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A GENERALISED QUASI-VARIATIONAL INEQUALITY WITHOUT UPPER SEMICONTINUITY

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In this note we deal with the following problem: given a nonempty closed convex subset X of $\mathbf{R}^n$ and two multifunctions $\Gamma : X \to 2^X$ and $\Phi : X \to 2^{\mathbf{R}^n}$, to find $(\widehat{x}, \widehat{z}) \in X \times \mathbf{R}^n$ such that

$$\widehat{x} \in \Gamma(\widehat{x}), \quad \widehat{z} \in \Phi(\widehat{x}) \quad \text{and} \quad \langle \widehat{z}, \widehat{x} - y \rangle \leqslant 0 \quad \text{for all} \quad y \in \Gamma(\widehat{x}).$$

We prove a very general existence result where neither Γ nor Φ are assumed to be upper semicontinuous. In particular, our result give a positive answer to an open problem raised by the first author recently.

1. INTRODUCTION

Given a nonempty closed convex subset X of $\mathbf{R}^n$ and two multifunctions $\Gamma : X \to 2^X$ and $\Phi : X \to 2^{\mathbf{R}^n}$, the *generalised quasi-variational inequality* problem associated with X, Γ and Φ (in short, $\mathrm{GQVI}(X, \Gamma, \Phi)$) is to find $(\widehat{x}, \widehat{z}) \in X \times \mathbf{R}^n$ such that

$$\widehat{x} \in \Gamma(\widehat{x}), \quad \widehat{z} \in \Phi(\widehat{x}) \quad \text{and} \quad \langle \widehat{z}, \widehat{x} - y \rangle \leqslant 0 \quad \text{for all} \quad y \in \Gamma(\widehat{x}),$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product of $\mathbf{R}^n$ which induces the Euclidean norm $\|\cdot\|$. This problem, which was introduced by Chan and Pang in [**3**], extends simultaneously the *generalised variational inequality* problem ($\Gamma(x) \equiv X$) and the *quasi-variational inequality* problem (where Φ is single-valued), both being generalisations of the classical variational inequality problem. The reader is referred to [**10**] for an excellent and detailed treatment of the basic facts in both theory and applications of finite dimensional variational inequalities and for very detailed references. For other applications of the problem $\mathrm{GQVI}(X, \Gamma, \Phi)$, we refer for instance to [**6, 7, 8**].

In the paper [**3**], Chan and Pang also established their classical existence result [**3**, Corollary 3.1], which remains undoubtedly one of the basic facts in the theory of variational inequalities. Chan and Pang's result is as follows (see also [**10**, Theorem 6.1]).

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Theorem 1.1. *Let $X \subseteq \mathbf{R}^n$ be nonempty, compact and convex, $\Gamma : X \to 2^X$ both lower and upper semicontinuous with nonempty closed convex values, $\Phi : X \to 2^{\mathbf{R}^n}$ upper semicontinuous with nonempty compact convex values. Then* $\mathrm{GQVI}(X, \Gamma, \Phi)$ *admits a solution.*

Recently, a general existence result for the problem $\mathrm{GQVI}(X, \Gamma, \Phi)$ was established by the first author in [4], which improves Theorem 1.1 in some directions. In particular, the multifunction Φ was allowed to lie in a larger class than the one of upper semicontinuous multifunctions. We now state such result.

Theorem 1.2. [**4**, Theorem 1]. *Let $X \subseteq \mathbf{R}^n$ be closed and convex, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^{\mathbf{R}^n}$ two multifunctions, $K \subseteq X$ a nonempty compact set. Assume that:*

(i) *the set $\Phi(x)$ is nonempty and compact for each $x \in X$, and convex for each $x \in X$, with $x \in \Gamma(x)$;*

(ii) *for each $y \in X - X$, the set $\{x \in X : \inf_{z \in \Phi(x)} \langle z, y \rangle \leqslant 0\}$ is closed;*

(iii) *Γ is a lower semicontinuous multifunction with closed graph and nonempty convex values;*

