A SIMPLE THEORY ON THE DYNAMICAL EFFECTS OF A STRATIFIED FLUID CORE UPON NUTATIONAL MOTION OF THE EARTH

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Abstract

The theory of Molodensky (1961) on dynamical effects of a stratified fluid outer core upon nutations and diurnal Earth tides is reconstructed on a new and probably much simpler ground. A theory equivalent to Molodensky's is well represented on the basis of two linear equations for angular-momentum balance of the whole Earth and the fluid outer core, which differ from the well-known equations of Poincaré (1910) only in the existence of products of inertia due to deformations of the whole Earth and fluid outer core. The products of inertia are characterized by four parameters which are easily computed for every Earth model by the usual Earth tide equations. A reciprocity relation exists between two of the parameters. The Adams-Wiliamson condition is not a necessary premise of the theory. Amplitudes of nutations and tidal gravity factors are computed for three Earth models. A dissipative core-mantle coupling is introduced into the theory qualitatively. The resulting equations are expressed in the same form as those of Sasao, Okamoto and Sakai (1977). Formulae for secular changes in the Earth-Moon system due to the core-mantle friction are derived as evidences of internal consistency of the theory.

1. Introduction

Study of the effects of the fluid outer core upon the nutational motion of the Earth based on realistic Earth models was first undertaken by Jeffreys and Vicente (1957a,b) with the aid of two simplified core models. Later, Molodensky (1961) developed a theory applicable to any stratified fluid core satisfying the Adams-Wiliamson condition. Recent investigation of free core oscillations and Earth tides by Shen and Mansinha (1976), who considered more general core flows than Molodensky (1961), yielded results identical with Molodensky's as

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far as the nutations and Earth tides are concerned, thus showing the general correctness of Molodensky's theory for those phenomena.

The theories mentioned above, being exactly frictionless, take into account inertial core-mantle coupling only, which is inferred to play a major role in the Earth's rotation. Effects of another coupling mechanism, a dissipative core-mantle coupling due to viscous and electromagnetic friction, may also be important because they may provide a possible means for the observational determination of the physical properties of the core. Hence attempts have been made to construct theories, with various degrees of simplifying assumptions, which take superposed effects of the inertial and dissipative couplings into account and which clarify their roles in the precession and nutations (Stacey, 1973; Toomre, 1974; Loper, 1975; Rochester, Jacobs, Smylie and Chong, 1975; Rochester, 1976; Sasao, Okamoto and Sakai, 1977). Sasao et al. (1977), in particular, succeeded in deriving a set of equations describing the nutational motion of the Earth, taking mantle elasticity, fluidity of the core, and core-mantle friction into account. However, their core model still was only a simple one (homogeneous and incompressible fluid with a point mass at the center) devised by Jeffreys and Vicente (1957a). Such incompleteness was imposed mainly because Molodensky's (1961) theory of the stratified fluid core, though rigorous and excellent, was too complicated in its original form to be extended to include the effects of the dissipative coupling. A purpose of this paper is to reconstruct Molodensky's (1961) theory on a new and probably much simpler basis in order to include the effects of the dissipative coupling.

2. Basic Assumptions

We use a reference system fixed to the mean principal axes of the mantle with the bases $\vec{i}_1$, $\vec{i}_2$ and $\vec{i}_3$ rotating with an angular velocity $\vec{\omega} = \Omega(m_1, m_2, 1+m_3)$. Let a state of hydrostatic equilibrium

$$\vec{\nabla} P_0 = \rho_0 \vec{\nabla} \phi_0 \quad , \tag{1}$$

be a basic state, in which the axially symmetric Earth is rotating around the $\vec{i}_3$-axis with a constant rotation rate Ω. Here P_0, ρ_0 and ϕ_0 denote the equilibrium pressure, density and gravitational plus centrifugal potential, respectively. Our basic assumptions, equivalent to those adopted by Molodensky (1961), are the following:

1) Equidensity and equipotential surfaces coincide in the basic state, i.e., $\rho_0(r,\theta) = \rho_0(r_0)$ and $\phi_0(r,\theta) = \phi_0(r_0)$, with

$$r_0 = r\left[1 + \frac{2}{3}\varepsilon(r)P_2(\cos\theta)\right] \quad , \tag{2}$$

where r and θ are geocentric distance and co-latitude, respectively, $\varepsilon(r)$ is the geometrical ellipticity of equipotential surfaces, and $P_2(\cos\theta)$ is the Legendre function of degree 2. Upper and lower boundary surfaces of the fluid outer core in the basic state are equipotential surfaces.

2) The velocity $\vec{v}_f$ of the core flow relative to the mantle, arising in the course of the nutational motion of the Earth, is composed of a dominant uniform-rotation term $\vec{\omega}_f \times \vec{r}$ and a small "correcting" term $\vec{v}$ which is caused by the non-sphericity and deformation of the equipotential surfaces and the compressibility of the fluid, i.e.,

$$\vec{v}_f = \vec{\omega}_f \times \vec{r} + \vec{v} \quad , \tag{3}$$

with $\vec{\omega}_f = \Omega(m_1^f, m_2^f, m_3^f)$, where $\vec{r}$ is a radius vector. The "correcting" term $\vec{v}$, consisting of terms of the order of $\varepsilon r|\vec{\omega}_f|$ and the time derivative of the elastic displacement in the solid parts of the Earth, can be neglected when we are concerned with the approximate calculations of the quasi-static deformation of the spherically symmetric model Earth. In the equations of angular-momentum balance, where terms of the order of $\varepsilon|\vec{\omega}_f|$ may, in general, play essential roles, $\vec{v}$ is neglected when it is multiplied by ε.

In the special case of the central particle model (Jeffreys and Vicente, 1975a), we have

$$\vec{v}_f = \vec{\omega}_f \times \vec{r} - \varepsilon(b)\Omega\vec{\nabla}(m_1^f yz - m_2^f xz) + \frac{1}{2}\vec{\nabla}\left[\frac{dq^{(x)}(b)}{dt}xz + \frac{dq^{(y)}(b)}{dt}yz\right] , \tag{4}$$

where $q^{(x)}(b)$ and $q^{(y)}(b)$ characterize the radial displacement of the core-mantle interface (Sasao et al., 1977), and, therefore, the second assumption is surely satisfied. Moreover, the assumption was justified by Shen and Mansinha (1976) from a more general point of view.

Since time variations of the third components of $\vec{\omega}$ and $\vec{\omega}_f$ are decoupled from those of the other two components in the first order theory, we will ignore m_3 and m_3^f until the last section. We do not assume the Adams-Williamson condition $\rho_0\vec{\nabla}\phi_0 = \lambda(r)\vec{\nabla}\rho_0/\rho_0$, where $\lambda(r)$ is Lame's elastic modulus, unlike Molodensky (1961).

