MONOTONE SEMIGROUPS OF OPERATORS ON CONES*

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In this paper we consider a special class of linear operators defined on a cone K in a Banach space X. This class of operators is the natural generalization of a class of operators which has applications in the theory of interpolation spaces. In particular, using the criteria developed in Theorem 1, it is possible to characterize those sequence spaces X such that every linear operator A of weak types (p, p) and (q, q) is a continuous mapping of X into itself. For details of this we refer the reader to [3].

We begin with a sequence of operators $\{E(m)\}$ each defined on K, and consider operators of the form $T = \Sigma\{t(m)E(m) : m = 1, \dots, \infty\}$, where $t(m) \geq 0$. Under the assumption that $\{E(m)\}$ forms a "monotone semigroup" we are able to establish conditions under which T will map K continuously into itself.

The method used allows us to give precise information about the spectral radius of T in terms of a number β associated with $\{E(m)\}$.

1. Preliminary remarks. We assume that X is a real Banach space and that $K \subset X$ is a closed normal cone in X so that $K + K \subset K$, $aK \subset K$ for $a \geq 0$, K is a closed subset of X, and there is an $\varepsilon > 0$ such that $x, y \in K$, $\|x\| \geq 1$, $\|y\| \geq 1$ imply $\|x + y\| \geq \varepsilon$.

The dual cone K' is the set of linear functionals $x' \in X'$ such that $\langle x, x' \rangle \geq 0$ for all $x \in K$. Since K is normal, $X' = K' - K'$ and if we define $p(x)$ for $x \in X$ by

(1) $$p(x) = \sup\{|\langle x, x'\rangle| : x' \in K', \|x'\| \leq 1\},$$

then p is a norm on X and there is $\gamma > 0$ such that

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(2) $$\gamma \|x\| \leq p(x) \leq \|x\| \quad \text{for all } x \in X$$

(see [5, pages 226-227]).

If T is a linear operator mapping K into itself, the partial norm and partial spectral radius of T are the numbers

(3) $$\|T\| = \|T\|_K = \sup\{\|Tx\| : x \in K, \ \|x\| \leq 1\},$$

(4) $$r(T) = r_K(T) = \lim_{n\to\infty} \|T^n\|_K^{1/n}.$$

These terms are due to Bonsall [1]. If $\|T\|_K < \infty$ we write $T \in [K]$.

Definition. A monotone semigroup of operators on K is a sequence $\{E(m)\}$ of non-zero linear operators leaving K invariant and satisfying

(a) $E(1)x = x$ for all $x \in K$

(b) $E(mn)x = E(m)E(n)x$ for $x \in K$, $m, n \in \mathbf{Z}^+ = \{1, 2, ..\}$

(c) $\langle E(m+1)x, x'\rangle \leq \langle E(m)x, x'\rangle$ for $x \in K$, $x' \in K'$, $m \in \mathbf{Z}^+$.

LEMMA 1. *Let $\{E(m)\}$ be a monotone semigroup on K and $h(m) = \|E(m)\|_K$. If $\beta = \sup\{-\log h(m)/\log m : m \in \mathbf{Z}^+\}$, then*
$$\beta = \lim_{m\to\infty} \{-\log h(m)/\log m\} \leq 0.$$

Proof. From (b) of the definition, we have $h(mn) \leq h(m)h(n)$ while from (c) we obtain $\gamma h(m) \leq h(n)$ for $m > n$, where γ is as in (2).