(iv) *$\Gamma(x) \cap K \neq \emptyset$ for all $x \in X$;*

(v) *for each $x \in X \setminus K$, with $x \in \Gamma(x)$, one has*

$$\sup_{y \in \Gamma(x) \cap K} \inf_{z \in \Phi(x)} \langle z, x - y \rangle > 0.$$

Then $\mathrm{GQVI}(X, \Gamma, \Phi)$ *has at least one solution.*

We recall that when the set X is compact, assumption (iii) above means exactly that Γ is both lower and upper semicontinuous with nonempty closed convex values. Moreover, it is easy to see that if the compact-valued multifunction Φ is upper semicontinuous, then it satisfies assumption (ii) of Theorem 1.2, while the converse is not necessarily true. Therefore, Theorem 1.2 strictly contains Theorem 1.1. We refer to [**15**] for the basic properties and a nice characterisation of the multifunctions satisfying assumptions (i) and (ii) of Theorem 1.2.

Very recently, in [**5**], the following problem was raised:

Problem A. Does Theorem 1.2 (hence, Theorem 1.1) remain true if we replace assumption (iii) by the following more general assumption

(iii)′ Γ is lower semicontinuous with nonempty convex values and the set $\{x \in X : x \in \Gamma(x)\}$ is closed ?

In the paper [**5**], it was shown that Problem A admits a positive answer (for compact X) if one assumes, in addition, that each set $\Gamma(x)$ has nonempty interior (see [**5**, Theorem 2.1]).

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Our goal in this paper is to give a complete positive answer to Problem A. That is, we show that the mentioned improvement of Theorem 1.2 is possible, without assuming any further restriction. In fact, we are able to prove the following result, which unifies all the above results and improves each of them.

THEOREM 1.3. *Let $X \subseteq \mathbf{R}^n$ be closed and convex, $K \subseteq X$ be a nonempty compact set, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^{\mathbf{R}^n}$ be two multifunctions. Assume that:*

- (i) *the set $\Phi(x)$ is nonempty and compact for each $x \in X$, and convex for each $x \in X$ with $x \in \Gamma(x)$;*
- (ii) *for each $w \in X - X$, the set $\{x \in X : \inf_{z \in \Phi(x)} \langle z, w \rangle \leqslant 0\}$ is closed;*
- (iii) *the multifunction Γ is lower semicontinuous with convex values and the set $\{x \in X : x \in \Gamma(x)\}$ is closed;*
- (iv) *$\Gamma(x) \cap K \neq \emptyset$ for all $x \in X$;*
- (v) *for each $x \in X \setminus K$ with $x \in \Gamma(x)$, and each $z \in \Phi(x)$, there exists $y \in \Gamma(x) \cap K$ such that $\langle z, x - y \rangle \geqslant 0$.*

Then $\mathrm{GQVI}(X, \Gamma, \Phi)$ *admits a solution.*

2. Preliminaries

Let S and V be two metric spaces, $\Psi : S \to 2^V$ a multifunction. We say that Ψ is *upper semicontinuous* in S if for each closed $\Omega \subseteq V$ the set $\Psi^-(\Omega) = \{s \in S : \Psi(s) \cap \Omega \neq \emptyset\}$ is closed in S. We say that Ψ is *lower semicontinuous* in S if for each open $\Omega \subseteq V$ the set $\Psi^-(\Omega)$ is open in S. We recall, in particular, that if for each $v \in V$ the set $\Psi^-(\{v\})$ is open in S, then Ψ is lower semicontinuous. In the sequel we shall write $\Psi^-(v)$ instead of $\Psi^-(\{v\})$. The graph of Ψ is the set $\{(s, v) \in S \times V : v \in \Psi(s)\}$. We briefly recall some basic implications. (For more details on multifunctions, the reader is referred to [**11**].)