3. Angular-Momentum Equation of the Fluid Outer Core

Let us ignore at first the dissipative forces and consider inertial coupling only. Then linearized hydrodynamical equations of motion and continuity are written in the rotating system

$$\rho_0\left(\frac{\partial\vec{v}_f}{\partial t} + \frac{d\vec{\omega}}{dt} \times \vec{r} + 2\Omega\vec{1}_3 \times \vec{v}_f\right) = -\vec{\nabla}P_1 + \rho_1\vec{\nabla}\phi_0 + \rho_0\vec{\nabla}\phi_1 \quad , \tag{5}$$

and

$$\frac{\partial\rho_1}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{v}_f + \vec{v}_f\cdot\vec{\nabla}\rho_0 = 0 \quad , \tag{6}$$

while the Poisson equation is

$$\nabla^2 \phi_d = \nabla^2 \phi_1 = -4\pi G \rho_1 \quad , \tag{7}$$

where ρ_1, P_1 and ϕ_1 are small perturbations of density, pressure and gravitational plus centrifugal potential, respectively, induced by the luni-solar tide-generating force and/or the departure of the instantaneous rotation axis of the mantle from the $\vec{i}_3$-axis. The perturbed potential ϕ_1 is composed of three terms:

$$\phi_1 = \phi_e + \phi_m + \phi_d \quad , \tag{8}$$

where

$$\phi_e = \frac{\Omega^2}{3} r^2 \operatorname{Re}(\tilde{\phi}\tilde{Y}_2^1) = \phi^{(x)} xz + \phi^{(y)} yz \tag{9}$$

is a luni-solar tide-generating potential,

$$\phi_m = -\frac{\Omega^2}{3} r^2 \operatorname{Re}(\tilde{m}\tilde{Y}_2^1) = -\Omega^2(m_1 xz + m_2 yz) \tag{10}$$

is a pole tide potential, and ϕ_d is a gravitational potential arising from the elastic deformation of the Earth. Here we introduced notations: $(\tilde{\ })$ - complex quantities; Re - real part; $\tilde{m} = m_1 + im_2$ - wobble, i.e., a geographical motion of the rotation axis of the mantle; $\tilde{\phi} = (\phi^{(x)} + i\phi^{(y)})/\Omega^2$ - non-dimensional complex coefficient of the tesseral mode of the tide-generating potential; $\tilde{Y}_n^m = P_n^m(\cos\theta)\exp(-im\lambda)$ - complex surface spherical harmonics of degree n and order m, where P_n^m is the associated Legendre function and λ is east longitude. Introducing a new quantity

$$\phi_f = -\frac{\Omega^2}{3} r^2 \operatorname{Re}(\tilde{m}_f\tilde{Y}_2^1) = -\Omega^2(m_1^f xz + m_2^f yz) \tag{11}$$

with $\tilde{m}_f = m_1^f + im_2^f$, and using equation (3), we write equation (5) in the form

$$\frac{\partial \vec{v}}{\partial t} + \frac{d(\vec{\omega}+\vec{\omega}_f)}{dt} \times \vec{r} + 2\Omega\vec{i}_3 \times \vec{v} + \Omega(\vec{i}_3 \times \vec{\omega}_f) \times \vec{r} = -\vec{\nabla}P - R\vec{\nabla}r_0 \quad , \tag{12}$$

where

$$P = \frac{P_1}{\rho_0} - \phi_1 - \phi_f \ , \quad \text{and} \quad R = \frac{P_1}{\rho_0^2}\frac{d\rho_0}{dr_0} - \frac{\rho_1}{\rho_0}\frac{d\phi_0}{dr_0} \quad . \tag{13}$$

Taking the vector product of equation (12) by $\rho_0\vec{r}$, integrating through the volume of the fluid outer core, and neglecting second and higher order terms, we have an equation for an angular momentum $\vec{H}_f$ of the fluid outer core

$$\frac{d\vec{H}_f}{dt} + \vec{\omega} \times \int_f \rho_0 \vec{r}\times\vec{v}dV - \vec{\omega}_f \times \vec{H}_f = -\int_f \rho_0 P\vec{r}\times d\vec{S} + \int_f \left(P\frac{d\rho_0}{dr_0} - \rho_0 R\right)\vec{r}\times\vec{\nabla}r_0 dV \ , \tag{14}$$

with

$$\vec{H}_f = A_f(\vec{\omega}+\vec{\omega}_f) + (C_f - A_f)\Omega\vec{i}_3 + c^f_{31}\Omega\vec{i}_1 + c^f_{32}\Omega\vec{i}_2 + \int_f \rho_0\vec{r}\times\vec{v}dV \quad , \tag{15}$$

where A_f, C_f, c^f_{31} and c^f_{32}, and $d\vec{S}$ are principal moments of inertia, products of inertia arising from the deformation, and a surface element at the boundaries, respectively, of the fluid outer core. A term $c^f_{33}\Omega\vec{i}_3$ is omitted because it makes no contribution to the first-order equations of nutation. Equation (2) gives

$$\vec{r}\times d\vec{S} = (\vec{r}\times\vec{\nabla}r_0)r^2 \sin\theta \, d\theta \, d\lambda \quad , \tag{16}$$

and

$$\vec{r}\times\vec{\nabla}r_0 = -\frac{2}{3} r\varepsilon(r)P^1_2(\cos\theta)(-\vec{i}_1 \sin\lambda + \vec{i}_2 \cos\lambda) \quad , \tag{17}$$

in the first order approximation with respect to ε. Hence only tesseral components of degree 2 in P and R may provide non-vanishing contributions to the right-hand-side integral of equation (14).

