_n \leq f_n(t) - f(t) \leq x_n, \text{ for all } n \in \mathbb{N} \text{ and } t \in T.$$

On the other hand, given $t \in T$ there exists $(y_{k,n}(t))_{(k,n) \in \mathbb{N} \times \mathbb{N}}$ a double sequence in X such that $y_{k,n}(t) \downarrow 0(k \to \infty, n \in \mathbb{N})$ and

$$-y_{k,n}(t) \leq g_{k,n}(t) - f_n(t) \leq y_{k,n}(t),$$

for all $(k, n, t) \in \mathbb{N} \times \mathbb{N} \times T$ with $(g_{k,n})_{(k,n) \in \mathbb{N} \times \mathbb{N}}$ a double sequence in $S(T, X)$.

Now by the diagonal property for every $t \in T$ there exists a sequence $(k_n(t))_{n \in \mathbb{N}}$ in $\mathbb{N}$ $(k_n(t) \leq k_{n+1}(t)$, $n \in \mathbb{N})$ such that

$$y_{k_n(t), n}(t) \xrightarrow{o} 0 \quad \text{and} \quad -y_{k_p(t), p}(t) \leq g_{k_p(t), p}(t) - f_p(t) \leq y_{k_p(t), p}(t)$$

for all $(p, t) \in \mathbb{N} \times T$. Hence $p\text{-}\lim g_n = f$ on T for the sequence $(g_n)_{n \in \mathbb{N}}$ in $S(T, X)$ defined by $g_n(t) := g_{k_n(t), n}(t)$, $t \in T$, $n \in \mathbb{N}$.

4. Integration

In what follows Z will denote a monotone complete p.o.v.s., V the monotone complete (so Archimedean) p.o.v.s. ordered by the generating cone Z^+ $(V = Z^+ - Z^+)$ and $\widehat{V}$ the Dedekind completion of V.

Assume also in this section that X is positively generated and there exists a bilinear, positive and separately order continuous function from $X \times Y$ into Z which we shall denote simply by juxtaposition.

Let $f \in E(T, X)$ (respectively $S(T, X)$). Then there exists a countable (respectively finite) partition $(A_n)_{n \in \mathbb{N}}$ (respectively $(A_i)_{1 \leq i \leq n}$) of T by elements of the σ-algebra (respectively algebra) $H_\sigma(T)$ (respectively $H(T)$) such that $f(t) = a_i - b_i$, whenever $t \in A_i$ with a_i, $b_i \geq 0$, $i \in \mathbb{N}$ (respectively $i = 1, 2, \ldots, n$).

Let also $m : H_\sigma(T) \to Y$ (respectively $m : H(T) \to Y$) be a σ-measure (respectively measure) on $H_\sigma(T)$ (respectively $H(T)$), that is, a positive and σ-additive (respectively finitely additive) with respect to order convergence in Y set function on $H_\sigma(T)$ (respectively $H(T)$) with $m(\varnothing) = 0$. Then f is $H(_\sigma(T), m)$-integrable on T if $o\text{-}\lim \sum_{i=1}^{n} a_i \cdot m(A_i)$ and $o\text{-}\lim \sum_{i=1}^{n} b_i \cdot m(A_i)$ exist in Z^+. In this case we put $\int_T f(t) \, dm(t) := o\text{-}\lim \sum_{i=1}^{n} (a_i - b_i) \cdot m(A_i)$, (respectively $\int_T f(t) \, dm(t) = \sum_{i=1}^{n} (a_i - b_i) \cdot m(A_i)$) and

$$I(T, X) := \left\{ f \in E(T, X) : \text{ there exists } \int_T f(t) \, dm(t) \right\}.$$

Obviously, $I(T, X)$ is a p.o.v. subspace of $E(T, X)$ and $S(T, X) \subsetneqq I(T, X)$.

It is not difficult to verify that the integral $\int_T f(t) \, dm(t)$ is well defined, is dependent only on f and is independent of the particular way in which f is written as an $H_\sigma(T)$-elementary (respectively $H(T)$-simple) function in $E(T, X)$ (respectively $S(T, X)$). Moreover it is independent of a rearrangement of the series $(\sum_{i=1}^{n} (a_i - b_i) \cdot m(A_i))_{n \in \mathbb{N}}$.

We set $\int_A f(t) \, dm(t) := \int_T c_A f(t) \, dm(t)$, whenever $f \in I(T, X)$ (respectively $S(T, X)$) with c_A the characteristic function of A in $H_\sigma(T)$

(respectively $H(T)$) and $c_A f$ the X-valued function from T defined by the formula $c_A f(t) := c_A(t) f(t)$, $t \in T$.

The following theorem can be easily proved.

THEOREM 4.1. (i) *The integral operator* $I: I(T, X) \to V$ *(respectively* $I : S(T, X) \to V$) *with* $I(f) := \int_T f(t) \, dm(t)$, $f \in I(T, X)$ *(respectively* $S(T, X)$) *is positive, linear and* (uo)-*continuous* $(o\text{-}\lim I(f_\delta) = 0$, *whenever* $(f_\delta)_{\delta \in \Delta}$ *is a net in* $I(T, X)$ *(respectively* $S(T, X)$) *with* $u\text{-}\lim f_\delta = 0$ *on* T).

(ii) $\int_T f(t) x \, d\mu(t) = \int_T f(t) \, d\mu_x(t)$, *where* $\mu: H_\sigma(T) \to X^\sim$ *(respectively* $\mu: H(T) \to X^\sim$) *is a* σ-*measure (respectively measure) and* $\mu_x: H_\sigma(T) \to \mathbb{R}$ *(respectively* $\mu_x: H(T) \to \mathbb{R}$) *with* $\mu_x(A) := \mu(A)(x)$, *whenever* $A \in H_\sigma(T)$ *(respectively* $A \in H(T)$), $f \in I(T, R)$ *(respectively* $f \in S(T, R)$), $x \in X^+$.

(iii) *The set function* $v: H_\sigma(T) \to V$ *(respectively* $w: H(T) \to V$) *with* $v(A) := \int_A f(t) \, dm(t)$ *(respectively* $w(A) := \int_A f(t) \, dm(t)$) *for every* $A \in H_\sigma(T)$ *(respectively* $A \in H(T)$) *is a* σ-*measure (respectively measure), whenever* $f \in I(T, X)^+$ *with* f o-*bounded (respectively* $f \in S(T, X)^+$).