Now define $g(k) = \log h(2^k)/\log 2$ for $k = 0, 1, 2, \ldots$ and notice that $g(k + \ell) \leq g(k) + g(\ell)$. Then, by a well-known result, if $\beta = -\inf g(k)/k$, then $\beta = \lim_{k\to\infty} (-g(k)/k)$ (see [4, page 244]). Given m, choose $k = [\log m/\log 2]$ so that $2^k \leq m < 2^{k+1}$. Then we have

(5) $$\gamma h(2^{k+1}) \leq h(m) \leq \gamma^{-1} h(2^k).$$

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Taking logarithms in (5) dividing by log m and letting $m \to \infty$, we obtain

$$(6) \qquad \lim_{m\to\infty} \log h(m)/\log m = -\beta .$$

To see that $\beta = \sup \{-\log h(m)/\log m\}$, note that just as for 2^k we have

$$\inf_k \log h(m^k)/\log m^k = \lim_{k\to\infty} \log h(m^k)/\log m^k = -\beta,$$

so that $\log h(m)/\log m \geq -\beta$.

Since $h(m) \geq m^{-\beta}$ we must have $\beta \geq 0$; for otherwise $h(m) \to \infty$ which contradicts $\gamma h(m) \leq h(1)$.

Now if $\mathfrak{T} = \{t(m)\}$ is a sequence of non-negative numbers, we define

$$(7) \qquad \zeta(s, \mathfrak{T}) = \sum_{m=1}^{\infty} t(m) m^{-s}, \text{ for real } s .$$

If the series diverges we write $\zeta(s, \mathfrak{T}) = \infty$, and since ζ is non-increasing we may define $\zeta(\pm\infty, \mathfrak{T})$ as the respective limits.

We define the abscissa of convergence of ζ by σ_o so

$$(8) \qquad \sigma_o = \sigma_o(\mathfrak{T}) = \inf \{s : \zeta(s, \mathfrak{T}) < \infty\} .$$

We may or may not have $\zeta(\sigma_o, \mathfrak{T}) < \infty$, but we do have $\zeta(\sigma_o, \mathfrak{T}) = \lim_{s \downarrow \sigma_o} \zeta(s, \mathfrak{T})$.

Note that ζ is continuous on (σ_o, ∞).

2. Statement of main results. Our main results give criteria for $T \in [K]$ where

$$(9) \qquad Tx = \sum_{m=1}^{\infty} t(m) E(m) x ,$$

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with domain the set of $x \in X$ for which the series converges in the weak $\langle X, X' \rangle$ topology.

Note that if $\beta < \infty$, the only situation in which we do not obtain an effective criterion for $T \in [K]$ is when $\zeta(\sigma_0, T) < \infty$, $\beta = \sigma_0$ and $\| E(m) \|_K \neq m^{-\beta}$ for an infinite set of m.

THEOREM 1. Let X be a real Banach space, K a closed normal cone, and $\{E(m)\}$ a monotone semigroup of operators on K. If $\mathfrak{J}$ is a sequence of non-negative numbers, define $T, \zeta, \sigma_0, \beta$ by (9), (8), (7), (6) respectively. Then

(a) if $\sum_{m=1}^{\infty} t(m) \| E(m) \|_K < \infty$, then $T \in [K]$ and

$$\| T \|_K \leq \sum_{m=1}^{\infty} t(m) \| E(m) \|_K ;$$

(b) if $\beta > \sigma_0$, then $T \in [K]$;

(c) if $T \in [K]$, then $\beta \geq \sigma_0$;

(d) if $T \in [K]$ and $\sigma_0 < \infty$, then $\zeta(\beta, \mathfrak{J}) < \infty$;

(e) if $\beta > \sigma_0$, then $\zeta(\beta, \mathfrak{J}) = r_k(T)$.

COROLLARY 1. If $\beta < \infty$ and $\zeta(\sigma_0, \mathfrak{J}) = \infty$, then the following are equivalent.

(a) $T \in [K]$

(b) $\zeta(\beta, \mathfrak{J}) < \infty$

(c) $\sum t(m) \| E(m) \|_K < \infty$.

COROLLARY 2. If $\beta < \infty$, and $\| E(m) \|_K = m^{-\beta}$ except for a finite set of m, then $T \in [K]$ if and only if $\zeta(\beta, \mathfrak{J}) < \infty$.