We now expand the velocity $\vec{v}_f$ in a series of orthogonal vectors

$$\vec{\tilde{R}}^m_n = \frac{\vec{r}}{r}\tilde{Y}^m_n \quad , \quad \vec{\tilde{S}}^m_n = r\vec{\nabla}\tilde{Y}^m_n \quad , \text{ and } \vec{\tilde{T}}^m_n = -\vec{r}\times\vec{\nabla}\tilde{Y}^m_n \quad , \tag{18}$$

satisfying the normalization relations

$$\oint \vec{\tilde{R}}^m_n\cdot\vec{\tilde{R}}^{m*}_k \sin\theta \, d\theta \, d\lambda = \frac{4\pi(n+m)!}{(2n+1)(n-m)!}\delta^k_n \quad ,$$

and

$$\oint \vec{\tilde{S}}^m_n\cdot\vec{\tilde{S}}^{m*}_k \sin\theta \, d\theta \, d\lambda = \oint \vec{\tilde{T}}^m_n\cdot\vec{\tilde{T}}^{m*}_k \sin\theta \, d\theta \, d\lambda = \frac{4\pi n(n+1)(n+m)!}{(2n+1)(n-m)!}\delta^k_n \quad , \tag{19}$$

where $(\tilde{\ })^*$ denotes complex conjugate. Since we are considering the nutations and diurnal Earth tides only, we put m=1. Then we have

$$\vec{v}_f = \mathrm{Re}\left\{\Omega r\tilde{m}_f\vec{\tilde{T}}^1_1 + \sum_n [\tilde{U}_n(r)\vec{\tilde{R}}^1_n + \tilde{V}_n(r)\vec{\tilde{S}}^1_n + \tilde{W}_n(r)\vec{\tilde{T}}^1_n]\right\} \quad , \tag{20}$$

where the coefficients $\tilde{U}_n$, $\tilde{V}_n$ and $\tilde{W}_n$ are functions of r only. We can now easily see from equation (20) that the integral $\int_f \rho_0\vec{r}\times\vec{v}dV$ in equations (14) and (15) may be put equal to zero by a suitable choice of $\vec{\omega}_f$. Indeed, since

$$\int_f\rho_0\vec{r}\times\vec{v}dV = \mathrm{Re}\sum_n[\int_f\rho_0(r)r^2\tilde{W}_n(r)\vec{\nabla}\tilde{Y}^1_n dV] = \frac{8\pi}{3}\mathrm{Re}[(\vec{i}_1-i\vec{i}_2)\int_c^b\rho_0(r)r^3\tilde{W}_1(r)dr] \quad , \tag{21}$$

where r=b and r=c are the upper and lower boundaries of the fluid outer core, we obtain

$$\int_f \rho_0\vec{r}\times\vec{v}dV = 0 \quad , \tag{22}$$

choosing $\tilde{m}_f$ in equation (20) in such a way that

$$\int_c^b \rho_0(r) r^3 \tilde{W}_1(r) dr = 0 \quad . \tag{23}$$

Putting

$$P = \mathrm{Re} \sum_n [\tilde{P}_n(r) \tilde{Y}_n^1] \quad , \quad \frac{P_1}{\rho_0} = \mathrm{Re} \sum_n [\tilde{Q}_n(r) \tilde{Y}_n^1] \quad \text{and} \quad \rho_1 = \mathrm{Re} \sum_n [\tilde{\rho}_n(r) \tilde{Y}_n^1] \quad , \tag{24}$$

we proceed to expand equation (12) in a series of orthogonal vectors. Straightforward calculations, where higher order terms with respect to ε are neglected, give the following components of interest:

$\vec{\tilde{T}}_1^1$-component:
$$\frac{\partial \tilde{W}_1}{\partial t} + r\Omega \frac{d(\tilde{m}+\tilde{m}_f)}{dt} + i\Omega^2 r\tilde{m}_f + \Omega\left(i\tilde{W}_1 - \frac{3}{5}\tilde{U}_2 - \frac{9}{5}\tilde{V}_2\right) = 0 \quad , \tag{25}$$

$\vec{\tilde{R}}_2^1$-component:
$$\frac{\partial \tilde{U}_2}{\partial t} + 2\Omega\left(i\tilde{V}_2 + \frac{1}{3}\tilde{W}_1 - \frac{16}{7}\tilde{W}_3\right) = -\frac{d\tilde{P}_2}{dr} - \tilde{R}_2 \quad , \tag{26}$$

and

$\vec{\tilde{S}}_2^1$-component:
$$\frac{\partial \tilde{V}_2}{\partial t} + \frac{\Omega}{3}\left(i\tilde{U}_2 + i\tilde{V}_2 + \tilde{W}_1 + \frac{32}{7}\tilde{W}_3\right) = -\frac{\tilde{P}_2}{r} \quad , \tag{27}$$

where

$$\tilde{R}_2 = \frac{1}{\rho_0}\left(\tilde{Q}_2 \frac{d\rho_0}{dr} - \tilde{\rho}_2 \frac{d\phi_0}{dr}\right) \quad . \tag{28}$$

Equations (26) and (27) show that the tesseral components of degree 2 in P/r and R have values of the order of $\Omega|\vec{v}|$, and, moreover, they are multiplied by ε in equation (14) in view of equations (16) and (17). Then, neglecting the right-hand-side terms in equation (14) on the basis of our second assumption, we obtain a very simple equation for the angular-momentum balance of the fluid outer core

$$\frac{d\vec{H}_f}{dt} - \vec{\omega}_f \times \vec{H}_f = 0 \quad , \tag{29}$$

where equation (22) is taken into account. If we denote the inertial core-mantle coupling torque and luni-solar torque upon the fluid core by $\vec{N}$ and $\vec{L}_f$, respectively, then a general angular-momentum equation

$$\frac{d\vec{H}_f}{dt} + \vec{\omega} \times \vec{H}_f = \vec{N} + \vec{L}_f \quad , \tag{30}$$

and equation (29) yield an expression for the inertial coupling torque

$$\vec{N} = (\vec{\omega}+\vec{\omega}_f) \times \vec{H}_f - \vec{L}_f \quad , \tag{31}$$

which is an extended version of Rochester's (1976) equation (27). Equations (15), (22) and (29) give

$$A_f\{D\tilde{m} + [D + i(1+e_f)\Omega]\tilde{m}_f\} + D\tilde{c}_3^f = 0 \quad , \tag{32}$$

where $D = d/dt$ is an operator of time derivative,

$$e_f = (C_f - A_f)/A_f \tag{33}$$

is the dynamical ellipticity of the fluid outer core, and $\tilde{c}_3^f = c_{31}^f + ic_{32}^f$.

The well-known angular-momentum equation of the whole Earth is

$$\frac{d\vec{H}}{dt} + \vec{\omega} \times \vec{H} = \vec{L} \quad , \tag{34}$$

with

$$\vec{H} = A\vec{\omega} + (C-A)\Omega\vec{i}_3 + A_f\vec{\omega}_f + c_{31}\Omega\vec{i}_1 + c_{32}\Omega\vec{i}_2 \quad , \tag{35}$$

where

$$\vec{L} = \int \rho_0\vec{r} \times \vec{\nabla}\phi_e dV \tag{36}$$

is the luni-solar torque upon the whole Earth, $A < C$, and c_{31} and c_{32} are principal moments of inertia and products of inertia arising from the deformation of the whole Earth. Here we neglected effects of the rotation of the solid inner core relative to the mantle because of its small moment of inertia. Equations (9), (34), (35) and (36) give

$$A(D-ie\Omega)\tilde{m} + (D+i\Omega)(A_f\tilde{m}_f + \tilde{c}_3) = -iAe\Omega\tilde{\phi} \quad , \tag{37}$$

where

$$e = (C-A)/A \tag{38}$$

is a dynamical ellipticity of the whole Earth, and $\tilde{c}_3 = c_{31} + ic_{32}$.