THEOREM 4.2. *Let* $(f_\delta)_{\delta \in \Delta}$ *be an approximating net in* $I(T, X)$ *(respectively* $S(T, X)$) *of* $f \in M(T, X)$ *(respectively* $f \in M_t(T, X)$). *Then* $u\text{-}\lim(f_\delta - f_{\delta'}) = 0$ *on* T *and the net* $(\int_T f_\delta(t) \, dm(t))_{\delta \in \Delta}$ *is* o-*fundamental in* V.

Using similar arguments to those in the proof of [50, Theorem 3.10] we obtain a proof of the preceding statement.

THEOREM 4.3. *Let* $f \in M(T, X)$ *(respectively* $f \in M_t(T, X)$). *Suppose that* $(f_\delta)_{\delta \in \Delta}$ *is an increasing approximating net in* $I(T, X)$ *(respectively* $S(T, X)$) *of* f. *Then there exists* $o\text{-}\lim \int_T f_\delta(t) \, dm(t)$ *in* V.

PROOF. A proof is similar to that of [50, Theorem 3.11].

COROLLARY 4.4. *Let* $f \in M(T, X)$ *(respectively* $M_T(T, X)$). *If* $(f_\delta)_{\delta \in \Delta}$, $(g_\theta)_{\theta \in \Theta}$ *are increasing approximating nets in* $I(T, X)$ *(respectively* $S(T, X)$) *of* f, *then* $o\text{-}\lim \int_T f_\delta(t) \, dm(t) = o\text{-}\lim \int_T g_\theta(t) \, dm(t)$ *in* V.

THEOREM 4.5. *Let* $f \in M(T, X)$ *(respectively* $M_t(T, X)$) *and* $(f_\delta)_{\delta \in \Delta}$ *an approximating net in* $I(T, X)$ *(respectively* $S(T, X)$) *of* f. *Then there exists the* $o\text{-}\lim \int_T f_\delta(t) \, dm(t)$ *in the Dedekind completion* $\hat{V}$ *of* V.

PROOF. Recall that V is Archimedean and may be identified with a.p.o.v. subspace of its Dedekind completion $\widehat{V}$. By Theorem 4.2 $(\int_T f_\delta(t)\,dm(t))_{\delta\in\Delta}$ is o-fundamental in V and hence o-converges in $\widehat{V}$.

COROLLARY 4.6. *Let* $f \in M(T, X)$ *(respectively* $M_t(T, X)$). *If* $(f_\delta)_{\delta\in\Delta}$ *and* $(g_\theta)_{\theta\in\Theta}$ *are approximating nets in* $I(T, X)$ *(respectively* $S(T, X)$) *of* f, *then* o-$\lim \int_T f_\delta(t)\,dm(t) = o$-$\lim \int_T g_\theta(t)\,dm(t)$ *in the Dedekind completion* $\widehat{V}$ *of* V.

The preceding results lead to the following definitions. The $(H_\sigma(T), m)$- (respectively $(H(T), m)-$) integral of an element f in $M(T, X)$ (respectively $M_t(T, X)$) is defined to be the element $\int_T f(t)\,dm(t)$ of $\widehat{V}$ with $\int_T f(t)\,dm(t) := o$-$\lim \int_T f_\delta(t)\,dm(t)$, where $(f_\delta)_{\delta\in\Delta}$ is an approximating net of f in $I(T, X)$ (respectively $S(T, X)$). We also denote by $\mathscr{L}^1(T, X)$ the set of $(H_\sigma(T), m)$-integrable functions f in $M(T, X)$. Clearly $\mathscr{L}^1(T, X)$ is a p.o.v. subspace of $M(T, X)$ and $M_t(T, x) \subseteq \mathscr{L}^1(T, X)$.

By Corollary 4.6, $\int_T f(t)\,dm(t)$ does not depend on the choice of an approximating net $(f_\delta)_{\delta\in\Delta}$ in $I(T, X)$ (respectively $S(T, X)$). Further we set $\int_A f(t)\,dm(t) := \int_T c_A(t)f(t)\,dm(t)$ whenever $f \in \mathscr{L}^1(T, X)$ (respectively $M_t(T, X)$) and $A \in H_\sigma(T)$ (respectively $A \in H(T)$).

On the other hand the propositions and theorems of the $(H_\sigma(T), m)$- (respectively $(H(T), m)$-) integral on the p.o.v.s. $I(T, X)$ (respectively $S(T, X)$) remain also valid for p.o.v.s. $\mathscr{L}^1(T, X)$ (respectively $M_t(T, X)$). In particular we state the following monotone-type convergence theorem. The proof is similar to that of [50, Theorem 3.13].

THEOREM 4.7. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathscr{L}^1(T, X)$ *(respectively* $M_t(T, X)$) *such that*

(i) *there exists* $x \in X$ *with* $0 \le f_{n+1}(t) \le f_n(t) \le x$, *for all* $t \in T$, $n \in \mathbb{N}$;

(ii) *there exists a sequence* $(A_n)_{n\in\mathbb{N}}$ *in* $H_\sigma(T)$ *(respectively* $H(T)$) *with* u-$\lim f_n = 0$ *on each* A_n, $n \in \mathbb{N}$, *and* o-$\lim m(T - A_n) = 0$;

(iii) $\widehat{V}$ *has the diagonal property.*

Then o-$\lim \int_T f_n(t)\,dm(t) = 0$.

5. Fichtengolts-Kantorovich-Hildebrandt-type theorems

Let X, Y be positively generated p.o.v.s. with Y monotone complete. Recall that a linear operator $U: Q(T, X) \to Y$, with $Q(T, X)$ a p.o.v.

subspace of $F(T, X)$, is (uo)-(respectively (po)- or (oo)-) continuous if
o-lim $U(f_\delta) = 0$ whenever $(f_\delta)_{\delta \in \Delta}$ is a net in $Q(T, X)$ with u-lim $f_\delta = 0$
on T (respectively p-lim $f_\delta = 0$ on T or o-lim $f_\delta = 0$).

We denote by $L_r(Q(T, X), Y)_{uo}$ (respectively $L_r(Q(T, X), Y)_{po}$ or
$L_r(Q(T, X), Y)_{oo}$) the p.o.v.s. of all differences of positive (uo)-(respec-
tively (po)-or (oo)-) continuous linear operators from $Q(T, X)$ into Y.

THEOREM 5.1. *Let* $U : M_t(T, X) \to Y$ *be a* (uo)-*continuous positive linear
operator. Then there exists a unique measure* $m : H(T) \to L_r(X, Y)$ *such
that* $U(f) = \int_T f(t)\, dm(t)$, *for every* f *in* $M_t(T, X)$.