If $\| E(m) \|_K = m^{-\beta}$ for all m, then $\| T \|_K = r_K(T) = \zeta(\beta, \mathfrak{J})$.

The proof of Theorem 1 is somewhat involved so we first indicate a number of examples.

3. Examples.

1. Let X be a Banach space of sequences $\{x(n)\}$ on which there is a function norm ρ of the type defined by Luxemburg [6]. In particular $\|x\| = \rho(|x|)$ and $|x(n)| \leq |y(n)|$ for all n implies $\rho(|x|) \leq \rho(|y|)$. The operators $E(m)$ defined by

$$(E(m)x)(n) = x(mn), \quad n \in \mathbf{Z}^{+} \tag{10}$$

clearly form a semigroup.

For the cone K, take the set of all non-negative, non-increasing sequences in X. Then $x(mn) \leq x(n)$ for all n so $\{E(m)\}$ is a monotone semigroup. This semigroup appears naturally in [3].

For illustration, let $\rho(|x|) = \{\Sigma|x(n)|^{p}\}^{1/p}$ so $x = \ell^{p}$. Then

$$\|E(m)\|_{K} = \sup\{\|E(m)x\| : \|x\| \leq 1,\ x \in K\} = m^{-1/p}. \tag{11}$$

Here Corollary 1 applies, so if $\{t(m)\}$ is a non-negative sequence

$$\|T\|_{K} = \sum_{m=1}^{\infty} t(m)m^{-1/p} .$$

Suppose, on the other hand, that for our cone we take P, the set of non-negative sequences in $X = \ell^{p}$, and let $\{t(m)\}$ be decreasing. Then, by using rearrangements of sequences one can see that

$$\|T\|_{P} = \|T\|_{K} .$$

However, $\|E(m)\|_{P} = 1$ for all m, so we do not have

$\|T\|_{P} = \sum_{m=1}^{\infty} t(m)\|E(m)\|_{P}$. The reason that our corollary does not apply here is that $\{E(m)\}$ is not monotone on P.

2. After examining Theorem 1, one might conjecture that $\zeta(\beta, \mathfrak{T}) < \infty$ would imply $T \in [K]$, and perhaps that $\zeta(\beta, \mathfrak{T}) = r(T)$, even when $\beta = \sigma_o$. However, this is not true even when E is defined by (10) as our next example shows.

Let $k \geq 16 > e^e$ and define h by

$$(12) \qquad h(m) = \begin{cases} 1 & , \text{ for } m = 1, 2, \ldots, k-1 \\ m^{-\beta} \log m & , \text{ for } m \geq k . \end{cases}$$

If we choose β so that

$$(13) \qquad \beta \geq \log \log k / \log k$$

then h can be seen to be non-increasing, and satisfy $h(mn) \leq h(m)h(n)$ for $m, n \in \mathbf{Z}^+$.

Now define a function norm ρ on sequences by

$$(14) \qquad \rho(|x|) = \sup \{ |x(n)| / h(n) : n \in \mathbf{Z}^+ \} .$$

If we take X to be the set of sequences with $\rho(|x|) < \infty$, and K as in Example 1, we can easily show that $\| E(m) \|_K = h(m)$.

Now take for $\mathfrak{T}$ the sequence defined by

$$(15) \qquad t(1) = 1 \text{ and } t(m) = m^{\beta - 1} (\log m)^{-2} \text{ for } m \geq 2 .$$

Then

$$(16) \qquad \sum_{m=1}^{\infty} t(m)h(m) = \infty \quad \text{and} \quad \sum_{m=1}^{\infty} t(m)m^{-\beta} < \infty .$$

But h is itself in X so the first part of (16) shows that $\| T \| = \infty$ and yet we have $\zeta(\beta, \mathfrak{T}) < \infty$. It is also clear that $r(T) = \infty \neq \zeta(\beta, \mathfrak{T})$. Because of (13) we can obtain examples for any $\beta > 0$ by choosing k sufficiently large.