4. Elastic Deformation of the Earth

For the calculation of the products of inertia, it is sufficient to use the familiar Earth-tide equations for the solid mantle and inner core (e.g. Saito, 1974), applicable to the quasi-static deformation of a spherically symmetric and self-gravitating elastic body, because of the smallness of the ellipticity and the ratio [tidal frequency]/[frequency of the free spheroidal oscillations of the Earth]. For the fluid core, neglecting the small "correcting" term in equations (26) and (27) on the basis of our second assumption, we have approximate equations

$$\frac{d\tilde{P}_2}{dr} + \tilde{R}_2 = 0 \quad , \tag{39}$$

and

$$\tilde{P}_2 = 0 \quad , \tag{40}$$

which can be interpreted as equations of the quasi-static deformation of the fluid outer core. Equations (28), (39) and (40) then give

$$\tilde{\rho}_2(r) + [(d\rho_0/dr)/g(r)]\tilde{Q}_2(r) = 0 \quad , \tag{41}$$

where $g(r) = -d\phi_0/dr$ is the gravitational acceleration. Now we introduce the familiar notations for the displacement, stress and perturbed potential devised by Alterman _et al._ (1959):

$$u_r = \mathrm{Re}[\tilde{y}_1(r)\tilde{Y}_2^1] \quad , \quad u_\theta = \mathrm{Re}[\tilde{y}_3(r)\frac{\partial \tilde{Y}_2^1}{\partial\theta}] \quad , \quad u_\lambda = \mathrm{Re}[\tilde{y}_3(r)\frac{\partial \tilde{Y}_2^1}{\sin\theta\partial\lambda}] \quad , \tag{42}$$

$$\sigma_{rr} = \mathrm{Re}[\tilde{y}_2(r)\tilde{Y}_2^1] \; , \; \sigma_{r\theta} = \mathrm{Re}[\tilde{y}_4(r)\frac{\partial \tilde{Y}_2^1}{\partial\theta}] \; , \; \sigma_{r\lambda} = \mathrm{Re}[\tilde{y}_4(r)\frac{\partial \tilde{Y}_2^1}{\sin\theta\partial\lambda}] \quad , \tag{43}$$

$$\phi_1 = \mathrm{Re}[\tilde{y}_5(r)\tilde{Y}_2^1] \quad , \tag{44}$$

and

$$\tilde{y}_6(r) = \frac{d\tilde{y}_5(r)}{dr} - 4\pi G\rho_0\tilde{y}_1(r) \quad , \tag{45}$$

where u and σ_{ij} are the elastic displacement and stress tensor, respectively. Equations (11), (13), (24), (40) and (44) then give

$$\tilde{Q}_2(r) = \tilde{y}_5(r) - \frac{\Omega^2}{3} r^2\tilde{m}_f \quad . \tag{46}$$

Substituting equations (41) and (46) into the Poisson equation (7), we have

$$\frac{d^2\tilde{Q}_2(r)}{dr^2} + \frac{2}{r}\frac{d\tilde{Q}_2(r)}{dr} - [4\pi G(d\rho_0/dr)/g(r) + \frac{6}{r^2}]\tilde{Q}_2(r) = 0 \quad , \tag{47}$$

coincident with equation (30) of Molodensky (1961). If we put $\tilde{m}_f \doteq 0$, the equation becomes identical with Takeuchi's (1950) well-known equation (214). It is interesting to note that equations (41) and (47) are derived directly from equations of the quasi-static momentum balance and the Poisson equation without any supplementary assumptions such as the Adams-Williamson condition. If the density of the outer core has jump discontinuities, we can calculate $\tilde{Q}_2(r)$ by a method of Saito (1974), instead of integrating equation (47). Thus the equations required turn out to be of the same form as those used in the familiar Earth tide calculations. The only difference arising from the relative rotation of the fluid outer core appears in the boundary conditions:

at the surface of the Earth, r=a,

$$\begin{aligned} &\tilde{y}_2(a) = \tilde{y}_4(a) = 0 \quad , \\ &\tilde{y}_6(a) + \frac{3}{a}\tilde{y}_5(a) = \frac{5}{3}\Omega^2 a(\tilde{\phi}-\tilde{m}) \quad , \end{aligned} \tag{48}$$

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at the core-mantle boundary, r=b,

$$\begin{aligned}
&\tilde{y}_2(b) - \rho_0(b-)g(b)\tilde{y}_1(b) + \rho_0(b-)\tilde{y}_5(b) = \frac{\Omega^2}{3}\rho_0(b-)b^2\tilde{m}_f \quad ,\\
&\tilde{y}_4(b) = 0 \quad ,\\
&\tilde{y}_5(b) - \tilde{Q}_2(b) = \frac{\Omega^2}{3} b^2 m_f \quad ,\\
&\tilde{y}_6(b) + 4\pi G\rho_0(b-)\tilde{y}_1(b) - \frac{d\tilde{Q}_2}{dr} = \frac{2}{3}\Omega^2 b\tilde{m}_f \quad ,
\end{aligned} \tag{49}$$

and at the outer core-inner core boundary, r=c,

$$\begin{aligned}
&\tilde{y}_2(c) - \rho_0(c+)g(c)\tilde{y}_1(c) + \rho_0(c+)\tilde{y}_5(c) = \frac{\Omega^2}{3}\rho_0(c+)c^2\tilde{m}_f \quad ,\\
&\tilde{y}_4(c) = 0 \quad ,\\
&\tilde{y}_5(c) - \tilde{Q}_2(c) = \frac{\Omega^2}{3} c^2\tilde{m}_f \quad ,\\
&\tilde{y}_6(c) + 4\pi G\rho_0(c+)\tilde{y}_1(c) - \frac{d\tilde{Q}_2}{dr} = \frac{2}{3}\Omega^2 c\tilde{m}_f \quad ,
\end{aligned} \tag{50}$$

whilst at the center of the Earth r=0, solutions must be finite. It is then convenient to represent the solutions as combinations of "static" and "dynamical" terms, say, which are obtained by putting formally $\tilde{\phi}-\tilde{m} = 1$, $\tilde{m}_f = 0$ and $\tilde{\phi}-\tilde{m} = 0$, $\tilde{m}_f = 1$, respectively, in the above boundary conditions. Thus, we have

$$\tilde{y}_i(r) = y_i^{(s)}(r)(\tilde{\phi}-\tilde{m}) + y_i^{(d)}(r)\tilde{m}_f$$

