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TWO RESULTS ABOUT H^∞ FUNCTIONAL CALCULUS ON ANALYTIC UMD BANACH SPACES

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Abstract

Let X be a Banach space with the analytic UMD property, and let A and B be two commuting sectorial operators on X which admit bounded H^∞ functional calculi with respect to angles θ_1 and θ_2 satisfying $\theta_1 + \theta_2 < \pi$. It was proved by Kalton and Weis that in this case, $A + B$ is closed. The first result of this paper is that under the same conditions, $A + B$ actually admits a bounded H^∞ functional calculus. Our second result is that given a Banach space X and a number $1 \leq p < \infty$, the derivation operator on the vector valued Hardy space $H^p(\mathbb{R}; X)$ admits a bounded H^∞ functional calculus if and only if X has the analytic UMD property. This is an 'analytic' version of the well-known characterization of UMD by the boundedness of the H^∞ functional calculus of the derivation operator on vector valued L^p-spaces $L^p(\mathbb{R}; X)$ for $1 < p < \infty$ (Dore-Venni, Hieber-Prüss, Prüss).

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1. Introduction and main statements

This paper deals with two questions concerning H^∞ functional calculus of sectorial operators, as introduced by McIntosh on Hilbert spaces (see [23]) and then developed in the Banach space setting by Cowling, Doust, McIntosh and Yagi in [6]. These questions are both closely related to the pioneering work of Dore and Venni [9] concerning the sum of commuting operators with bounded imaginary powers on UMD Banach spaces.

Let X be a Banach space and let A and B be two commuting sectorial operators on X, with respective types ω_1 and ω_2, and with respective domains $D(A)$ and $D(B)$. Then their sum $A + B : x \mapsto A(x) + B(x)$, with domain $D(A + B) = D(A) \cap D(B)$,

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is a closable operator. Assume that $\omega_1 + \omega_2 < \pi$. Then according to some earlier work of Da Prato and Grisvard [7, Section 3], the closure $\overline{A+B}$ is in turn a sectorial operator of type $\max\{\omega_1, \omega_2\}$. Now assume the stronger condition that A and B admit bounded imaginary powers, with the following estimates:

$$(1.1) \qquad \forall s \in \mathbb{R}, \quad \left\|A^{is}\right\| \leq K_1 e^{\theta_1 |s|} \quad \text{and} \quad \left\|B^{is}\right\| \leq K_2 e^{\theta_2 |s|},$$

for some constants $K_1, K_2 > 0$ and θ_1, θ_2 in $(0, \pi)$ such that $\theta_1 + \theta_2 < \pi$. It was proved in [9] (in the invertible case) and then in [13] and [26] (in the general case) that if X is a UMD Banach space, then $A + B$ is closed. Furthermore, it was proved in [26] and [10] that under these conditions, $A + B$ admits bounded imaginary powers.

This led to the following two natural questions. Assume that

(H) A has a bounded $H^\infty(\Sigma_{\theta_1})$ functional calculus, B has a bounded $H^\infty(\Sigma_{\theta_2})$ functional calculus, and $\theta_1 + \theta_2 < \pi$.

For which Banach spaces X does this imply that $A + B$ is closed and for which ones does this imply that $\overline{A+B}$ admits a bounded H^∞ functional calculus? This amounts to consider the following two possible properties (P1) and (P2) of a Banach space X.

(P1) Whenever A and B are commuting sectorial operators on X satisfying (H) for some $\theta_1, \theta_2 \in (0, \pi)$, the sum $A + B$ is closed.

(P2) Whenever A and B are commuting sectorial operators on X satisfying (H) for some $\theta_1, \theta_2 \in (0, \pi)$, the operator $\overline{A+B}$ admits a bounded H^∞ functional calculus.

The above questions were first tackled in [19] where it is shown for example that Banach lattices, or Banach spaces with Pisier's property (α) satisfy (P1) and (P2). On the other hand, UMD Banach spaces obviously satisfy (P1) by the above mentioned Dore-Venni Theorem. However the problem whether (P2) is satisfied by all UMD Banach spaces was left open in [19]. Our first result (Theorem 1.1 below) solves this question. We will actually be able to consider the larger class of Banach spaces with the so-called property (Δ), defined by the inequality (1.4) below.