4. Proofs of the main results. We begin by introducing the sequences $\mathfrak{T}^k = \{t_k(m)\}$, $\mathfrak{R}_\lambda = \{r(m,\lambda)\}$ corresponding to a sequence $\mathfrak{T} = \{t(m)\}$; $\mathfrak{T}^k$ is defined formally by

$$\zeta(s,\mathfrak{T}^k) = \zeta(s,\mathfrak{T})^k, \ k = 0, 1, \ldots . \tag{17}$$

and $\mathfrak{R}_\lambda$ by

$$r(m,\lambda) = \sum_{k=0}^{\infty} \lambda^{-k-1} t_k(m) \ \text{(possibly } \infty) . \tag{18}$$

If $r(m,\lambda) < \infty$ for all m, we denote by R_λ the operator $\Sigma r(m,\lambda)E(m)$. The following lemma gives the pertinent information about R_λ.

LEMMA 2. (a) If $\sigma_o(\mathfrak{T}) < \infty$, then the series (18) converges for all $m \in \mathbf{Z}^+$, if $\lambda > t(1)$.

(b) If $T \in [K]$, then $r_k(T) \geq t(1)$.

(c) $R_\lambda \in [K]$ if and only if $\lambda > r_k(T)$ and in this case $R_\lambda = (\lambda - T)^{-1}$.

(d) For $\lambda > t(1)$, let $\sigma_1 = \sigma_o(\mathfrak{R}_\lambda)$. Then σ_1 is the unique solution of $\zeta(s,\mathfrak{T}) = \lambda$ if $\lambda < \zeta(\sigma_o,\mathfrak{T})$ or else $\sigma_1 = \sigma_o(\mathfrak{T})$ if $\lambda \geq \zeta(\sigma_o,\mathfrak{T})$. Furthermore,

$$\zeta(s,\mathfrak{R}_\lambda) = (\lambda - \zeta(s,\mathfrak{T}))^{-1} \ \text{for} \ s > \sigma_o(\mathfrak{R}_\lambda) . \tag{19}$$

Proof. (a) By formula (17), if $s > \sigma_o$ and $\lambda > \zeta(s,\mathfrak{T})$, then

$$(\lambda - \zeta(s,\mathfrak{T}))^{-1} = \sum_{k=0}^{\infty} \lambda^{-k-1} \zeta(s,\mathfrak{T})^k = \sum_{m=1}^{\infty} r(m,\lambda)m^{-s} = \zeta(s,\mathfrak{R}_\lambda) \tag{20}$$

which shows that $r(m,\lambda) < \infty$ for $\lambda > \zeta(s,\mathfrak{T})$ and hence for $\lambda > \lim_{s\to\infty} \zeta(s,\mathfrak{T}) = t(1)$.

(b) If $T \in [K]$, then for $x \in K$, $x' \in K'$ we have

$$\langle T^k x, x' \rangle \geq t_k(1) \langle E(1)x, x' \rangle = t(1)^k \langle x, x' \rangle .$$

Now applying (2) we obtain $\| T^k \| \geq t(1)^k \gamma$ from which $r(T) \geq t(1)$ follows.

(c) If $\lambda > r(T)$, the Neumann series for $(\lambda - T)^{-1} x$ converges and it is clearly equal to $R_\lambda x$. Conversely if both $R_\lambda \in [K]$, $T \in [K]$ then a direct computation gives $TR_\lambda x = R_\lambda Tx = \lambda R_\lambda x$ which shows that $R_\lambda = (\lambda - T)^{-1}$, and hence $\lambda > r(T)$. (See [1, Theorems 5 and 6].)

(d) If $\zeta(\sigma_o, \mathfrak{T}) > \lambda > t(1)$, and s satisfies $\lambda > \zeta(s, \mathfrak{T})$, then Formula (20) is valid, and shows that $\zeta(s, \mathfrak{R}_\lambda) \to \infty$ as s decreases to to the solution σ_1 of $\lambda = \zeta(s, \mathfrak{T})$, so $\sigma_o(R_\lambda) = \sigma_1$.

