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THE K-THEORY OF THE COTANGENT SPHERE BUNDLE OF $\mathbb{R}P^n$

BY

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ABSTRACT. We calculate the topological K-theory of the cotangent sphere bundle of $\mathbb{R}P^n$ and show the manner in which it is detected by the eta invariant.

1. **Introduction**. Throughout this paper K-theory will mean $Z/2$-graded, complex topological K-theory [1].

If $\tau_{\mathbb{R}P^n}$ denotes the tangent bundle of $\mathbb{R}P^n$ and $\tau^*_{\mathbb{R}P^n}$ denotes the cotangent bundle we will denote by Y_n and X_n, respectively, the associated sphere bundles,

(1.1) $$Y_n = S(\tau_{\mathbb{R}P^n}) \text{ and } X_n = S(\tau^*_{\mathbb{R}P^n}).$$

Being homeomorphic, Y_n and X_n have isomorphic K-theory.

Using (pseudo-) differential operators, Gilkey [2, 3] has constructed a homomorphism, the eta invariant, defined on $K^*(S(\tau^*_M))$, for M a smooth, closed manifold. The computations given below arose in order to understand the eta invariant when $M = \mathbb{R}P^n$.

Before stating our result, we gather some well-known facts.

1.2. Let W_n denote either X_n or Y_n of (1.1) and let $\pi : W_n \to \mathbb{R}P^n$ denote the bundle projection. Let H denote the (complex) Hopf bundle on $\mathbb{R}P^n$ and $\sigma = H - 1 \in \tilde{K}^0(\mathbb{R}P^n)$. From [1, p. 107] we have

$$\tilde{K}^0(\mathbb{R}P^n) \cong Z/2^m, \quad \text{if } n = 2m \text{ or } 2m+1,$$

generated by σ (where $2\sigma + \sigma^2 = 0$) and

$$K^1(\mathbb{R}P^n) = \begin{cases} Z \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even.} \end{cases}$$

If $H_{\mathbb{R}}$ is the real Hopf bundle we have bundle isomorphisms.

(1.3) $$\tau_{\mathbb{R}P^n} \oplus \mathbb{R} \cong (n+1)H_{\mathbb{R}} \cong \tau^*_{\mathbb{R}P^n} \oplus \mathbb{R}.$$

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The second isomorphism in (1.3) results from the isomorphism of $H_{\mathbb{R}}$ and $H^*_{\mathbb{R}}$. If TE denotes the Thom space of a vector bundle, E, then (1.3) yields homeomorphisms.

$$(1.4) \qquad \Sigma T(\tau_{\mathbb{R}P^n}) \cong T(n+1)H_R) \cong \Sigma T(\tau^*_{\mathbb{R}P^n}).$$

We have a spherical ($W_n = X_n$ or Y_n) fibration

$$(1.5) \qquad S^{n-1} \to W_n \to \mathbb{R}P^n.$$

Using (1.3)–(1.5) we will prove the following

THEOREM 1.6. *Let* $W_n = X_n$ *or* Y_n *of* (1.1) ($n \geq 1$).

(i) $\pi^* : K^t(\mathbb{R}P^n) \to K^t(W_n)$ *is injective.*
(ii) $\tilde{K}^0(W_{2n+1}) \cong Z/2^n \oplus Z/2^n \oplus Z$
$\tilde{K}^1(W_{2n+1}) \cong Z \oplus Z.$

(iii) $\tilde{K}^0(W_{2n}) = \begin{cases} Z/4 \text{ if } n = 1 \\ Z/2^n \oplus Z/2^n \text{ if } n \geq 2 \end{cases}$
$\tilde{K}^1(W_{2n}) \cong Z$

REMARK 1.7. §1.6(iii) is the more subtle of these calculations and, in fact, we derive a little of the ring structure in that case (see (3.8) and (3.9)). We prove §1.6(ii) in §2 and §1.6(iii) in §3.

We close this section with a proof (included for the reader's convenience) of a well-known property of (1.5).