and

$$\tilde{Q}_2(r) = Q_2^{(s)}(r)(\tilde{\phi}-\tilde{m}) + Q_2^{(d)}(r)\tilde{m}_f \quad , \tag{51}$$

with $i = 1,\ldots,6$. The "static" terms are nothing but the usual solutions of the quasi-static Earth tide equations (e.g. Saito, 1974). Love's numbers are

$$h = h_0 + h_1[\tilde{m}_f/(\tilde{\phi}-\tilde{m})] \quad , \quad \ell = \ell_0 + \ell_1[\tilde{m}_f/(\tilde{\phi}-\tilde{m})]$$

and

$$k = k_0 + k_1[\tilde{m}_f/(\tilde{\phi}-\tilde{m})] \quad , \tag{52}$$

where

$$h_0 = \frac{3g(a)}{\Omega^2 a^2} y_1^{(s)}(a) \quad , \quad \ell_0 = \frac{3g(a)}{\Omega^2 a^2} y_3^{(s)}(a) \quad , \quad k_0 = \frac{3}{\Omega^2 a^2} y_5^{(s)}(a) - 1 \quad ,$$

and

$$h_1 = \frac{3g(a)}{\Omega^2 a^2}\, y_1^{(d)}(a) \quad , \quad \ell_1 = \frac{3g(a)}{\Omega^2 a^2}\, y_3^{(d)}(a) \quad , \quad k_1 = \frac{3}{\Omega^2 a^2}\, y_5^{(d)}(a) \quad . \tag{53}$$

h_0, ℓ_0 and k_0 are so-called static Love's numbers. MacCullagh's theorem gives

$$\tilde{c}_3 = -A[\kappa(\tilde{\phi}-\tilde{m}) - \xi\tilde{m}_f] \quad , \tag{54}$$

where

$$\kappa = (k_0 a^5 \Omega^2)/(3GA) \quad \text{and} \quad \xi = -(k_1 a^5 \Omega^2)/(3GA) \quad , \tag{55}$$

whereas a general formula

$$\tilde{c}_3^f = -\int_f \rho_1 (xz+iyz)dV - \oint_f \rho_0 \vec{u}\cdot(xz+iyz)d\vec{S} = \frac{1}{4\pi G}\int_f \nabla^2\phi_1 (xz+iyz)dV$$
$$- \oint_f \rho_0 \vec{u}\cdot(xz+iyz)d\vec{S}$$

yields

$$\tilde{c}_3^f = -A_f[\gamma(\tilde{\phi}-\tilde{m}) - \beta\tilde{m}_f] \quad , \tag{56}$$

where

$$\gamma = -\frac{1}{5GA_f}\{r^4[y_6^{(s)}(r) - 2y_5^{(s)}(r)/r]\}\Big|_c^b$$

and

$$\beta = -\frac{1}{5GA_f}\{r^4[y_6^{(d)}(r) - 2y_5^{(d)}(r)/r]\}\Big|_c^b \quad . \tag{57}$$

Equations (32), (37), (54) and (56) now form a closed set of equations, which reduce to the classical equations of Poincaré (1910) and to the equations for the central particle model (Sasao et al., 1977) if we put $\tilde{c}_3 = \tilde{c}_3^f = 0$ and $\tilde{c}_3^f = -A_f\tilde{q}/2$ with $\tilde{q} = 3\tilde{y}_1(b)/b$, respectively.

5. A Reciprocity Relation

The "static" and "dynamical" terms of $\tilde{y}_i(r)$ $(i = 1,\ldots,6)$, being solutions of the same equations, satisfy an identity

$$\frac{d}{dr}\{r^2[(y_1^{(s)}y_2^{(d)} - y_2^{(s)}y_1^{(d)}) + 6(y_3^{(s)}y_4^{(d)} - y_4^{(s)}y_3^{(d)})$$
$$+ \frac{1}{4\pi G}(y_5^{(s)}y_6^{(d)} - y_6^{(s)}y_5^{(d)})]\} = 0 \quad , \tag{58}$$

which is another expression of Betti's reciprocity theorem (Saito, 1974). For $Q_2^{(s)}(r)$ and $Q_2^{(d)}(r)$, on the other hand, another identity

$$\frac{d}{dr}[r^2(Q_2^{(s)}\frac{dQ_2^{(d)}}{dr} - \frac{dQ_2^{(s)}}{dr}Q_2^{(d)})] = 0 \tag{59}$$

holds. Applying these identities successively to the mantle, fluid outer core and solid inner core and taking account of the respective boundary conditions, we have

$$5a^3 y_5^{(d)}(a) = \{r^4 [y_6^{(s)}(r) - 2y_5^{(s)}(r)/r]\}|_c^b \quad . \tag{60}$$

It is evident from equations (53), (55) and (57) that equation (60) implies

$$A\xi = A_f \gamma \quad . \tag{61}$$

This reciprocity relation may serve as a useful mean for the computation check.

6. Equivalence of our Equation (32) to Equation (39) of Molodensky (1961)

From equation (45) and the continuity of the normal component of the velocity, we have at r=b- and r=c+

$$\dot{\tilde{y}}_6(r) = \dot{\tilde{y}}_5'(r) - 4\pi G\rho_0 \frac{r^2 \dot{\tilde{\eta}}_2(r)}{\phi_0'} \quad , \tag{62}$$

where the dot and prime denote differentiation with respect to time and radial distance, respectively, and $\dot{\tilde{\eta}}_2$ is defined by

$$(\vec{\omega}_f \times \vec{r} + \vec{v}) \cdot \vec{\nabla}\phi_0 = \mathrm{Re} \sum_n [r^n \dot{\tilde{\eta}}_n(r) \tilde{Y}_n^1] \quad . \tag{63}$$

Since $A_f \cong (8\pi/3) \int_c^b \rho_0(r) r^4 dr$, we obtain from equations (32), (56) and (62)

$$(10 \int_c^b \rho_0 r^4 dr)[\dot{\tilde{m}} + \dot{\tilde{m}}_f + i(1+e_f)\Omega\tilde{m}_f] + \frac{3}{4\pi G}$$
$$\times \{r^4 [r^2 (\frac{\dot{\tilde{y}}_5}{r^2})' - 4\pi G\rho_0 \frac{r^2\dot{\tilde{\eta}}_2}{\phi_0'}]\}|_c^b = 0 \quad . \tag{64}$$