- (a) if Ψ has closed graph, then each set $\Psi(s)$ is closed;
- (b) if Ψ has closed graph and V is compact, then Ψ is upper semicontinuous;
- (c) if Ψ is upper semicontinuous with nonempty closed values, then the graph of Ψ is closed;
- (d) if $V = S$ and the graph of Ψ is closed, then the set $\{s \in S : s \in \Psi(s)\}$ is also closed.

If $A \subseteq \mathbf{R}^n$, we denote by $\overline{A}$ the closure of A. Also, we denote by $\mathrm{ri}(A)$ the relative interior of A (that is, the interior of A in its affine hull), while $\mathrm{span}(A)$ will denote the linear subspace of $\mathbf{R}^n$ spanned by A. We recall that each nonempty convex subset of $\mathbf{R}^n$ has nonempty relative interior.

If $x \in \mathbf{R}^n$ and $r > 0$, we put $B(x, r) = \{v \in \mathbf{R}^n : \|x - v\| < r\}$, $B_r = B(0_{\mathbf{R}^n}, r)$ and $B_r^c = \{v \in \mathbf{R}^n : \|v\| \leqslant r\}$. The following fact follows at once from [**16**, Theorem

3.8.6]. For the sake of completeness, we give a short proof.

Proposition 2.1. *Let D be a metric space, $\Gamma : D \to 2^{\mathbf{R}^n}$ be a multifunction, and $r > 0$. Assume that:*

(i) *Γ is lower semicontinuous with convex values;*

(ii) *$\Gamma(x) \cap B_r \neq \emptyset$ for all $x \in D$.*

Then the multifunction $x \to \Gamma(x) \cap B_r^c$ is lower semicontinuous in D.

Proof: Since B_r is open, by (i) and (ii) the multifunction $x \to \Gamma(x) \cap B_r$ is lower semicontinuous in D with nonempty values, hence by [**11**, Proposition 7.3.3] the multifunction $x \to \overline{\Gamma(x) \cap B_r}$ is lower semicontinuous in D. Now, let us show that for each $x \in D$ one has

$$\overline{\Gamma(x) \cap B_r} = \overline{\Gamma(x) \cap B_r^c}. \tag{1}$$

Fix $x \in D$, and let $y \in \overline{\Gamma(x) \cap B_r^c}$. Let (y_k) be a sequence in $\Gamma(x) \cap B_r^c$ such that $(y_k) \to y$. Choose $v \in \Gamma(x) \cap B_r$ (such v exists by (ii)) and, for each $k \in \mathbf{N}$, put $u_k = (1 - 1/k)y_k + (1/k)v$. It is easy to see that the whole sequence (u_k) lies in $\Gamma(x) \cap B_r$. Since $(u_k) \to y$, we get $y \in \overline{\Gamma(x) \cap B_r}$. Therefore, we have shown that $\overline{\Gamma(x) \cap B_r^c} \subseteq \overline{\Gamma(x) \cap B_r}$ for all $x \in D$. Since the converse inclusion trivially holds, (1) holds. Therefore, the multifunction $x \to \overline{\Gamma(x) \cap B_r^c}$ is lower semicontinuous in D. Again by [**11**, Proposition 7.3.3] we have that the multifunction $x \to \Gamma(x) \cap B_r^c$ is lower semicontinuous in D; that is our claim. ∎

3. Results

Theorem 1.3 will follow from the following more general result.

Theorem 3.1. *Let $X \subseteq \mathbf{R}^n$ be closed and convex, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^{\mathbf{R}^n}$ be two multifunctions. Assume that:*

(i) *the set $\Phi(x)$ is nonempty and compact for each $x \in X$, and convex for each $x \in X$, with $x \in \Gamma(x)$;*

(ii) *for each $w \in X - X$, the set $\{x \in X : \inf_{z \in \Phi(x)} \langle z, w \rangle \leqslant 0\}$ is closed;*

(iii) *the multifunction Γ is lower semicontinuous with convex values and the set $\{x \in X : x \in \Gamma(x)\}$ is closed.*

Moreover, assume that there exists $r > 0$ such that the following conditions hold:

(iv) *$X \cap B_r \neq \emptyset$ and $\Gamma(x) \cap B_r \neq \emptyset$ for all $x \in X \cap B_r^c$;*

(v) *for each $x \in X$, with $x \in \Gamma(x)$ and $\|x\| = r$, and each $z \in \Phi(x)$, there exists $y \in \Gamma(x)$, with $\|y\| < r$, such that $\langle z, x - y \rangle \geqslant 0$.*

Then there exists $(\widehat{x}, \widehat{z}) \in X \times \mathbf{R}^n$, with $\|\widehat{x}\| \leqslant r$, which solves GQVI$(X, \Gamma, \Phi)$.

Before proving Theorem 3.1, we consider the case where the set X is compact. That is, we first prove the following result.

THEOREM 3.2. *Let X be a nonempty compact and convex subset of $\mathbf{R}^n$, $\Gamma : X \to 2^X$ and $\Phi : X \to 2^{\mathbf{R}^n}$ be two multifunctions. Assume that:*

(i) *the set $\Phi(x)$ is nonempty and compact for each $x \in X$, and convex for each $x \in X$ with $x \in \Gamma(x)$;*

(ii) *for each $w \in X - X$, the set $\{x \in X : \inf_{z \in \Phi(x)} \langle z, w \rangle \leqslant 0\}$ is closed;*

(iii) *the multifunction Γ is lower semicontinuous with nonempty convex values;*

(iv) *the set $\{x \in X : x \in \Gamma(x)\}$ is closed.*

Then $\mathrm{GQVI}(X, \Gamma, \Phi)$ *admits a solution.*

PROOF: Define $F : X \to 2^X$ by

$$F(x) = \{y \in \Gamma(x) : \inf_{z \in \Phi(x)} \langle z, x - y \rangle > 0\}$$

for each $x \in H$. We shall prove that F is lower semicontinuous. We first define a set-valued mapping $F_1 : X \to 2^X$ by $F_1(x) = \{y \in X : \inf_{z \in \Phi(x)} \langle z, x - y \rangle > 0\}$ for each $x \in X$. Then F_1 has convex values and its graph is open by the Lemma 2.2 of [**14**, p.376]. Note that Γ is lower semicontinuious and $F(x) = F_1(x) \cap \Gamma(x)$ for each $x \in X$. It is easy to see that F is lower semicontinous by the definition of the lower semicontinuity (for example, see [**16**, p.141]).

Now, observe that by [**12**, Theorem 3.1‴] there exists a continuous $g : X \to X$ such that $g(x) \in \Gamma(x)$ for all $x \in X$. By Brouwer's classical fixed point theorem, (see, for instance, [**1**, Theorem 7.1], there exists $\bar{x} \in X$ such that $\bar{x} = g(\bar{x}) \in \Gamma(\bar{x})$. Therefore, the set $H := \{x \in X : x \in \Gamma(x)\}$ is nonempty. Consider the multifunction $G : X \to 2^X$ defined by putting

$$G(x) = \begin{cases} F(x) & \text{if } x \in H \\ \Gamma(x) & \text{if } x \in X \setminus H. \end{cases}$$

We claim that there exists $x \in H$ such that $F(x) = \emptyset$. Arguing by contradiction, assume that $F(x) \neq \emptyset$ for all $x \in H$. Thus, the multifunction G has nonempty convex values. Moreover, since H is closed by assumption, F and Γ are lower semicontinuous and $F(x) \subseteq \Gamma(x)$ for all $x \in X$, it is easily seen (see, for instance, [**13**, Lemma 2.3]) that the multifunction G is lower semicontinuous. Again by [**12**, Theorem 3.1‴] and Brouwer's fixed point theorem, there exists $x^* \in X$ such that $x^* \in G(x^*)$. Since $G(x) \subseteq \Gamma(x)$ for all $x \in X$, we get $x^* \in H \cap F(x^*)$. In particular, we have $\inf_{z \in \Phi(x^*)} \langle z, x^* - x^* \rangle > 0$, which is absurd. Consequently, there exists $\widehat{x} \in H$ such that $F(\widehat{x}) = \emptyset$. That is, $\widehat{x} \in \Gamma(\widehat{x})$ and $\inf_{z \in \Phi(\widehat{x})} \langle z, \widehat{x} - y \rangle \leqslant 0$ for all $y \in \Gamma(\widehat{x})$. By [**2**, Proposition 1] we get