PROOF. Define the measure $m : H(T) \to L_r(X, Y)$ by the formula

$$m(A)(x) := U(c_A x) \quad \text{for all } A \in H(T) \text{ and } x \in X.$$

Now let $f \in M_t(T, X)$ and take an approximating net $(f_\delta)_{\delta \in \Delta}$ of f in
$S(T, X)$. Since $U(f_\delta) = \int_T f_\delta(t)\, dm(t)$ for every $\delta \in \Delta$ we get

$$U(f) = o\text{-}\lim U(f_\delta) = o\text{-}\lim \int_T f_\delta(t)\, dm(t) = \int_T f(t)\, dm(t).$$

The uniqueness of m is an evident fact.

Similarly we can prove

THEOREM 5.2. *Let* $U : \mathscr{L}^1(T, X) \to Y$ *be a* (po)-*continuous positive lin-
ear operator. Then there exists a unique* σ-*measure* $m : H_\sigma(T) \to L_r(X, Y)$
such that $U(f) = \int_T f(t)\, dm(t)$, *for every* f *in* $\mathscr{L}^1(T, X)$.

Next let us denote by $\mathbf{O}(H(T), L_r(X, Y))$ the p.o.v.s. of all differences
of $L_r(X, Y)$-valued measures on the algebra $H(T)$.

The following assertion is an immediate consequence of Theorem 5.1.

THEOREM 5.3. $L_r(M_t(T, X), Y)_{uo}$ *is isomorphic to* $\mathbf{O}(H(T), L_r(X, Y))$.

6. Approximation and measurability in partially ordered normed spaces

In this section, X will denote a partially ordered normed space, $\|\cdot\|$ the
corresponding norm, τ the norm (strong) topology of X and $H_\sigma(X)$ the
σ-algebra of Borel subsets of X.

An element $e > 0$ of X is called an orderunit norm if $\|x\| \leq k$ implies
$-k \cdot e \leq x \leq k \cdot e$, $k \geq 0$, $x \in X$.

The norm $\|\cdot\|$ is order continuous (respectively σ-continuous) on X if o-$\lim x_\delta = x$ (respectively o-$\lim x_n = x$) implies o-$\lim x_\delta = x$ (respectively o-$\lim x_n = x$) whenever $(x_\delta)_{\delta \in \Delta}$ (respectively $(x_n)_{n \in \mathbb{N}}$) is a net (respectively sequence) in X and $x \in X$.

The norm $\|\cdot\|$ is called absolute monotone on X if $-y \le x \le y$ implies $\|x\| \le \|y\|$, whenever $x, y \in X$.

Let $H_\sigma(T)$ be a σ-algebra of subsets of T and $m: H_\sigma(T) \to Y$ a σ-measure. A function $f: T \to X$ is called $(H_\sigma(T), m)$-partitionable if for every neighborhood G of 0 there is a countable partition $(A_n)_{n \in \mathbb{N}}$ of T in $H_\sigma(T)$ such that $m(T - \bigcup_{n \in \mathbb{N}} A_n = 0$ and $f(A_n) - f(A_n) \subseteqq G$, for all $n \in \mathbb{N}$.

Also f is called totally bounded (respectively $(H_\sigma(T), H_\sigma(X))$-measurable) if its range $f(T)$ is a totally bounded subset of X (respectively $f^{-1}(F) \in H_\sigma(T)$, whenever $F \in H_\sigma(X))$.

THEOREM 6.1. *Let* $f: T \to X$ *be an* $(H_\sigma(T), H_\sigma(X))$-*measurable function. Assume that* X *is almost Archimedean, separable and possesses an orderunit norm* e. *Then* $f \in M(T, X)$.

PROOF. This can be proved analogously to [50, Theorem 4.1].

THEOREM 6.2. *Let* $f \in M(T, X)$. *Assume that* X *is almost Archimedean and the norm* $\|\cdot\|$ *is absolute monotone. Then*

  (i) f *is* $(H_\sigma(T), m)$-*partitionable,*

  (ii) f *is* $(H_\sigma(T), H_\sigma(X))$-*measurable.*

PROOF. This is proved similarly to [50, Theorem 4.2].

THEOREM 6.3. *Let* $f \in M_t(T, X)$. *Suppose that the norm* $\|\cdot\|$ *of* X *is absolute monotone and order continuous. Then* f *is totally bounded.*

PROOF. There exists nets $(f_\delta)_{\delta \in \Delta}$ and $(x_\delta)_{\delta \in \Delta}$ in $S(T, X)$ and X^+, respectively, such that $-x_\delta \le f_\delta(t) - f(t) \le x_\delta$, for every $t \in T$ and $x_\delta \downarrow 0$. Hence the assertion follows from [40, page 70].

THEOREM 6.4. *Let* $f: T \to X$ *be a totally bounded and* $(H_\sigma(T), H_\sigma(X))$-*measurable function. Assume that* X *is almost Archimedean and has an orderunit norm* e. *Then* $f \in M_t(T, X)$.

PROOF. Clearly, given $n \in \mathbb{N}$ there exist a finite partition (A_i), $i = 1, 2, \ldots, k_n$ of T and a sequence $(f_n := \sum_{i=1}^{k_n} c_{A_i} a_i)_{n \in \mathbb{N}}$ in $S(T, X)$ such

that $-\frac{1}{n}e \leq f_n(t) - f(t) \leq \frac{1}{n}e$, whenever $(n, t) \in \mathbb{N} \times T$. Consequently, the desired result follows.

We shall now turn our attention to the following cases. Let T be a normal (respectively locally compact) topological space and $C_{tb}(T, X)$ (respectively $C_{00}(T, X)$) the partially ordered normed space of continuous totally bounded (respectively with compact support) functions from T into X endowed with pointwise ordering and uniform topology τ_u. In particular we write $C_{tb}(T)$ (respectively $C_{00}(T)$) for $C_{tb}(T, R)$ (respectively $C_{00}(T, R)$).

THEOREM 6.5. *Suppose that X is almost Archimedean and has an orderunit norm e. Then*

(i) *for every f in $C_{tb}(T, X)$ (respectively $C_{oo}(T, X)$) there exists a sequence $(g_n := \sum_{i=1}^{k_n} f_i x_i)_{n \in \mathbb{N}}$ with $f_i \in C_{tb}(T)$ (respectively $f_i \in C_{oo}(T)$), $x_i \in X$, $i = 1, 2, \ldots, k_n$, and $n \in \mathbb{N}$ such that $u\text{-}\lim g_n = f$ on T.*

(ii) $C_{tb}(T, X) \subseteq M_t(T, X)$ *(respectively $C_{oo}(T, X) \subseteq M_t(T, X)$).*

PROOF. (i) Given $f \in C_{tb}(T, X)$ (respectively $f \in C_{oo}(T, X)$) and $n \in \mathbb{N}$ there exist $x_i \in X$, $i = 1, 2, \ldots, k_n$, with

$$T = \bigcup_{i=1}^{k_n} f^{-1}(B_{1/n}(x_i))$$

where $B_{1/n}(x_i)$ denotes the open sphere with center x_i and radius $1/n$.