PROPOSITION 1.8. *In* (1.1), *if* $n \geq 2$, *the action of*

$$\pi_1(\mathbb{R}P^n) \cong Z/2 \text{ on } Z \cong \pi_{2n-1}(S^{2n-1}) \cong H^{2n-1}(S^{2n-1}) \cong K^1(S^{2n-1})$$

is non-trivial if and only if n is even.

PROOF. Let $u(t) = (\cos(\pi t), \sin(\pi t), 0, 0, \ldots) \in S^n (n \geq 2)$. Then u induces $\hat{u} : I/\partial I \to \mathbb{R}P^n$ which generates $\pi_1(\mathbb{R}P^n)$. If $f(t, z) = \hat{u}(t)$ we must find a lifting H in the diagram

$$(1.9) \qquad \begin{array}{ccc} (0) \times S^{n-1} & \longrightarrow & Y_n \\ \downarrow & \overset{H}{\nearrow} & \downarrow \pi \\ I \times S^{2n-1} & \underset{f}{\longrightarrow} & \mathbb{R}P^n \end{array}$$

Now $Y_n = \{(a, b) \in S^n \times S^n | a \perp b\}/\approx$ where $(a, b) \approx (-a, -b)$.
Define $\hat{H} : I \times S^{n-1} \to S^n \times \mathbb{R}^{n+1}$ by

$$\hat{H}(t, (x_1, \ldots, x_n)) = (u(t), (-\sin(\pi t)x_1, \cos(\pi t)x_1, x_2, \ldots x_n)).$$

Then $\hat{H}$ induces the required H. However

$$\begin{aligned} H(1, (x_1, \ldots, x_n)) &= [(-1, 0, 0, \ldots), (0, -x_1, x_2, \ldots, x_n))] \\ &= [(1, 0, 0, \ldots), (0, x_1, -x_2, -x_3, \ldots, -x_n)] \end{aligned}$$

in terms of the tangent space of $(1, 0, \ldots, 0)$. On S^{n-1} the map which changes the sign of all but one coordinate has degree $(-1)^{n-1}$, as required.

2. **Proof of Theorem 1.6(ii)**. From (1.5) we have a spectral sequence, with simple coefficients,

$$E_2^{s,t} = H^s(\mathbb{R}P^{2n+1}; K^t(S^{2n})) \Rightarrow K^{s+t}(W_{2n+1}).$$

This spectral sequence collapses since $E_2^{s,t} = 0$ if $t \equiv 1(2)$ and

$$E_2^{s,0} = \begin{cases} Z \oplus Z & \text{if } s = 0 \text{ or } s = 2n+1 \\ Z/2 \oplus Z/2 & \text{if } s = 2, 4, 6, \ldots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

If $F^sK^t = \ker(K^t(W_{2n+1}) \to K^t(\pi^{-1}(\mathbb{R}P^{s-1}))$ then $E_2^{s,t-s} \cong F^sK^t/F^{s+1}K^t$ from which it is clear that the 2-primary torsion is killed by 2^n. However if D_n is the disc bundle of $\tau_{\mathbb{R}P^n}$ or $\tau^*_{\mathbb{R}P^n}$ then, by (1.4), we have

$$\begin{aligned} K^\alpha(D_{2n+1}, W_{2n+1}) &\cong K^{\alpha+1}((n+1)H) \\ &= K^{\alpha+1}(\mathbb{R}P^{2n+1}) \end{aligned}$$

by the Thom isomorphism. The exact sequence of (D_{2n+1}, W_{2n+1}) easily yields an exact sequence

$$\begin{array}{ccccccc} 0 \to \tilde{K}^0(\mathbb{R}P^{2n+1}) & \overset{\pi^*}{\to} & \tilde{K}^0(W_{2n+1}) & \to & K^0(RP^{2n+1}) & \to 0 \\ \downarrow \cong & & & & \downarrow \cong & \\ Z/2^n & & & & Z \oplus Z/2^n & \end{array}$$

From this we see that Tors $(\tilde{K}^0(W_{2n+1})) \cong Z/2^n \oplus Z/2^n$ and, from the spectral sequence, Theorem 1.6(ii) and half of §1.6(i) follows immediately.

3. **The proof of Theorem 1.6(iii)**. Let W_{2n} be X_{2n} or Y_{2n}.