From the well-known equation

$$e_f \int_c^b \rho_0 r^4 dr = \frac{1}{5} \{[\rho_0 \varepsilon(r) r^5]|_c^b - \int_c^b \rho_0' \varepsilon(r) r^5 dr\} \quad , \tag{65}$$

and equations (46) and (47), we find that equation (64) can be written as

$$[\frac{3}{\phi_0'} \rho_0 r^6 \dot{\tilde{\eta}}_2 - \frac{i\Omega\tilde{m}_f}{\phi_0'} \rho_0 r^4 K_1 - \frac{1}{4\pi G} (\frac{3\dot{\tilde{Q}}_2 + i\Omega\tilde{m}_f K_1}{r^2})' \; r^6]|_c^b$$
$$= 10(\dot{\tilde{m}} + \dot{\tilde{m}}_f + i\Omega\tilde{m}_f) \int_c^b \rho_0 r^4 dr \quad , \tag{66}$$

where the function

$$K_1(r) = 2r\varepsilon(r)\phi_0' \tag{67}$$

is a particular solution of equation (47) (Molodensky, 1961). If we consider, following Molodensky (1961), a circular motion $\propto \exp(i\sigma t)$ with $|(\sigma+\Omega)/\Omega| \ll 1$, we see that our equation (66) coincides with Molodensky's (1961) equation (39). It is worthwhile to note that equation (66) can be derived rather directly from the Poisson equation and equations (6) and (25), which have not been used explicitly in our treatment.

7. Dissipative Coupling Torque

We now introduce the dissipative coupling torque in a qualitative manner, assuming: 1) our second assumption holds in most of the fluid outer core, because the effects of core viscosity and magnetic field are important only in the boundary layer, which is much thinner than $b\varepsilon(b)$ (Rochester, 1976); 2) for a similar reason, the hydrodynamical equation (5) and subsequent equations (25)-(27) still hold in the fluid outer core except in the thin boundary layer; and 3) modifications of the elastic deformations imposed by the effects of the frictional forces are small and can be neglected. Then, introduction of the dissipative terms into equation (5) results in adding the dissipative coupling torque $\vec{\Gamma}$ to equation (29)

$$\frac{d\vec{H}_f}{dt} - \vec{\omega}_f \times \vec{H}_f = \vec{\Gamma} \quad . \tag{68}$$

As a general expression for $\vec{\Gamma}$ we adopt

$$\vec{\Gamma} = -(K+K'\vec{i}_3\times)(\vec{\omega}_f-\omega_3^f\vec{i}_3) - K^*\omega_3^f\vec{i}_3 \quad , \tag{69}$$

where K, K' and K^* are coupling constants. Here we introduced the third component of $\vec{\omega}_f$ for generality. For the case of a rigid mantle and incompressible core, Toomre (1974), Loper (1975) and Rochester (1976) gave explicit expressions for K and K' in terms of the core viscosity and magnetic field on the basis of the theory of the Eckman-Hartmann layer. A ratio $\eta = K'/K$, in particular, was shown to be nearly zero in viscous coupling and unity in electromagnetic coupling. Then, we have, instead of equation (32)

$$A_f\{D\tilde{m} + [D + i(1+e_f)\Omega]\tilde{m}_f\} + D\tilde{c}_3^f = -K(1+i\eta)\tilde{m}_f \quad . \tag{70}$$

8. Numerical Computations

Results of numerical computations for three Earth models are listed in Table 1. The moments of inertia are calculated by formulae

$$A = \frac{8\pi}{3} \int \rho_0 r^4 dr - \frac{8\pi}{9} \int \rho_0 r^4 \varepsilon(r) dr - \frac{8\pi}{45} \int \rho_0 r^5 \frac{d\varepsilon}{dr} dr \quad , \qquad (71)$$

and

$$C = \frac{8\pi}{3} \int \rho_0 r^4 dr + \frac{16\pi}{9} \int \rho_0 r^4 \varepsilon(r) dr + \frac{16\pi}{45} \int \rho_0 r^5 \frac{d\varepsilon}{dr} dr \quad , \qquad (72)$$

where $\varepsilon(r)$ is obtained by solving the well-known Clairaut equation. The reciprocity relation (61) holds satisfactorily.

Since e, e_f, β, γ, ξ and κ are small compared with unity, equations (37), (70), (54) and (56) can be further simplified, and become identical with equations (34) and (35) of Sasao et al. (1977). Differences are only in the numerical parameters. Hence, all the discussions in Sasao et al. (1977) on the search for the observable consequences of

Table 1. Numerical Parameters

	Wang	Bullen & Haddon	Gutenberg & Bullen-A
A (g cm^2)	8.012752*10^{44}	8.017676*10^{44}	8.090656*10^{44}
A_f (g cm^2)	9.151935*10^{43}	8.988057*10^{43}	8.608042*10^{43}
e	3.244940*10^{-3}	3.249863*10^{-3}	3.275467*10^{-3}
e_f	2.525403*10^{-3}	2.522115*10^{-3}	2.563954*10^{-3}
γ	1.971477*10^{-3}	2.019606*10^{-3}	1.957810*10^{-3}
β	6.270286*10^{-4}	6.436952*10^{-4}	5.865066*10^{-4}
κ	1.045419*10^{-3}	1.047138*10^{-3}	1.032208*10^{-3}
ξ	2.251765*10^{-4}	2.264039*10^{-4}	2.083010*10^{-4}
$A\xi$ (g cm^2)	1.8042835*10^{41}	1.8152330*10^{41}	1.6852912*10^{41}
$A_f\gamma$ (g cm^2)	1.8042833*10^{41}	1.8152329*10^{41}	1.6852913*10^{41}
h_0	0.608470	0.613919	0.607119
ℓ_0	0.085859	0.084623	0.082941
k_0	0.300284	0.300963	0.299372
h_1	-0.128311	-0.130915	-0.121139
ℓ_1	0.003807	0.004113	0.004182
k_1	-0.064679	-0.065072	-0.060414
σ_1/Ω	1/402.7	1/403.1	1/398.4
n_0/Ω	-1/466.6	-1/472.7	-1/451.9

Models: Wang (1972); Bullen & Haddon (1967) - base Earth model B_1; Gutenberg & Bullen-A (e.g. Alterman et al., 1961).

the dissipative coupling and damping of the free core nutation can be readily reproduced in the case of the stratified fluid core. We have the Chandler frequency, ignoring oceanic effects,

$$\sigma_1 \cong \frac{A}{A_m} (e-\kappa)\Omega \quad , \tag{73}$$

with $A_m = A - A_f$, and the frequency of the nearly diurnal wobble associated with the free core nutation

$$\tilde{\sigma}_2 = -\Omega + \tilde{n}_0 = -\Omega + n_0 + i\tilde{\alpha}_2 \quad , \tag{74}$$

where

$$n_0 \cong -\frac{A}{A_m} (e_f - \beta)\Omega \tag{75}$$

is the frequency of the free core nutation in the dissipationless limit, and

$$\tilde{\alpha}_2 = \frac{A}{A_f A_m} K(1+i\eta) = \alpha_2(1+i\eta) \tag{76}$$

is the damping coefficient of the free core nutation. Values of σ_1 and n_0 are also shown in Table 1.