Consider a continuous partition of unity (f_i) subordinated to the preceding finite open covering of T, $i = 1, 2, \ldots, k_n$.

Therefore

$$-\frac{1}{n}e \leq \sum_{i=1}^{k_n} f_i(t)x_i - f(t) \leq \frac{1}{n}e,$$

whenever $t \in T$, which implies the desired result.

(ii) The assertions follow from Theorem 6.4.

COROLLARY 6.6. *Suppose that X is almost Archimedean and has an orderunit norm e. Then*

(i) *for every f in $C_{tb}(T, X)^+$ (respectively $C_{oo}(T, X)^+$) there exists a sequence $(g_n := \sum_{i=1}^{k_n} f_i \cdot x_i)_{n \in \mathbb{N}}$ with $f_i \in C_{tb}(T)^+$ (respectively $f_i \in C_{oo}(T)^+$), $x_i \in X^+$, $i = 1, 2, \ldots, k_n$, and $n \in \mathbb{N}$ such that*

$$u\text{-}\lim g_n = f \text{ on } T.$$

(ii) $C(T, X)^+ \subseteq M_t(T, X)^+$ *(respectively $C_{oo}(T, X)^+ \subseteq M_t(T, X)^+$).*

7. Some classes of operators in partially ordered normed spaces

Let X be a partially ordered normed space and Y a p.o.v.s. A linear operator $U: X \to Y$ is said to be (τo)-continuous if $o\text{-}\lim U(x_\delta) = 0$ whenever $(x_\delta)_{\delta \in \Delta}$ is a net in X that τ-converges to 0 $(\tau\text{-}\lim x_\delta = 0)$.

We denote by $L_r(X, Y)_{\tau o}$ the p.o.v.s of all differences of positive (τo)-continuous linear operators from X into Y. In particular we write $X_\tau^\sim$ for $L_r(X, R)_{\tau o}$.

As is easily verified, if $U \in L_r(X, Y)_{\tau o}$ and X, Y are positively generated then U' (the restriction of $U^\sim$ on $Y_\gamma^\sim$) maps $Y_\gamma^\sim$ into $X_\tau^\sim$. Moreover $U'' : (X_\tau^\sim)_\gamma^\sim \to (Y_\gamma^\sim)_\gamma^\sim$ is order continuous.

Throughout this section, assume that T denotes a Hausdorff topological space and $Q(T, X)$ a p.o.v. subspace of $F(T, X)$ with $f(T)$ norm bounded for every f in $Q(T, X)$, endowed with the uniform topology τ_u and the induced pointwise ordering from $F(T, X)$.

Now we proceed to discuss connections between $(\tau_u o)$- and (uo)-continuous operators from $Q(T, X)$ into Y.

THEOREM 7.1. *Let $U: Q(T, X) \to Y$ be a linear (uo)-continuous operator. Assume that X is almost Archimedean and has an orderunit norm e. Then U is $(\tau_u o)$-continuous.*

PROOF. Let $(f_\delta)_{\delta \in \Delta}$ be a net in $Q(T, X)$ with $\tau_u\text{-}\lim f_\delta = 0$. Thus there exists a real decreasing net $(r_\delta)_{\delta \in \Delta}$ such that $\|f_\delta(t)\| \leq \sup\{\|f_\delta(t)\| : t \in T\} \leq r_\delta$, for every $(\delta, t) \in \Delta \times T$ and $r_\delta \downarrow 0$. Hence $u\text{-}\lim f_\delta = 0$ on T, which implies $o\text{-}\lim U(f_\delta) = 0$.

Using similar arguments we can prove the following

THEOREM 7.2. *Let $U: Q(T, X) \to Y$ be a $(\tau_u o)$-continuous linear operator. Suppose that the norm $\|\cdot\|$ of X is absolute monotone and order continuous. Then U is (uo)-continuous.*

8. Alexandroff-type integral representation theorems

In this section, we assume that X is a positively generated almost Archimedean partially ordered normed space with an orderunit norm e and Y a γ-separative (respectively σ-separative) positively generated p.o.v.s.

We denote by $\phi(H(T), X_\tau^\sim)$ the p.o.v.s. of all differences of $X_\tau^\sim$-valued measures on the Borel algebra $H(T)$ of a normal topological space T such

that $\mu_x : H(T) \to R$ with $\mu_x(A) := \mu(A)(x)$ is a real-valued regular Borel measure, whenever $x \in X^+$, $A \in H(T)$ and $\mu \in \phi(H(T), X_\tau^\sim)^+$.

We also denote by

$$\psi(H(T), L_r(X, (Y_\gamma^\sim)_{\gamma_{\tau_0}}^\sim) \quad (\text{respectively } \psi(H(T), L_r(X, (Y_\sigma^\sim)_{\gamma_{\tau_0}}^\sim))$$

the p.o.v.s. of all differences of

$$L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau_0-} \quad (\text{respectively } L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau_0-})$$

valued measures on the Borel algebra $H(T)$ such that

$$m_{y^\sim} \in \phi(H(T), X_\tau^\sim) \quad \text{with } m_{y^\sim}(A) := m(A)(x)(y^\sim),$$

whenever $x \in X$, $y^\sim \in Y_\gamma^\sim$ (respectively $y^\sim \in Y_\sigma^\sim$) $y^\sim \geq 0$ and

$$m \in \psi(H(T), L_r(X, (Y_\gamma^\sim)_{\gamma_{\tau_0}}^\sim)) \quad (\text{respectively } m \in \psi(H(T), (Y_\sigma^\sim)_{\gamma_{\tau_0}}^\sim)).$$

THEOREM 8.1. *Let* w *be a positive element of* $C_{tb}(T, X)_{\tau_u}^\sim$. *Then there exists exactly one Borel measure* μ *in* $\phi(H(T), X_\tau^\sim)$ *such that* $w(f) = \int_T f(t) \, d\mu(t)$, *for every* f *in* $C_{tb}(T, X)$.

PROOF. Let $x \in X^+$. Define $C_{tb}^x(T, X) := \{h \cdot x : h \in C_{tb}(T)\}$. Clearly $C_{tb}^x(T, X)$ is algebraic and order isomorphic to $C_{tb}(T)$. Therefore we identify $C_{tb}^x(T, X)$ with $C_{tb}(T)$.