In case $\lambda \geq \zeta(\sigma_o, \mathfrak{T})$, the relation (20) shows that $\sigma_o(\mathfrak{R}_\lambda) \leq \sigma_o(\mathfrak{T})$. However, since $\lambda > 0$, $r(m, \lambda) \geq \lambda^{-1} t(m)$ so that $\zeta(s, \mathfrak{T}) \leq \lambda \zeta(s, \mathfrak{R}_\lambda)$ which shows $\sigma_o(\mathfrak{R}_\lambda) \geq \sigma_o(\mathfrak{T})$.

Proof of Theorem 1. (a) For $x \in K$, $x' \in X'$, and $\|x\| \leq 1$, $\|x'\| \leq 1$ we have $\langle E(m)x, x' \rangle \leq \| E(m) \|_K$. Thus, we obtain $\langle Tx, x' \rangle = \Sigma\, t(m) \langle E(m)x, x' \rangle \leq \Sigma\, t(m) \| E(m) \|_K$, which proves (a) on taking supremums first over x', then over x.

(b) By definition of β, given $\varepsilon > 0$, there is an $m_o(\varepsilon)$ so that $m^{-\beta} \leq \| E(m) \|_K \leq m^{-\beta+\varepsilon}$ for $m \geq m_o(\varepsilon)$. Choose $\varepsilon > o$ so that $\beta - \varepsilon > \sigma_o$ and $\zeta(\beta - \varepsilon, \mathfrak{T}) < \infty$, and then apply part (a).

(c) We first note that the monotone condition $\langle E(m+1)x, x' \rangle < \langle E(m)x, x' \rangle$ implies the following inequality if $T \in [K]$ and $\| E(m) \|_K = h(m)$.

$$h(2^{k+1})\ 2^{ks} \sum_{m=2^k}^{2^{k+1}-1} m^{-s} t(m) \leq c_s \| T \|_K \tag{21}$$

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where $c_s = \gamma^{-1}$ or $\gamma^{-1}2^{-s}$ according to whether $s \leq 0$ or $s \geq 0$. To see this for $s \geq 0$, let $x \in K$, $x' \in K'$ with $\|x\| \leq 1$, $\|x'\| \leq 1$. Then, since $(2^k/m)^s \leq 1$,

$$\|T\| \geq \langle Tx, x'\rangle \geq \sum_{m=2^k}^{2^{k+1}-1} t(m)\langle E(m)x, x'\rangle$$

$$\geq \langle E(2^{k+1})x, x'\rangle \; 2^{ks} \sum_{m=2^k}^{2^{k+1}-1} t(m)m^{-s} .$$

Now, if $s < \sigma_o$ and $s + \varepsilon < \sigma_o$, then $\zeta(s+\varepsilon, \mathfrak{J}) = \infty$. Using this, one can show that there is a sequence $\mathfrak{a}$ of values of k for which

$$\sum_{m=2^k}^{2^{k+1}-1} m^{-s} t(m) \to \infty .$$

But, then from (21) we have $2^{-k(\beta - s)} \leq 2^{\beta} h(2^{k+1}) 2^{ks} \to 0$ as $k \to \infty$, through $\mathfrak{a}$. This shows $\beta > s$ and since $s < \sigma_o$ is arbitrary that $\beta \geq \sigma_o$.