Nutation amplitudes normalized by the rigid-body value and the tidal gravity factor (1+h-3k)/2 are calculated in the dissipationless limit ($\alpha_2 = 0$) by means of equations (37) and (38) of Sasao et al. (1977):

$$\tilde{m} = \left(1 - \frac{\kappa}{e}\frac{n}{\Omega}\right) \tilde{m}_R + \frac{A_f}{A} \frac{n}{\Omega-n} \tilde{m}_f \quad , \tag{77}$$

and

$$\tilde{m}_f = \frac{A}{A_m} \frac{\Omega-n}{n-\tilde{n}_0} \left(1 - \frac{\gamma}{e} + \frac{\gamma-\kappa}{e}\frac{n}{\Omega}\right) \tilde{m}_R \quad , \tag{78}$$

with

$$\tilde{m}_R = \frac{e\Omega}{\Omega-n} \tilde{\phi} \quad , \tag{79}$$

and by equations (52), where n denotes the frequency of the circular nutation $\propto \exp[i(-\Omega+n)t]$. The results are shown in Table 2 and Figures 1 and 2.

9. Secular Changes in the Earth-Moon System Due to the Core-Mantle Friction

We now consider briefly the problem of secular changes in the Earth-Moon system due to core-mantle friction, which was studied in Sasao et al. (1977) only in the rigid-mantle case. The rotational energy of the Earth is written

Table 2. Amplitude of Nutations and Tidal Gravity Factors

[Nutation Amplitude]/[That of Rigid Earth]

Period (mean solar day)	-13.7	-182.6	-365.3	-6798	6798	365.3	182.6	13.7
Wang	1.073	1.085	1.236	0.9963	1.0032	1.028	1.035	1.028
Bullen & Haddon	1.070	1.079	1.213	0.9965	1.0031	1.026	1.033	1.026
Gutenberg & Bullen-A	1.070	1.082	1.253	0.9966	1.0029	1.026	1.032	1.026
Jeffreys & Vicente a	1.0768	1.0895		0.9964	1.0036		1.0350	1.0269
Jeffreys & Vicente b	1.067	1.142		0.99889	1.00120		0.9707	1.0266
Molodensky I	1.080	1.0884	1.2472	0.99617	1.00333	1.0286	1.0359	1.026
Molodensky II	1.071	1.0786	1.2448	0.99666	1.00291	1.0273	1.0316	1.022

Tidal Gravity Factor (1+h-3k)/2

Tide Component	OO_1	ϕ_1	ψ_1	K_1		S_1	P_1	O_1	Static Value
Wang	1.159	1.172	1.235	1.135	1.138	1.149	1.152	1.157	1.158
Bullen & Haddon	1.163	1.176	1.238	1.139	1.142	1.153	1.156	1.162	1.1625
Gutenberg & Bullen-A	1.159	1.172	1.245	1.136	1.139	1.149	1.152	1.157	1.158
Jeffreys & Vicente a	1.224	1.196		1.182	1.184		1.209	1.221	1.152
Jeffreys & Vicente b	1.210	1.263		1.185			1.172	1.211	1.188
Molodensky I	1.161	1.174		1.136	1.139		1.154	1.159	1.160
Molodensky II	1.165	1.178		1.142	1.145		1.158	1.164	1.1646

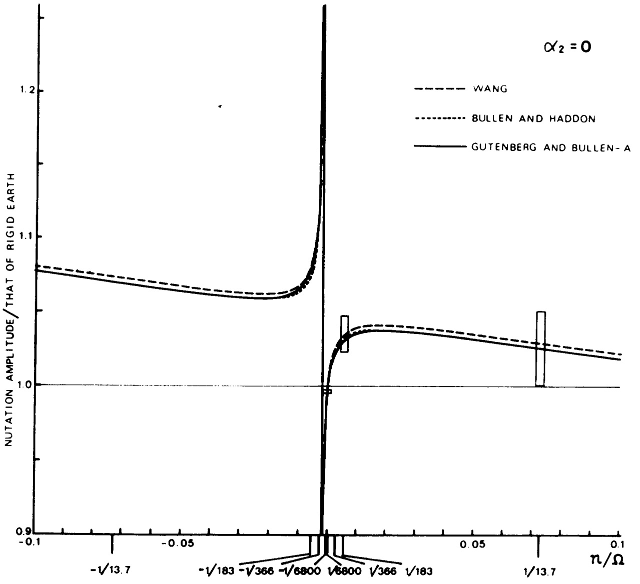

Fig. 1. Ratio of the nutation amplitude to that of the rigid Earth in the dissipationless limit (α_2=0). Curves show the frequency-dependence of the ratio for the Earth models of Wang, Bullen and Haddon, and Gutenberg and Bullen-A. Results of current astronomical observations are contained within the boxed areas (Sasao et al., 1977). It is evident that the model-dependence of the curves is rather insignificant compared with the accuracy of the observations.

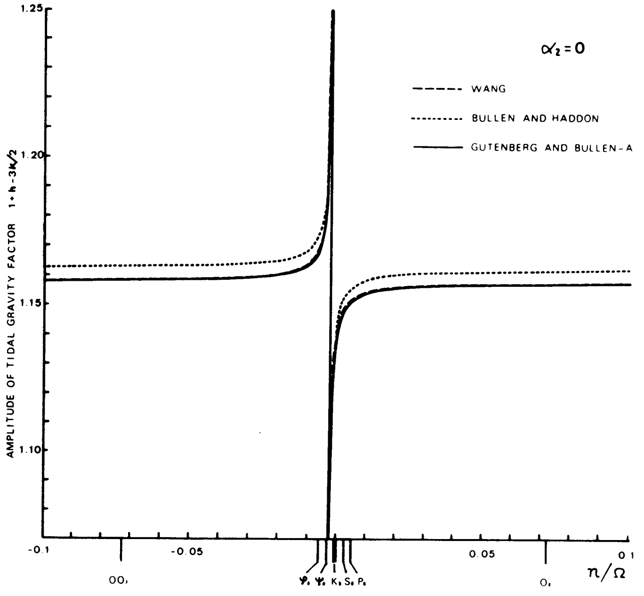

Fig. 2. Frequency-dependence of the tidal gravity factor (1+h-3k)/2.