Thus by the scalar Alexandroff representation theorem [4, page 577] there exists a unique regular real-valued measure $\mu_x : H(T) \to R$ such that

$$(1) \qquad w_x(h) = w(h \cdot x) = \int_T h(t) \, d\mu_x(t), \quad \text{for every } h \in C_{tb}(T),$$

where w_x is the restriction of w to $C_{tb}(T)$.

Extend (1) to be true for arbitrary $x \in X$ with $x = x_1 - x_2$, $x_i \geq 0$, $i = 1, 2$, by the formula

$$w_x(h) = w_{x_1 - x_2}(h) := \int_T h(t) \, d\mu_{x_1}(t) - \int_T h(t) \, d\mu_{x_2}(t),$$

for every $h \in C_{tb}(T)$. Clearly $w_x \in C_{tb}(T)_{\tau_u}^\sim$.

Next define a measure $\mu : H(T) \to X_\tau^\sim$ by the relation

$$\mu(A)(x) := \mu_x(A), \qquad A \in H(T), \ x \in X.$$

On the other hand, for each $h \in C_{tb}(T)$ and $x \in X$, we have

$$w(h \cdot x) = \int_T h(t)x \, d\mu(t) = \int_T h(t) \, d\mu_x(t),$$

obtained by using an approximating sequence in $S(T, R)$ of $h \in C_{tb}(T)$ and by applying Theorem 4.1 (ii).

Moreover Theorem 6.5 guarantees that μ represents w on the whole space $C_{tb}(T, X)$. Its uniqueness is evident.

THEOREM 8.2. *Let* U *be a positive element in* $L_r(C_{tb}(T, X), Y)_{\tau_u o}$. *Then there exists exactly one measure* $m \in \psi(H(T), L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau o})$ (*respectively* $m \in \psi(H(T), L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau o})$) *such that* $U(f) = \int_T f(t) dm(t)$, *for every* f *in* $C_{tb}(T, X)$.

PROOF. Let $y^\sim \in Y_\gamma^\sim$ (respectively $y^\sim \in Y_\sigma^\sim$) with $y^\sim \geq 0$. Consider the positive linear τ_u-continuous functional $U_{y^\sim}: C_{tb}(T, X) \to R$, defined by the formula $U_{y^\sim}(f) = y^\sim(U(f))$, for every $f \in C_{tb}(T, X)$.

Theorem 8.1 guarantees the existence of exactly one measure m_y in $\varphi(H(T), X_\tau^\sim)$ such that $U_{y^\sim}(f) = \int_T f(t) dm_{y^\sim}(t)$, for all f in $C_{tb}(T, X)$.

Now we define a measure $m: H(T) \to L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau o}$ (respectively $m: H(T) \to L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau o}$) by the relation

$$m(A)(x)(y^\sim) := m_{y^\sim}(A)(x),$$

whenever $A \in H(T)$, $x \in X$, and the corresponding integral operator

$$U_1: M_t(T, X) \to (Y_\gamma^\sim)_\gamma^\sim \quad \text{(respectively } U_1: M_t(T, X) \to (Y_\gamma^\sim)_\gamma^\sim \text{)}$$

by the relation $U_1(f) := \int_T f(t) dm(t)$, whenever $f \in M_t(T, X)$.

Since $C_{tb}(T, X) \subseteq M_t(T, X)$ we easily establish that

$$U_1(f) = U(f) = \int_T f(t) dm(t),$$

for every f in $C_{tb}(T, X)$.

To prove uniqueness let l be another measure in

$$\psi(H(T), L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau o}) \quad \text{(respectively } \psi(H(T), L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau o}))$$

such that

$$U(f) = \int_T f(t) dm(t) = \int_T f(t) dl(t),$$

whenever $f \in C_{tb}(T, X)$. Consequently $\int_T f(t) dm_{y^\sim}(t) = \int_T f(t) dl_{y^\sim}(t)$, for every $f \in C_{tb}(T, X)$, which implies $m = l$.

THEOREM 8.3. $C_{tb}(T, X)_{\tau_u}^\sim$ *is isomorphic to* $\varphi(H(T), X_\tau^\sim)$.

PROOF. By the integration process, each μ in $\phi(H(T), X_\tau^\sim)$ defines a positive functional w in $C_{tb}(T, X)_{\tau_u}^\sim$ and the correspondence $\xi: \phi(H(T), X_\tau^\sim) \to C_{tb}(T, X)_{\tau_u}^\sim$ defined by $\xi(\mu) := w$ is positive and linear.

Again, from the scalar Alexandroff representation theorem it is an easy matter to prove that ξ is one-to-one.

On the other hand Theorem 8.1 guarantees that ξ is onto. This completes the proof of the assertion.

THEOREM 8.4. *Suppose that Y is order reflexive. Then $L_r(C_{tb}(T, X), Y)_{\tau_u o}$ is isomorphic to $\psi(H(T), L_r(X, Y)_{\tau o})$.*

PROOF. This is proved analogously to Theorem 8.3.

9. Riesz-type integral representation theorems

Let X be a partially ordered normed space and Y a p.o.v.s. provided with the properties of the foregoing section.

We denote by $\phi_\sigma(H_\sigma(T), X_\tau^\sim)$ the p.o.v.s. of all differences of $X_\tau^\sim$-valued σ-measures on the Borel σ-algebras $H_\sigma(T)$ of a locally compact topological space T such that $\mu_x: H_\sigma(T) \to R$ with $\mu_x(A) := \mu(A)(x)$ is a real-valued regular Borel measure, whenever $x \in X^+$, $A \in H_\sigma(T)$ and $\mu \in \phi_\sigma(H_\sigma(T), X_\tau^\sim)^+$.

We also denote by

$$\psi_\sigma(H_\sigma(T), L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau o}) \quad \text{(respectively } \psi_\sigma(H_\sigma(T), L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau o}))$$

the p.o.v.s. of all differences of

$$L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau o}- \quad \text{(respectively } L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau o}-)$$

valued σ-measures on the Borel σ-algebra $H_\sigma(T)$ such that

$$m_{y^\sim} \in \phi_\sigma(H_\sigma(T), X_\tau^\sim) \quad \text{with } m_{y^\sim}(A) := m(A)(x)(y^\sim) ,$$

whenever $x \in X$, $y^\sim \in Y_\gamma^\sim$ (respectively $y^\sim \in Y_\sigma^\sim$), $y^\sim \geq 0$ and

$m \in \psi_\sigma(H_\sigma(T), L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau o})$ (respectively $m \in \psi_\sigma(H_\sigma(T), (Y_\sigma^\sim)_\gamma^\sim)_{\tau o})$.