Let $(\varepsilon_i)_{i\geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$. That is, the ε_i's are pairwise independent random variables on Ω and $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$ for any $i \geq 1$. Then for any finite family $x_1, \dots, x_n$ in X, we let

$$(1.2) \qquad \left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\|_{\mathrm{Rad}(X)} = \int_\Omega \left\|\sum_{i=1}^{n} \varepsilon_i(\omega) x_i\right\|_X d\mathbb{P}(\omega).$$

Now let $(\varepsilon'_j)_{j\geq 1}$ be another Rademacher sequence on $(\Omega, \mathbb{P})$ and assume that $(\varepsilon_i)_{i\geq 1}$ and $(\varepsilon'_j)_{j\geq 1}$ are mutually independent. Then for any finite family $(x_{ij})_{1\leq i,j\leq n}$ in X, we let

$$\left\| \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j' x_{ij} \right\|_{\mathrm{Rad}^2(X)} = \int_\Omega \left\| \sum_{i,j=1}^{n} \varepsilon_i(\omega)\varepsilon_j'(\omega) x_{ij} \right\|_X d\mathbb{P}(\omega). \tag{1.3}$$

By definition we say that X satisfies (Δ) if there is a constant $C > 0$ such that for any finite family $(x_{ij})_{1\le i,j\le n}$ in X, we have

$$\left\| \sum_{1\le i\le j\le n} \varepsilon_i \varepsilon_j' x_{ij} \right\|_{\mathrm{Rad}^2(X)} \le C \left\| \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j' x_{ij} \right\|_{\mathrm{Rad}^2(X)}. \tag{1.4}$$

This property was explicitly defined by Kalton and Weis in [18]. It goes back to a paper of Haagerup and Pisier [15] where it is implicitly shown that any analytic UMD Banach space (AUMD in short), hence any UMD Banach space satisfies (Δ) (see also [18, Proposition 3.2]). We refer to [11], [15, Section 4] and Section 2 below for the definition of AUMD Banach spaces and relevant information. Kalton and Weis showed in [18, Corollary 6.4] that any Banach space with property (Δ) satisfies (P1) above. Our first result says that (P2) is satisfied as well for this class.

THEOREM 1.1. *Let A and B be two commuting sectorial operators on a Banach space X and let θ_1, θ_2 in $(0,\pi)$ such that $\theta_1 + \theta_2 < \pi$. Assume that A has a bounded $H^\infty(\Sigma_{\theta_1})$ functional calculus, B has a bounded $H^\infty(\Sigma_{\theta_2})$ functional calculus and X has property (Δ). Then $A + B$ is closed and for any $\theta > \max\{\theta_1, \theta_2\}$, $A + B$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

In [9], Dore and Venni were mainly interested in applications to L^p-maximal regularity for generators of bounded analytic semigroups on UMD Banach spaces. For that purpose, they needed one of the operators A or B in (1.1) to be a derivation operator. Indeed, they proved the following result [9, Theorem 3.1]: given a number $1 < p < \infty$ and a UMD Banach space X, the derivation operator d/dt on $L^p(\mathbb{R}; X)$, with domain $W^{1,p}(\mathbb{R}; X)$, admits bounded imaginary powers. This result was strengthened in [16] where it is proved that in this case, d/dt has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $L^p(\mathbb{R}; X)$ for any $\theta > \pi/2$. Shortly after the Dore-Venni paper appeared, Prüss showed the following converse to their result. If the derivation operator d/dt on $L^p(\mathbb{R}; X)$ admits bounded imaginary powers, then X is UMD (see [25, Section 8.1]).