$$E_r = \frac{1}{2}\int_s \rho|\vec{\omega}\times\vec{r}|^2 dV + \frac{1}{2}\int_f \rho|(\vec{\omega}+\vec{\omega}_f)\times\vec{r}+\vec{v}|^2 dV$$

$$= \frac{1}{2}\left[C^s_{ij}\omega_i\omega_j + C^f_{ij}(\omega_i + \omega^f_i)(\omega_i + \omega^f_j)\right] \quad , \qquad (80)$$

where C^s_{ij} and C^f_{ij} are moment-of-inertia tensors of the solid part and fluid outer core, respectively, of the Earth, and a term proportional to v^2 and rotation of the solid inner core relative to the mantle are neglected. Then, taking into account equations (34), (36), (68) and (69), we have

$$\frac{dE_r}{dt} = \omega_i\frac{dH_i}{dt} + \omega^f_i\frac{dH^f_i}{dt} - \frac{1}{2}\frac{dC_{ij}}{dt}\omega_i\omega_j - \frac{dC^f_{ij}}{dt}\omega_i\omega^f_j$$

$$= \Omega^2\left[-K|\tilde{m}_f|^2 - K^*m^{f\,2}_3 + Ae\Omega\,\mathrm{Im}(\tilde{m}^*\tilde{\phi}) + L_3/\Omega - \mathrm{Re}(\tilde{m}^*D\tilde{c}_3) - \mathrm{Re}\left(\tilde{m}^*_f D\tilde{c}^f_3\right)\right] \quad , \qquad (81)$$

in the second order of $|\tilde{m}|$, $|\tilde{m}_f|$, m^f_3 and $|\tilde{\phi}|$, where $C_{ij} = C^s_{ij} + C^f_{ij}$ is a moment-of-inertia tensor of the whole Earth and L_3 is the third component of the luni-solar torque caused by the deformation of the Earth:

$$L_3 = -\Omega^2\,\mathrm{Im}\left(\tilde{c}^*_3\,\tilde{\phi}\right) \quad . \qquad (82)$$

Im denotes imaginary part. Equating the secular loss of the mechanical energy of the Earth-Moon system to the energy dissipated at the core-mantle boundary (Loper, 1975), we have

$$\overline{\frac{d(E_{\oplus+☾} + E_r)}{dt}} = -\Omega^2\sum_{☾}\left(K|\tilde{m}_f|^2 + K^*m^{f\,2}_3\right) \quad , \qquad (83)$$

ere $E_{\oplus+☾}$ is the orbital kinetic plus potential energy of the Earth-on system but the rotational energy of the Moon is neglected. A mbol $\Sigma_{☾}$ implies summation over all the lunar nutation terms. We neg-ected in equation (83) losses of the elastic and gravitational ener-ies associated with the deformation, because they are proportional to he square of the amplitude of the small elastic displacement, which aries very slowly. Considering circular nutations $\propto \exp[i(-\Omega+n)t]$, we obtain from equations (81)-(83)

$$\overline{\frac{dE_{\oplus+☾}}{dt}} = \sum_{☾}\Omega^2\mathrm{Im}\left\{-Ae\Omega\tilde{m}^*\tilde{\phi} + (\Omega-n)\left[\tilde{c}^*_3(\tilde{\phi}-\tilde{m}) - \tilde{c}^{f*}_3\tilde{m}_f\right] + n\tilde{c}^*_3\tilde{\phi}\right\} \quad . \qquad (84)$$

Substituting equations (54) and (56), together with the reciprocity relation (61), into equation (84), and using equations (77)-(79), we have

$$\overline{\frac{dE_{\oplus+☾}}{dt}} = -A_f\Omega^3\sum_{☾}\frac{ne}{\Omega-n}\left(1 - \frac{\Upsilon}{e} + \frac{\Upsilon-\kappa}{e}\frac{n}{\Omega}\right)\mathrm{Im}(\tilde{m}^*_f\tilde{\phi}) = \sum_{☾}\frac{n}{\Omega-n}K\Omega^2|\tilde{m}_f|^2 \quad , \qquad (85)$$

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i.e., the same form as that in the rigid-mantle case (Sasao *et al.*, 1977). Equation (85) and Kepler's third law lead to equation (53) of Sasao *et al.* (1977) for the secular change of the orbital speed of the Moon.

On the other hand, the third components of equations (34) and (68), together with equations (54), (56) and (61), give

$$\frac{d}{dt}\left(Cm_3 + C_f m_3^f\right) = A_f \Omega \mathrm{Im}[\tilde{m}_f^* \tilde{m} - \gamma \tilde{m}_f^*(\tilde{\phi} - \tilde{m})] \quad , \tag{86}$$

and

$$\frac{d}{dt}\left(C_f m_3 + C_f m_3^f\right) = A_f \Omega \mathrm{Im}[\tilde{m}_f^* \tilde{m} - \gamma \tilde{m}_f^*(\tilde{\phi} - \tilde{m})] - K^* m_3^f \quad . \tag{87}$$

Elimination of m_3 from these equations yields

$$\frac{dm_3^f}{dt} + \alpha^* m_3^f = \frac{A_f}{C_f} \Omega \mathrm{Im}[\tilde{m}_f^*(\tilde{m} - \gamma\tilde{\phi})] \quad , \tag{88}$$

where $\alpha^* = CK^*/(C_f C_m)$ and we neglected γ compared with unity. Equations (77)-(79) and (88) give the rate of a secular westward drift of the core due only to the precession and nutations:

$$\overline{\omega_3^f} = \frac{C_m A_f}{CK^*} \Omega^2 \sum \frac{e\Omega}{\Omega - n}\left(1 - \frac{\gamma}{e} + \frac{\gamma - \kappa}{e}\frac{n}{\Omega}\right) \mathrm{Im}(\tilde{m}_f^* \tilde{\phi}) = -\frac{C_m}{C} \sum \frac{\Omega^2}{\Omega - n} \frac{K}{K^*} |\tilde{m}_f|^2 \ , \tag{89}$$

where summation is taken over all the nutation terms. Substituting equation (89) into equations (86) and (87), we obtain the secular change of length of day

$$\overline{\frac{d(\ell.o.d.)}{dt}} = -(\ell.o.d.)\overline{\frac{dm_3}{dt}} = \frac{(\ell.o.d.)}{C} \sum \frac{\Omega}{\Omega - n} K |\tilde{m}_f|^2 \quad , \tag{90}$$

which is identical with equation (58) of Sasao *et al.* (1977) for the rigid-mantle case.

It is thus concluded, following Sasao *et al.* (1977), that the secular changes due to the core-mantle friction are less efficient than those due to the ocean-tide braking. Equations (85) and (90) are quite simp and appear to be reasonable. This situation seems to suggest self-consistency of our treatment in spite of the simplifying assumptions.

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