Now we can easily obtain proofs of the following theorems by using similar arguments to those of the corresponding theorems of the preceding section.

THEOREM 9.1. *Let w be a positive element of $C_{oo}(T, X)_{\tau_u}^\sim$. Then there exists exactly one Borel σ-measure in $\phi_\sigma(H_\sigma(T), X_\tau^\sim)$ such that $w(f) = \int_T f(t)\, d\mu(t)$, for every f in $C_{oo}(T, X)$.*

THEOREM 9.2. *Let U be a positive element in $L_r(C_{oo}(T, X), Y)_{\tau_u o}$. Then there exists exactly one σ-measure $m \in \psi_\sigma(H_\sigma(T), L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau o})$ (respectively $m \in \psi_\sigma(H_\sigma(T), L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau o}))$ such that $U(f) = \int_T f(t)\, dm(t)$, for every f in $C_{oo}(T, X)$.*

THEOREM 9.3. $C_{oo}(T, X)_{\tau_u}^\sim$ *is isomorphic to* $\phi_\sigma(H_\sigma(T), X_\tau^\sim)$.

THEOREM 9.4. *Suppose that* Y *is order reflexive. Then*

$$L_r(C_{oo}(T, X), Y)_{\tau_u o}$$

is isomorphic to $\psi_\sigma(H_\sigma(T), L_r(X, Y)_{\tau o})$.

10. Multiplicative operators

We start with the following assumptions. Let X be a partially ordered normed algebra positively generated by a normal cone X^+ with a separately order continuous multiplication, T a normal topological space and $H(T)$ its Borel algebra.

Then $C_{tb}(T, X)$ is also a partially ordered normed algebra positively generated by a normal cone under the usual uniform topology τ_u and pointwise algebraic-order structure.

As is also well known, $C_{tb}(T, X)^\sim_{\tau_u} = C_{tb}(T, X)'$ (the topological dual of $C_{tb}(T, X)$). Therefore $C_{tb}(T, X)^\sim_{\tau_u}$ is a separating p.o.v. subspace of $C_{tb}(T, X)^\sim$.

Let Y be another partially ordered algebra. Assume that Y is monotone complete γ- (respectively $\sigma-$) separative and positively generated.

A positive operator U in $L_r(C_{tb}(T, X), Y)$ is called multiplicative if $U(f \cdot g) = U(f) \cdot U(g)$, whenever $f, g \in C_{tb}(T, X)$.

We continue with a useful criterion for multiplicativity.

LEMMA 10.1. *Let* U *be a positive element in* $L_r(C_{tb}(T, X), Y)_{\tau_u o}$. *Then* U *is multiplicative if and only if* $U''(A \cdot B) = U''(A) \cdot U''(B)$, *for all* A, B *in* $(C_{tb}(T, X)^\sim_{\tau_u})^\sim_\gamma$.

PROOF. In order to prove the sufficiency, take any A, B in $(C_{tb}(T, X)^\sim_{\tau_u})^\sim_\gamma$ and $y^\sim \in Y^\sim_\gamma$ (respectively $y^\sim \in Y^\sim_\sigma$), $y^\sim \geq 0$. Thus $U''(A \cdot B)(y^\sim) = (A \cdot B)(U'(y^\sim)) = A \cdot (B \cdot U'(y^\sim))$, $(U''(A) U''(B))(y^\sim) = A \cdot U'(U''(B)(y^\sim))$. Also if $f \in C_{tb}(T, X)$ we have

$$(B \cdot U'(y^\sim))(f) = B \cdot (U'(y^\sim))_f \quad \text{and} \quad U'(U''(B)(y^\sim))(f) = B(U'(y^\sim_{U(f)})).$$

Furthermore for $g \in C_{tb}(T, X)$ we deduce that

$$U'(y^\sim)_f(g) = y^\sim(U(f \cdot g)) \text{ and } U'(y^\sim_{U(f)})(g) = y^\sim(U(f \cdot g)).$$

Hence $U''(A \cdot B) = U''(A) \cdot U''(B)$.

Necessity follows from $C_{tb}(T, X) \subseteq (C_{tb}(T, X)_{\tau_u}^\sim)_\gamma^\sim$.

On the other hand, in view of preceding definitions and results, the following lemmas can be easily established.

LEMMA 10.2. *Let* $A \in H(T)$ *and* $x \in X$. *Then the function* $[c_A x]$: $C_{tb}(T, X)_{\tau_u}^\sim = \phi(H(T), X_\tau^\sim) \to R$ *defined by the formula*

$$[c_A x](m) := \int_T c_A(t) x \, dm(t),$$

whenever $m \in \phi(H(T), X_\tau^\sim)$, *belongs in* $(C_{tb}(T, X)_{\tau_u}^\sim)_\gamma^\sim$.

LEMMA 10.3. *The representing measure* $m \in \psi(H(T), L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau_0})$ *(respectively* $m \in \psi(H(T), L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau_0}))$ *of a positive operator* $U \in L_r(C_{tb}(T, X), Y)_{\tau_u o}$ *is given by the formula* $m(A)x = U''([c_A x])$, *whenever* $A \in H(T)$ *and* $x \in X$.

LEMMA 10.4. $([c_{A_1} x_1]) \cdot ([c_{A_2} x_2]) = [c_{A_1 \cap A_2}(x_1 \cdot x_2)]$, *for all* A_1, A_2 *in* $H(T)$ *and* x_1, x_2 *in* X.

Now we state and prove the main results of this section.

THEOREM 10.5. *A positive element* U *in* $L_r(C_{tb}(T, X), Y)_{\tau_u o}$ *is multiplicative if and only if the representing measure* m *in*

$$\psi(H(T), L_r(X, (Y_\gamma^\sim)_\gamma^\sim)_{\tau_0})$$

(respectively $\psi(H(T), L_r(X, (Y_\sigma^\sim)_\gamma^\sim)_{\tau_0}))$ *satisfies*

$$m(A_1 \cap A_2)(x_1 \cdot x_2) = (m(A_1)(x_1)) \cdot (m(A_2)(x_2)),$$

for all x_1, x_2 *in* X *and* A_1, A_2 *in* $H(T)$.

PROOF. First suppose that U is multiplicative. Then it follows easily by Lemmas 10.2 and 10.3 that

$$m(A_1 \cap A_2)(x_1 \cdot x_2) = (m(A_1)(x_1)) \cdot (m(A_2)(x_2)),$$

whenever x_1, $x_2 \in X$ and A_1, $A_2 \in H(T)$.
Conversely suppose that the preceding condition holds. Then for arbitrary f, g in $C_{tb}(T, X)$, a similar argument to that in the last part of the proof

of [15, Theorem 22] indicates that

$$\int_T f(t)g(t)\,dm(t) = \int_T f(t)\,dm(t) \cdot \int_T g(t)\,dm(t).$$

This completes the proof.