$$\inf_{z \in \Phi(\widehat{x})} \sup_{y \in \Gamma(\widehat{x})} \langle z, \widehat{x} - y \rangle \leqslant 0.$$

Since the function $z \to \sup\limits_{y \in \Gamma(\widehat{x})} \langle z, \widehat{x} - y\rangle$ is lower semicontinuous (see [**9**, p.10]) and the set $\Phi(\widehat{x})$ is compact, there exists $\widehat{z} \in \Phi(\widehat{x})$ such that

$$\sup_{y \in \Gamma(\widehat{x})} \langle \widehat{z}, \widehat{x} - y\rangle = \inf_{z \in \Phi(\widehat{x})} \sup_{y \in \Gamma(\widehat{x})} \langle z, \widehat{x} - y\rangle \leqslant 0.$$

The proof is complete. ∎

PROOF OF THEOREM 3.1: Put $X_r = X \cap B_r^c$. By Proposition 2.1, the multifunction $\Gamma_r : X_r \to 2^{X_r}$ defined by putting $\Gamma_r(x) = \Gamma(x) \cap B_r^c$ is lower semicontinuous in X_r with nonempty convex values. Moreover, the set $\{x \in X_r : x \in \Gamma_r(x)\} = \{x \in X : x \in \Gamma(x)\} \cap B_r^c$ is closed by (iii). Finally, we observe that for each $w \in X_r - X_r$ the set $\{x \in X_r : \inf\limits_{z \in \Phi(x)} \langle z, w\rangle \leqslant 0\}$ is closed. Therefore, by Theorem 3.2, there exists $\widehat{x} \in \Gamma(\widehat{x})$, with $\|\widehat{x}\| \leqslant r$, and $\widehat{z} \in \Phi(\widehat{x})$ such that

$$\langle \widehat{z}, \widehat{x} - y\rangle \leqslant 0 \quad \text{for all} \quad y \in \Gamma(\widehat{x}) \cap B_r^c. \tag{3}$$

We claim that $(\widehat{x}, \widehat{z})$ solves $\mathrm{GQVI}(X, \Gamma, \Phi)$. To see this, let $v \in \Gamma(\widehat{x})$. We distinguish two cases.

(a) $\|\widehat{x}\| < r$. In this case, there exists $\lambda \in (0,1)$ such that $v_\lambda := (1-\lambda)\widehat{x} + \lambda v \in \Gamma(\widehat{x}) \cap B_r$. By (3), we get

$$0 \geqslant \langle \widehat{z}, \widehat{x} - v_\lambda\rangle = \lambda\langle \widehat{z}, \widehat{x} - v\rangle,$$

hence $\langle \widehat{z}, \widehat{x} - v\rangle \leqslant 0$, as desired.

(b) $\|\widehat{x}\| = r$. By assumption (v) there exists $u \in \Gamma(\widehat{x}) \cap B_r$ such that $\langle \widehat{z}, \widehat{x} - u\rangle \geqslant 0$. By (3), we get

$$\langle \widehat{z}, \widehat{x} - u\rangle = 0. \tag{4}$$

Let $b \in (0,1)$ be such that $x_b := (1-b)u + bv \in \Gamma(\widehat{x}) \cap B_r$. By (3) and (4) we get

$$0 \geqslant \langle \widehat{z}, \widehat{x} - x_b\rangle = b\langle \widehat{z}, \widehat{x} - v\rangle,$$

hence $\langle \widehat{z}, \widehat{x} - v\rangle \leqslant 0$, as desired. The proof is complete. ∎

REMARK 3.2. (i) Theorem 1.3 follows at once from Theorem 3.1 by choosing any $r > 0$ in such a way that $K \subseteq B_r$. Moreover, it is worth noticing that also the coercivity condition (v) of Theorem 1.3 is weaker than the corresponding assumption (v) in Theorem 1.2.