LEMMA 3.1.

$$H^j(W_{2n}; Z) \cong \begin{cases} Z/4 \text{ if } j = 2n \\ Z \text{ if } j = 0 \text{ or } 4n-1 \\ Z/2 \text{ if } 2 \le j \le 4n-2; j \text{ even}; j \ne 2n \\ 0 \text{ otherwise.} \end{cases}$$

PROOF. Consider the Serre spectral sequence for $H^*(-; \wedge)$ of (1.5),

$$E_2^{s,t}(\wedge) = H^s(\mathbb{R}P^{2n}; H^t(S^{2n-1}; \wedge)) \Rightarrow H^{s+t}(W_{2n}; \wedge).$$

When $\wedge = Z, E_2^{s,t} = 0$ except for

$$E_2^{s,0}(Z) \cong \begin{cases} Z \text{ if } s = 0 \\ Z/2 \text{ if } 2 \le s \le 2n,\ s \text{ even} \end{cases}$$

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and, by §1.8,

$$E_2^{s,2n-1}(Z) \cong \begin{cases} Z \text{ if } s = 2n, \\ Z/2 \text{ if } 1 \le s \le 2n-1, \ s \text{ odd.} \end{cases}$$

For dimensional reasons $\{E^{s,t}(Z)\}$ collapses, so that we have only to determine the extension $Z/2 \to H^{2n}(W_{2n};Z) \twoheadrightarrow Z/2$. However this extension is resolved by showing that $H^{2n-1}(W_{2n};Z/2) \cong Z/2$. This is seen as follows. We have a classifying diagram for sphere bundles

$$\begin{array}{ccccc} S^{2n-1} & \to & W_{2n} & \xrightarrow{\pi} & \mathbb{R}P^{2n} \\ \| & & \downarrow \rho & & \downarrow \tau \\ S^{2n-1} & \to & BO(2n-1) & \xrightarrow{\pi'} & BO(2n). \end{array}$$

Hence if $0 \neq x \in H^1(\mathbb{R}P^{2n};Z/2)$,

$$\begin{aligned} \pi^*(x^{2n}) &= \pi^* t^*(w_{2n}), \text{ by (1.3)}, \\ &= \rho^*(\pi')^*(w_{2n}) \\ &= \rho^*(0) = 0. \end{aligned}$$

Therefore $d_{2n}^{0,2n-1}: E_{2n}^{0,2n-1}(Z/2) \to E_{2n}^{2n,0}(Z/2) \equiv H^{2n}(\mathbb{R}P^{2n};Z/2)$ is an isomorphism and from $\{E_r^{s,t}(Z/2)\}$ we see that $H^{2n-1}(W_{2n};Z/2) \equiv Z/2$.

LEMMA 3.2. *There is an epimorphism*

$$\tilde{K}^0(W_{2n})/\pi^*(\tilde{K}^0(\mathbb{R}P^{2n})) \twoheadrightarrow Z/2^n.$$

PROOF. *First proof:* The eta invariant [2, 3] is surjective, $\tilde{K}^0(W_{2n}) \to Z/2^n$, and annihilates the image of π^*.

Second proof: The sphere bundle, $S(2n+1)H_{\mathbb{R}})$, is $(S^{2n} \times S^{2n})/(Z/2)$ (with the antipodal involution on each factor).

We have a Mayer-Vietoris diagram of the following form.

$$(3.3) \qquad \begin{array}{ccccc} & & (D^{2n+1} \times S^{2n})/(Z/2) & & \\ & \nearrow & & \searrow & \\ S((2n+1)H_{\mathbb{R}}) & & & & \mathbb{R}P^{4n+1} \\ & \searrow & & \nearrow & \\ & & (S^{2n} \times D^{2n+1})/(Z/2) & & \end{array}$$

Also, from the homeomorphism of sphere bundles in (1.3), we have another Mayer-Vietoris diagram.

$$(3.4) \qquad \begin{array}{ccccc} & & W_{2n} \times I & & \\ & \nearrow & & \searrow & \\ W_{2n} \times \partial I & & & & S((2n+1)H_{\mathbb{R}}) \\ & \searrow & & \nearrow & \\ & & D_{2n} \times \partial I & & \\ & & \downarrow \simeq & & \\ & & \mathbb{R}P^{2n} \times \partial I & & \end{array}$$

Here D_{2n} is the disc bundle associated to W_{2n}, as in §2.