When T is a locally compact topological space and $H_\sigma(T)$ the Borel σ-algebra of subsets of T, using similar arguments, we can prove the following.

THEOREM 10.6. *A positive element* U *in* $L_r(C_{oo}(T,X),Y)_{\tau_{u^o}}$ *is multiplicative if and only if the representing* σ-*measure* m *in*

$$\psi_\sigma(H_\sigma(T),\ L_r(X,(Y_\gamma^{\sim})_\gamma^{\sim})_{\tau o})$$

(*respectively* $\psi_\sigma(H_\sigma(T),\ L_r(X,(Y_\sigma^{\sim})_\gamma^{\sim})_{\tau o})$) *satisfies*

$$m(A_1 \cap A_2)(x_1 \cdot x_2) = (m(A_1)(x_1)) \cdot (m(A_2)(x_2)),$$

for all x_1, x_2 *in* X *and* A_1, A_2 *in* $H_\sigma(T)$.

11. Applications

Our results found significant and important applications in partially ordered vector spaces of operators acting usually on a Hilbert space H. We shall briefly outline some definitions and principal results concerning such kinds of operator spaces.

For more details in this direction consult [2, 3, 10, 18, 23–24, 29, 30, 32, 55, 59, 61, 65–67, 70, 74, 80–82].

(i) C^*-*algebras.* Let $\mathscr{A}$ be a C^*-algebra with identity e, H its universal representation space, A the self-adjoint part of $\mathscr{A}$ and $L(H,H)$ the bounded linear operators acting on H.

We recall that A is a partially ordered Banach space with positive cone $\{a^* \cdot a : a \in \mathscr{A}\}$ and $\mathscr{A}$ may be regarded as a subalgebra of $L(H,H)$. Also A is an orderunit space with orderunit and orderunit norm e; moreover the orderunit norm on A coincides with the C^*-norm.

We call $\mathscr{A}$ monotone (respectively σ-monotone) complete if A is monotone (respectively σ-monotone) complete. Each W^*-algebra and each AW^*-algebra of type I is monotone complete.

Let A_s be a subalgebra of $\mathscr{A}$ of self-adjoint elements.

Our presented results can be immediately applied in the following cases:

$$X = A, \qquad Y = A', \qquad Z = R; \quad \text{or} \quad X = Y = Z = A_s.$$

(ii) *Jordan algebras.* Let J be a Jordan algebra of selfadjoint operators acting on H, that is, a real vector space of such operators closed under the product $U \odot V := \frac{1}{2}(U \cdot V + V \cdot U)$.

By a Banach Jordan algebra J we mean a real Banach space and a Jordan algebra such that $\|U \odot V\| \leq \|U\| \cdot \|V\|$, whenever U, $V \in J$. It is called a JB-algebra if moreover J has a unit $\|U^2\| = \|U\|^2$ and $\|U^2\| \leq \|U^2 + V^2\|$, for all U, $V \in J$.

A JC-algebra is a uniformly closed Jordan algebra with unit.

Alfsen, Shultz and Størmer proved in[2] that if J is a JB-algebra with identity I then the set $\{U^2 : U \in J\}$ is a (proper) closed convex cone under which J becomes a partially ordered Banach space with I as an orderunit norm and $-I \leq U \leq I$ implies $0 \leq U^2 \leq I$, whenever $U \in J$.

It is well known that every JC-algebra is a JB-algebra but the converse is false.

Our included results are directly applied to the cases:

$$X = J, \qquad Y = J', \qquad Z = R; \quad \text{and} \quad X = Y = Z = J.$$

(iii) *Semifields and O^*-algebras.* An associative, commutative partially ordered algebra E is called a semifield if it satisfies the following conditions.

(s_1) E contains a subset K such that $K + \overline{K} \subseteq K$ and $K \cdot K \subseteq K$ with $\overline{K} := \{x \in E : \text{there exists a sequence } (x_n)_{n \in \mathbb{N}} \text{ in } K \text{ such that } x = o\text{-}\lim x_n\}$.

(s_2) $E = K - K$.

(s_3) There exists $\sup\{x_n : n \in \mathbb{N}\}$ in E for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in E.

(s_3) E has more than one element and the equation $a \cdot x = b$ has at least one solution which is an element of K whenever a, $b \in K$.

(s_5) $\overline{K}$ coincides with the set of all non-negative elements of E.

We note that the class of semifields defined here is much wider than the class of topological semifields studied in [3] and is a subclass of K_σ-spaces developed and investigated by L. V. Kantorovich, B. Z. Vulikh and A. G. Pinsker in [37, 38, 74].

Now a $*$-involutory (complex) algebra $\mathscr{A}$ with unit e is called an O^*-algebra if there is defined a partial ordering on the vector space A of self-adjoint elements of $\mathscr{A}$, compatible with the algebraic operations and satisfying the following axioms:

(o_1) $A^+ + A^+ \subseteq A^+ := \{a \in A : a \geq 0\}$ and $(a^* \cdot a) \in A^+$ for every $a \in A$;

(o_2) for every majorised increasing net $(a_\delta)_{\delta \in \Delta}$ in A there exists $a := \sup\{a_\delta : \delta \in \Delta\}$ in A such that $a \cdot b = b \cdot a$ provided $a_\delta \cdot b = b \cdot a_\delta$, whenever $\delta \in \Delta$ and $b \in A$;

(o_3) every maximal commutative $*$-subalgebra of $\mathscr{A}$ is a (complex) semifield.

We provided important applications of our results in the following cases: $X = E$, $Y = E^\sim$, $Z = R$ or $X = Y = Z = E$, with E a semifield and $X = A^+ - A^+$, $Y = A'$, $Z = R$ or $X = Y = Z = A_s$, whenever A is the

self-adjoint part of an O^*-algebra $\mathscr{A}$ and A_s an algebra of elements of A.

(iv) *Quantum probability theory.* E. B. Davies and J. T. Lewis, in order to describe repeated measurements in quantum mechanical systems, have defined in [20] the concrete concept of quantum instrument.

More precisely, let (X, X^+, e) be a triple consisting of a real Banach space X, (according to the terminology of [20] a state space), positively generated by the (strong) closed cone X^+ and a positive linear functional $e : X \to R$ such that $e(x) = \|x\|$, whenever $x \in X^+$.

