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PARACOMPACTNESS FOR ORDERED SUMS

BY

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Let us say that an order embedding of an uncountable regular cardinal in a linearly ordered set is *continuous* if it preserves the suprema (for all smaller limit ordinals). This makes the embedding a homeomorphism for the two order topologies; and if the image has no supremum, it is a closed subspace. Since uncountable regular cardinals fail to be paracompact, a linearly ordered set can be paracompact only if it admits no such embedding or anti-embedding. Conversely, Gillman and Henriksen have shown that this suffices (Trans. A.M.S. **77** (1954) pp. 352 ff). This has led Ostaszewski to show that the lexicographic product of two sets admitting no such (anti-)embedding enjoys the same property, and at the end of Colloq. Math. **30** (1974) 121–125 to point out that more generally, the lexicographic sum of such sets over such a set does, provided the summands are alike with regard to having or not having first or last elements. The complete result is furnished in the following.

By the lexicographic sum of non-void linearly ordered sets Y_x over the linearly ordered index set X, we understand their disjoint union Z, linearly ordered by the rule that elements belonging to a single Y_x receive the order of that Y_x, whilst elements from different Y_x's are ordered as are the indices in X of the Y_x's to which they belong. It comes to the same to have Z an ordered set with an order preserving surjection on X such that the interval mapped on each x is order isomorphic to Y_x. An order preserving map is either terminally constant or an embedding of a cofinal subset (which may be chosen to contain all its suprema). Thus every subset of Z is either terminally in one of the Y_x's or has a cofinal subset embedded in X (whence by surjectivity, with preservation of the suprema in Z which it contains). The subset has a supremum if: it is bounded in its terminal Y_x, then just when it has a supremum in that Y_x; it is unbounded in its terminal Y_x, or contained in no terminal Y_x, then just when the indices x whose Y_x it meets have a next largest index, say x', for which $Y_{x'}$ has a first element. Recalling that a cofinal – a fortiori a terminal – subset of a regular cardinal is order isomorphic with it, we obtain from the following, in conjunction with its dual, a criterion for paracompactness of the sum:

In order that a regular cardinal always have a supremum whenever it is

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continuously embedded in a lexicographic sum, it is necessary and sufficient that this be so for its continuous embeddings into bounded subsets of each Y_x, that it admit no continuous embedding onto an unbounded subset of a Y_x unless that index x has a successor x' whose $Y_{x'}$ has a first element, and that this be so for those of its continuous embeddings into X for which suprema are taken on only at indices x' whose $Y_{x'}$ have first elements.

Apart from enabling us to give examples of paracompact sums in which the Y_x do not uniformly have or fail to have the extreme elements, even for the case that they do enjoy this property (in particular for products) the result furnishes the following necessary and sufficient conditions for the paracompactness of a sum: If none of the summands Y_x has a first or last element, they must all be paracompact while X can be arbitrary; if each Y_x has a (last) first element then X must have (infima) suprema for continuously (anti-)embedded uncountable regular cardinals (e.g. if each Y_x has both a first and last element, X must be paracompact) and each Y_x must be paracompact except insofar as its index x has an immediate (predecessor) successor, in which case that Y_x need not have (infima) suprema for unbounded continuously (anti-)embedded uncountable regular cardinals.

As an application, we see that $X \times Y$ can be paracompact in the lexicographic order topology without either X or Y having to be. Thus if Y were paracompact without first or last element, X could be any set; if X were the integers, Y could be an uncountable regular cardinal.

The point of view adopted here also yields a somewhat more efficient proof, for the paracompactness of a lexicographic product of paracompact ordered sets indexed by an arbitrary ordinal, than that appearing on pp. 69–73 of M. J. Faber, *Metrizability in generalized ordered spaces*, Math. Centrum Amsterdam 1974 (which was kindly transmitted to me by Ostaszewski). By the latter's Theorem 1 and transfinite induction it suffices to consider limit ordinals for which the subproducts over proper inital segments have paracompact order topology. Restricted to a continuously embedded uncountable regular cardinal, the projections on these subproducts are either all terminally constant, or cofinal continuous embeddings into the subproducts over sufficiently large initial segments – in either event, each projected image has a supremum which is preserved by projection from the subproduct over a larger, on that over a smaller (still sufficiently large) initial segment; whence they determine a supremum in the full product (technically: the latter is the inverse limit of these subproducts).

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