The Mayer-Vietoris K-theory sequences of (3.3) and (3.4) easily establish that

(a) $\pi^*:\tilde{K}^0(\mathbb{R}P^{2n}) \to \tilde{K}^0(W_{2n})$ is injective (which is §1.6(i)) and that

(b) $\tilde{K}^0(W_{2n})/\text{im}(\pi^*) \cong Z/2^n$.

3.5. Completion of the proof of Theorem 1.6. The Atiyah-Hirzebruch spectral sequence (in which $E_2^{s,\text{odd}} = 0$) $E_2^{s,0} = H^s(W_{2n};Z) \Rightarrow K^s(W_{2n})$ collapses, by §3.1. Let G^sK^t be the associated filtration of $K^t(W_{2n})$ so that $G^sK^t/G^{s+1}K^t \cong E_2^{s,t-s}$.

We also have the K-theory Serre spectral sequence of (1.5)

$$\hat{E}_2^{s,t} = H^s(\mathbb{R}P^{2n}; K^t(S^{2n-1})) \Rightarrow K^{s+t}(W_{2n}).$$

The E_2-term of this spectral sequence (remembering that $t \in Z/2$) is the same as $E^{s,t}(Z)$ in §3.1. By §3.2 this spectral sequence also collapses and if $z \subset \tilde{K}^0(W_{2n})$ is represented by a generator $[z] \in \hat{E}_2^{1,1}$ then z has 2-primary order which is at least 2^n. From the $K^0(\mathbb{R}P^{2n})$-module structure on this spectral sequence $\sigma^j z$ is represented by the generator of $\hat{E}_2^{2j+1,1}$. Since $2^{n-1}z \in \text{im}(\pi^*)$ each non-zero $2^\alpha z$ must be represented in $\hat{E}_2^{*,1}$. This means that $2^n z$ is either zero or it lies in the lowest filtration ($\cong \hat{E}_2^{2n,0}$) so that either

$$2^n z = 0 \quad \text{or} \quad 2^n z = \pi^*(\sigma^n). \tag{3.6}$$

We can rule out the second alternative in (3.6) by inspecting G^s-filtrations, except in the case $n = 1$ when §1.6(iii) is clear from the Atiyah-Hirzebruch spectral sequence. Since σ^n is represented in $E_2^{2n,0}$ $\sigma^n \in G^{2n}K^0 - G^{2n+1}K^0$. However z is represented by a generator of $E_2^{2n,0} \cong Z/4$ so that $2^n z \in G^{4n-1}K^0$. If $n > 1$, $G^{4n-2} \subset G^{2n+1}$, so that $2^n z = \pi^*(\sigma^n)$ is impossible.

3.7. From Theorem 1.6 we see that the exact sequence for (D_{2n}, W_{2n}) yields an exact sequence of $K^0(\mathbb{R}P^{2n})$-modules. $0 \to K^0(RP^{2n}) \xrightarrow{\pi^*} K^0(W_{2n}) \xrightarrow{\beta} \tilde{K}^0(\mathbb{R}P^{2n}) \to 0$.

Therefore $\beta(2z + \sigma z) = 0$ and from the Atiyah-Hirzebruch spectral sequence there is an equation of the form

$$\pi^*(\sigma^n) = 2z + \sum_{1 \le j} \lambda_j \sigma^j z.$$

Since $\beta\pi^*(\sigma^n) = 0$ and $\beta(\sigma z + 2z) = 0$ we must have

$$0 = 2 - 2\lambda_1 + 4\lambda_2 - \ldots \in Z/2^n.$$

Hence we obtain the following relation

$$\pi^*(\sigma)^n = 2z + \pi^*(\sigma)z \in K^0(W_{2n}). \tag{3.8}$$

Note that $z \in G^{2n}K^0$ so that

$$z^2 = 0 \in K^0(W_{2n}), \tag{3.9}$$

Since it lies in $G^{4n}K^0 = 0$.

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