(ii) It is easy to construct an example of a situation where Theorem 3.2 (hence, Theorems 1.3 and 3.1) applies but Theorems 1.1 and 1.2 do not. To see this, take $X = [0,1]$, $\Phi(x) = \{x\}$ and

$$\Gamma(x) = \begin{cases} [0,1] & \text{if } x \neq \dfrac{1}{2} \\ \left\{\dfrac{1}{2}\right\} & \text{if } x = \dfrac{1}{2}. \end{cases}$$

In fact, such Γ is lower semicontinuous with nonempty closed convex values, but the graph of Γ is not closed, Γ is not upper semicontinuous. However, we have $\{x \in X : x \in \Gamma(x)\} = [0,1]$, hence Theorem 3.2 applies. We also note that the interior of $\Gamma(1/2)$ is empty, hence [**5**, Theorem 2.1] cannot be applied (see the Introduction).

References

[1] J.P. Aubin, *Optima and equilibria* (Springer-Verlag, Berlin, Heidelberg, New York, 1993).

[2] H. Brézis, L. Nirenberg and G. Stampacchia, 'A remark on Ky Fan's minimax principle', *Boll. Un. Mat. Ital.* **6** (1972), 293–300.

[3] D. Chan and J.S. Pang, 'The generalized quasi-variational inequality problem', *Math. Oper. Res.* **7** (1982), 211–222.

[4] P. Cubiotti, 'An existence theorem for generalized quasi-variational inequalities', *Set-Valued Anal.* **1** (1993), 81–87.

[5] P. Cubiotti, 'A note on Chan and Pang's existence theorem for generalized quasi-variational inequalities', *Appl. Math. Lett.* **9** (1996), 73–76.

[6] P. Cubiotti, 'An application of quasi-variational inequalities to linear control systems', *J. Optim. Theory Appl.* **89** (1996), 101–113.

[7] M. De Luca and A. Maugeri, 'Quasi-variational inequalities and applications to equilibrium problems with elastic demand', in *Nonsmooth Optimization and Related Topics*, (F.H. Clarke, V.F. Dem'yanov and F. Giannessi, Editors), Ettore Majorana International Science Series (Plenum Press, 1989).

[8] M. De Luca and A. Maugeri, 'Discontinuous quasi-variational inequalities and applications to equilibrium problemsin', in *Nonsmooth Optimization. Methods and Applications*, (F. Giannessi, Editor) (Gordon and Breach Sc. Publ., London, Tokyo, 1992).

[9] I. Ekeland and R. Temam, *Analyse convexe et problemes variationnels* (Dunod, Paris, 1974).

[10] P.T. Harker and J.S. Pang, 'Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications', *Math. Programming* **48** (1990), 161–220.

[11] E. Klein and A.C. Thompson, *Theory of correspondences* (John Wiley and Sons, New York, 1984).

[12] E. Michael, 'Continuous selections I', *Ann. of Math.* **63** (1956), 361–382.

[**13**] O. Naselli Ricceri, 'On the covering dimension of the fixed point set of certain multifunctions', *Comment. Math. Univ. Carolin.* **32** (1991), 281–286.

[**14**] E. Tarafdar and X. Z. Yuan, 'Non-compact generalized quasi-variational inequalities in locally convex topological vector spaces', *Nonlinear World* **1** (1994), 373–383.

[**15**] N.D. Yen, 'On an existence theorem for generalized quasi-variational inequalities', *Set-Valued Anal.* **3** (1995), 1–10.

[**16**] X.Z. Yuan, *Contributions to nonlinear analysis*, Ph.D. dissertation (Department of Mathematics, Statistics and Computing Science, Dalhousie University, 1994).

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