(d) By part (c), we have $\beta \geq \sigma_o$, so that if $\zeta(\sigma_o, \mathfrak{J}) < \infty$ there is nothing to prove. Hence assume $\zeta(\sigma_o, \mathfrak{J}) = \infty$. Let $\lambda > r(T)$, and σ_1 be the solution of $\zeta(s, \mathfrak{J}) = \lambda$. Then by Lemma 2(d), $\sigma_1 = \sigma_o(\mathfrak{R}_\lambda)$. Since $R_\lambda \in [K]$, (c) implies that $\beta \geq \sigma_o(\mathfrak{R}_\lambda) = \sigma_1$, and hence

(22) $$\zeta(\beta, \mathfrak{J}) \leq \zeta(\sigma_1, \mathfrak{J}) = \lambda < \infty .$$

(e) We assume $\beta > \sigma_o$ so $T \in [K]$ and we wish to show $r(T) = \zeta(\beta, \mathfrak{J})$. If $\zeta(\sigma_o, \mathfrak{J}) = \infty$, (22) shows that $\zeta(\beta, \mathfrak{J}) \leq r(T)$ since $\lambda > r(T)$ is arbitrary. In case $\zeta(\sigma_o, \mathfrak{J}) < \infty$ we can again derive (22) provided $\lambda \leq \zeta(\sigma_o, \mathfrak{J})$ and hence we have $\zeta(\beta, \mathfrak{J}) \leq r(T)$ always.

On the other hand, $\beta > \sigma_o$ implies $\zeta(\beta, \mathfrak{T}) < \zeta(\sigma_o, \mathfrak{T})$ (unless $T = t(1)I$ which can be handled directly). Let $\varepsilon > 0$ be chosen so $\zeta(\beta, \mathfrak{T}) + \varepsilon = \lambda < \zeta(\sigma_o, \mathfrak{T})$ and let $\sigma_1 = \sigma_o(\mathfrak{R}_\lambda)$. By Lemma 2(d) we see that $\sigma_1 < \beta$. But by part (b) of the theorem this implies $R_\lambda \in [K]$ and hence $\lambda > r(T)$, or since $\varepsilon > 0$ is arbitrary that $\zeta(\beta, \mathfrak{T}) \geq r(T)$, completing the proof of (e).

Proof of Corollary 1. Since $\zeta(\sigma_o, \mathfrak{T}) = \infty$, we have $\zeta(\beta, \mathfrak{T}) < \infty$, if and only if $\beta > \sigma_o$ so the equivalence of (a), (b), (c) follows from parts (a) and (d) of Theorem 1.

Proof of Corollary 2. If $\zeta(\beta, \mathfrak{T}) < \infty$ then $\Sigma\, t(m) \| E(m) \|_K < \infty$ and hence $T \in [K]$. Conversely if $T \in [K]$ and $\sigma_o < \infty$, then $\zeta(\beta, \mathfrak{T}) < \infty$ by Theorem 1 (d). If $\sigma_o = \infty$, $T \notin [K]$ since this would imply $\beta \geq \sigma_o$ (by Theorem 1 (c)), contradicting $\beta < \infty$.

If $\| E(m) \|_K = m^{-\beta}$ for all β, then Theorem 1 (a), and the fact that $r(T) \leq \| T \|$ gives

(23) $$r(T) \leq \| T \| \leq \zeta(\beta, \mathfrak{T}).$$

Thus for $\beta > \sigma_o$, Theorem 1 (d) gives $r(T) = \zeta(\beta, \mathfrak{T})$ which proves the required result. For $\beta = \sigma_o$, we can prove that $r(T) \geq \zeta(\beta, \mathfrak{T})$ by assuming the contrary and choosing λ with $r(T) < \lambda < \zeta(\beta, \mathfrak{T})$. The argument leading to (22) then goes through as before and completes the proof.

Remarks. 1. The proof of Corollary 2 shows that the relation $r(T) \geq \zeta(\beta, \mathfrak{T})$ is always valid with equality in case $\beta \neq \sigma_o$. In view of the second example of Section 3, this is all that can be claimed in general.

2. The assumption that $E(1)x = x$ for all $x \in K$ is unnecessary for the results of Theorem 1 as one sees by replacing the cone K by $K_1 = E(1)K$.

3. Extensions to semigroups of the form $E(s)E(t) = E(st)$, $s, t \in \mathbb{R}^+$ can be made. A particular case was discussed in [2], and improvements of the results given there can be made along the lines of the proofs given here.

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