Assume moreover that $\|x\| = \inf\{\|x_1\| + \|x_2\| : x = x_1 - x_2, \; x_i \in X^+, \; i = 1, 2\}$ for every $x \in X$.

As is easily verified, X is monotone complete, the cone X^+ is normal and $e \in X_y^{\sim}$. Also every positive operator in $L_r(X, X)$ is continuous, and hence $L_r(X, X) = L_y(X, X)$.

Let $H_\sigma(T)$ be a σ-algebra of subsets of a space T. In the terminology of [20] an observable is a positive set function $a : H_\sigma(T) \to X'$ satisfying

$(ob)_1$ $0 \le a(A) \le a(T)$, for all $A \in H_\sigma(T)$ and $a(T) = e$,

$(ob)_2$ a is σ-additive with respect to the weak $*$-topology of X'. We say a is multiplicative if $a(A_1 \cap A_2) = a(A_1) \cdot a(A_2)$, whenever $A_1, A_2 \in H_\sigma(T)$. On the other hand an instrument is a positive set function $\mathscr{E} : H_\sigma(T) \to L_r(X, X)$ such that

(in)$_1$ ” $e(\mathscr{E}(T)(x)) = e(x)$, for every $x \in X$,

(in)$_2$ ” $\mathscr{E}$ is σ-additive with respect to strong operator topology.

It is an easy matter to prove that $\mathscr{E}$ (respectively a) is an $L_r(X, X)$-σ-measure (respectively X'-measure) on $H_\sigma(T)$. Furthermore, it has been shown in [20] that given an instrument $\mathscr{E}$ there is a unique observable a such that $e(\mathscr{E}(A)(x)) = a(A)(x)$, for all $x \in X$ and $A \in H_\sigma(T)$. Also every observable a is determined in such a way by at least one instrument $\mathscr{E}$.

Recall that the example most significant to quantum mechanics is to take X the self-adjoint trace class operators $\mathscr{T}_s(H)$ on a separable Hilbert space H, e the usual trace on X, T a separable locally compact topological space and $H_\sigma(T)$ the Borel σ-algebra of T.

As we may know, a corresponding integration process of X-, X'- or $L_r(X, X)$-valued random variables in a similar sense of E. Rowecka [54] and H. Umegaki and A. T. Bharucha-Reid [73] with respect to an $L_r(X, X)$-(respectively X-) instrument (respectively observable) is lacking from this operational approach to quantum probability theory.

So our presented results can be immediately applied to cover this missing process.

(v) *The operator Feynman-Kac formula.* Let X be a partially ordered Banach space, $L(X, X)$ the space of bounded linear operators acting on X,

T a locally compact Hausdorff space, $H_\sigma(T)$ the σ-algebra of Borel or Baire subsets of T and a σ-measure $m: H_\sigma(T) \to L(X, X)$.

Let $f: T \to R$ be a Borel or Baire function. Define the operator $m(f): X \to X$ with $m(f)(x) := \int_T f(t)\,dm(t)(x)$, for every $x \in X$, whenever the right-hand side exists.

Given such a function f, we assume that the operator $m(f)$ is the infinitesimal generator of a C_0-semigroup $Q(u)$, $u \in R^+$ of linear operators on X (cf. [34, 69]).

We also consider another C_0-semigroup $W(u)$, $u \in R^+$, of linear operators on X, whose infinitesimal generator is the operator A.

Now we are looking for a third semigroup $U(u)$, $u \in R^+$, satisfying the initial value problem

$$(11.1) \qquad U'(s) = AU(s) + m(f)U(s), \qquad s > 0, \qquad U(0) = I.$$

In many cases there are difficulties in deriving the semigroup $U(u)$, $u \in R^+$, since the preceding initial-value problem (11.1) does not in general have classical or mild solutions.

So we try to construct $U(u)$, $u \in R^+$, using an operator formula of the Feynman-Kac type. For this purpose let $s \in R^+$ and let T_s be the set of all continuous paths $p: [0, s] \to T$ based on the closed interval $[0, s]$. Let also P_s be the semialgebra of sets

$$A := \{p \in T_s: p(s_j) \in B_j, j = 1, 2, \ldots, k\}$$

for arbitrary $k = 1, 2, \ldots$,

$$0 \leq s_1 < s_2 < \cdots < s_{k-1} < s_k \leq s \quad \text{and} \quad B_j \in H_\sigma(T), \qquad j = 1, 2, \ldots, k.$$

Then a measure (respectively σ-measure) $m_s: H_\sigma(T_s) \to L(X, X)$ such that

$$m_s(A) := W(s - s_k)m(B_k)W(s_k - s_{k-1})m(B_{k-1}) \cdots W(s_2 - s_1)m(B_1)W(s_1),$$

for every $A \in P_s$, will be called a (W, m, s)-measure, where $H_\sigma(T_s)$ denotes the σ-algebra generated by P_s.

Now the semigroup $U(u)$, $u \in R^+$, can be interpreted in the following operator formula of the Feynman-Kac type (cf. [42, 43]):

$$(11.2) \qquad U(s) = \int_{T_s} \left[\exp\left(\int_0^s f(p(r))\,dr \right) \right] dm_s(p).$$

We recall that a Riemann-Stieltjes or Bochner-type integration process of R. G. Bartle, N. Dunford, J. Schwartz [5], R. G. Bartle [6], J. Batt and E. J. Berg [7], J. K. Brooks and P. W. Lewis [12], J. K. Brooks and N. Dinculeanu [13], N. Dinculeanu [22], I. Dobrakov [25, 26], N. Dunford [27], R. J. Easton and D. H. Tucker [28], J. R. Edwards and S. G. Wayment [31],

R. K. Goodrich [34], A. Shuchat [64], K. Swong [68] and U. J. Uherka [72], cannot be applied to define the right hand side of (11.2) since m_s is not in general of finite variation (cf. [43]).

On the other hand, our presented integration theory does not presuppose finite variation of the corresponding measures (respectively σ-measures) so the obtained integrals can be immediately applied to define the desired semigroup $U(u)$, $u \in R^+$, from formula (11.2).

REMARK. In mathematical physics, the C_o-semigroup $W(u)$, $u \in R^+$, usually describes the motion of a free quantum mechanical particle with n degrees of freedom $(T = \mathbb{R}^n$, $n \geq 1)$ in a force field with a given potential, and the "solutions" $u(s) = U(s)x_0$ of (11.1) characterize the states of such a particle at time $s > 0$, with initial state x_0 in $X = \mathscr{L}^2(\mathbb{R}^n)$ and superposition of $Q(u)$, $W(u)$ the C_o-semigroup $U(u)$, $u \in R^+$.

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