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DEFERRED CORRECTIONS FOR EQUATIONS OF THE SECOND KIND

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Abstract

A deferred correction procedure for the approximate solution of the second-kind equation is introduced, compared with an extrapolation procedure, and illustrated for integral and differential equations.

1. Degenerate kernel method

Consider the second-kind equation

$$y = f + ky, \tag{1}$$

where f and y belong to a Banach space E, and k is a compact linear operator in E. It is assumed that 1 is not an eigenvalue of k, in which case a unique solution y exists for any $f \in E$. An example of such an equation is the integral equation

$$y(t) = f(t) + \int_0^1 K(t, s)y(s)\, ds \tag{2}$$

considered in the Banach space C of continuous functions.

The exact equation (1) is approximated by

$$y_n = p_n f + p_n k_n y_n, \tag{3}$$

where k_n is a bounded linear operator in E, and p_n is a bounded linear projection, with the property $\|k - p_n k_n\| \to 0$ as $n \to \infty$. It then follows that the inverse $(I - p_n k_n)^{-1}$ exists as a bounded linear operator in E if n is sufficiently large. Moreover

$$\|y - y_n\| \leqslant O(\|ky - p_n k_n y\| + \|f - p_n f\|).$$

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An example of (3) is the degenerate kernel method

$$y_n(t_i) = f(t_i) + \sum_{j=0}^{n} y_n(t_j) \sum_{k=0}^{n} K(t_i, t_k) \int_0^1 e_k e_j \, ds, \quad i = 0, \ldots, n, \tag{4}$$

where $t_i = i/n$, the basis e_i is a piecewise linear function which equals one at t_i and zero at t_j ($j \neq i$), and

$$p_n f(t) = \sum_{i=0}^{n} f(t_i) e_i(t), \quad k_n y(t) = \int_0^1 \sum_{k=0}^{n} K(t, t_k) e_k(s) y(s) \, ds. \tag{5}$$

Then, if $K \in C^4$, we have

$$\|k - p_n k_n\| = O(n^{-2}), \qquad \|(I - p_n)(k - k_n)\| = O(n^{-4}). \tag{6}$$

We consider here the natural iteration

$$\bar{y}_n = f + k_n y_n, \tag{7}$$

which has been observed in [4] and has the property $p_n \bar{y}_n = y_n$ as in [2], [3], [8]. Then introduce a correction term x_n defined from y_n by

$$x_n = p_n r_n + p_n k_n x_n, \qquad r_n = k_n \bar{y}_n + k y_n + 2f - 2y_n. \tag{8}$$

It should be emphasized that for the degenerate kernel operator, (5),

$$k_n \bar{y}_n(t) = k_n f(t) + \sum_{k=0}^{n} \sum_{j=0}^{n} K(t, t_k) \int_0^1 K(s, t_j) e_k(s) \, ds \int_0^1 e_j y_n \, ds.$$

Direct calculation leads to

THEOREM 1. *If $y_n, \bar{y}_n$ and x_n satisfy* (3), (7) *and* (8), *then*

$$(I - p_n k_n)(y - \bar{y}_n - x_n) = (k - p_n k_n)(y - y_n) + (I - p_n)(k - k_n) y_n.$$

COROLLARY 2. *for the degenerate kernel method* (4) *of integral equation* (2) *we have, if $K \in C^4$,*

$$\|y - \bar{y}_n - x_n\| \leq O\left(n^{-2} \|y - y_n\|\right).$$

It is worth stating the following extrapolation result, which can be shown by a direct calculation.

THEOREM 3. *If $\bar{y}_n$ satisfies* (7), *then*

$$(I - k)(3y - 4\bar{y}_{2n} + \bar{y}_n) = k(I - p_n)(\bar{y}_{2n} - \bar{y}_n) + k(3I - 4p_{2n} + p_n)\bar{y}_{2n}$$
$$+ (k - k_n)(y_{2n} - y_n) + (3k - 4k_{2n} + k_n) y_{2n}.$$

COROLLARY 4. *For the degenerate kernel method* (4) *we have, if $K(t, s)y(s) \in C^4$,*

$$\|3y - 4\bar{y}_{2n} + \bar{y}_n\|_c \leq O(n^{-4}).$$

We remark that the general theory for the corrections and extrapolations due to Stetter [9], Pereyra [6] and Baker [1] are based on the asymptotic expansion argument which requires higher regularity on K and y.

2. Projection-type method

The usual projection-type method takes $k_n = k$ in (3). Then the theorems 1 and 3 reduce to

THEOREM 5. *If*

$$y_n = p_n f + p_n k y_n, \qquad \bar{y}_n = f + k y_n, \tag{9}$$

$$x_n = p_n k(\bar{y}_n - y_n) + p_n k x_n, \tag{10}$$

then

$$(I - p_n k)(y - \bar{y}_n - x_n) = (I - p_n)k(y - y_n).$$

THEOREM 6 [5]. *If* y_n *and* $\bar{y}_n$ *satisfy* (9), *then* $(I - k)(3y - 4\bar{y}_{2n} + \bar{y}_n) = k(I - p_n)k(y_{2n} - y_n) + k(3I - 4p_{2n} + p_n)\bar{y}_{2n}$.

We call $\bar{y}_n$ the iterated projection solution, which has been studied extensively and in depth in Sloan [7], Chandler [2], Chatelin [3], and particularly in [8].

To illustrate the applications of Theorems 5 and 6 we consider here the problem

$$-y'' = f + ay, \tag{11}$$

$$y(0) = y(1) = 0. \tag{12}$$

Let $\mathring{S}_n$ be the piecewise linear function space satisfying (12) and y_n be the finite element solution in $\mathring{S}_n$ defined by

$$(y_n', w') = (f + ay_n, w) \quad \text{for } w \in \mathring{S}_n. \tag{13}$$

Direct calculation shows

$$y_n(t_i) = \int_0^{t_i} y_n'(s)(1 - t_i)\,ds - \int_{t_i}^1 y_n'(s)t_i\,ds = \int_0^1 y_n'(s)G'(t_i, s)\,ds.$$

The Green function $G(t, s)$ of (11) and (12) at nodes t_i $(i = 0, \ldots, n)$ form a basis in $\mathring{S}_n$. Then (13) is equivalent to

$$y_n(t_i) = \int_0^1 (f + ay_n)G(t_i, s)\,ds \tag{14}$$

or (9), with

$$ky = \int_0^1 a(s)G(t, s)y(s)\,ds$$

and p_n the interpolatory projection.

Let $\bar{y}_n$ be the iterated finite element solution defined by

$$-\bar{y}_n'' = f + ay_n, \qquad \bar{y}_n(0) = \bar{y}_n(1) = 0. \tag{15}$$

It follows from (14) that

$$\bar{y}_n(t_i) = y_n(t_i).$$

Let x_n be the correction term defined from y_n by (10), or equivalently let x_n be the piecewise linear function that satisfies

$$(x_n', w') = \big(a(\bar{y}_n - y_n) + ax_n, w\big) \quad \text{for } w \in \mathring{S}_n. \tag{16}$$

Then Theorems 5 and 6 lead to

COROLLARY 7. *If $y_n, \bar{y}_n$ and x_n satisfy* (13), (15) *and* (16), *then*

$$\|y - \bar{y}_n - x_n\| \leqslant O\big(n^{-2}\|y - y_n\|\big).$$

COROLLARY 8. *If y_n and $\bar{y}_n$ satisfy* (13) *and* (15), *and $y \in C^4$, then*

$$\|3y - 4\bar{y}_{2n} + \bar{y}_n\| \leqslant O(n^{-4}).$$

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