Our second result (Theorem 1.2 below) says that similarly, the AUMD property characterizes those Banach spaces X such that the derivation operator has a bounded H^∞ functional calculus (or bounded imaginary powers) on X-valued Hardy spaces $H^p(\mathbb{R}; X)$. We note that contrary to the above mentioned results, the value $p = 1$ can be included in our analytic setting. We will assume that the reader is familiar with classical (= scalar valued) Hardy spaces on the real line $\mathbb{R}$ and on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and we refer to the monographs [17] and [12] for the necessary background.

Vector-valued Hardy spaces on the real line are defined as follows. We let X be a Banach space. Given any $f \in L^1(\mathbb{R}; X)$, we let $\widehat{f} : \mathbb{R} \to X$ denote its Fourier transform defined by

$$\widehat{f}(\xi) = \int f(t) e^{-i\xi t}\, dt, \quad \xi \in \mathbb{R}.$$

By definition, $H^1(\mathbb{R}; X)$ is the (closed) subspace of $L^1(\mathbb{R}; X)$ of all functions f such that $\widehat{f}(\xi) = 0$ for any $\xi \leq 0$. Now let $1 < p < \infty$. Then we define $H^p(\mathbb{R}; X) \subset L^p(\mathbb{R}; X)$ as the closure of $H^1(\mathbb{R}; X) \cap L^p(\mathbb{R}; X)$ in $L^p(\mathbb{R}; X)$. Equivalently, $H^p(\mathbb{R}; X)$ is the subspace of all functions $f \in L^p(\mathbb{R}; X)$ whose Poisson integral on the upper half-plane of $\mathbb{C}$ is analytic. In the case when $X = \mathbb{C}$, these spaces coincide with the classical Hardy spaces $H^p(\mathbb{R})$.

Given $1 \leq p < \infty$, let $(T_t)_{t \geq 0}$ denote the isometric translation semigroup on $L^p(\mathbb{R}; X)$, defined by

$$T_t(f)(s) = f(s - t), \quad f \in L^p(\mathbb{R}; X),\ t \geq 0,\ s \in \mathbb{R}.$$

Then $H^p(\mathbb{R}; X)$ is an invariant subspace of $(T_t)_{t \geq 0}$. Indeed, for any $f \in H^1(\mathbb{R}; X)$, for any $t \geq 0$, and for any $\xi \leq 0$,

$$\widehat{T_t(f)}(\xi) = e^{-i\xi t}\widehat{f}(\xi) = 0.$$

The negative generator of $(T_t)_{t \geq 0}$ on $L^p(\mathbb{R}; X)$ is equal to the derivation operator d/dt, with domain $W^{1,p}(\mathbb{R}; X)$. We will use the same notation d/dt to denote its restriction to $H^p(\mathbb{R}; X)$, with domain $W^{1,p}(\mathbb{R}; X) \cap H^p(\mathbb{R}; X)$. Of course the latter coincides with the negative generator of the restriction of $(T_t)_{t \geq 0}$ to $H^p(\mathbb{R}; X)$. We do not refer to either p or X in this notation, but the space on which we consider d/dt should be clear from the context.

We now turn to analogous definitions on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We assume that $\mathbb{T}$ is equipped with its normalized Haar measure. That is, if we identify $\mathbb{T}$ with $[-\pi, \pi)$ in the usual way, then the associated measure on this interval is $dt/2\pi$. Given a Banach space X and $f \in L^1(\mathbb{T}; X)$, we define its (X-valued) Fourier coefficients by

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt, \quad k \in \mathbb{Z}.$$

By definition, $H^p(\mathbb{T}; X)$ (respectively $H_0^p(\mathbb{T}; X)$) is the subspace of $L^p(\mathbb{T}; X)$ of all functions f such that $\widehat{f}(k) = 0$ for any $k < 0$ (respectively $k \leq 0$). We simply write $H^p(\mathbb{T})$ and $H_0^p(\mathbb{T})$ in the case when $X = \mathbb{C}$. Again we may define derivation

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operators in this context. Indeed, for any $1 \le p < \infty$, let $(T_t)_{t\ge 0}$ be the isometric semigroup on $L^p(\mathbb{T}; X)$ defined by letting $T_t(f)(s) = f(s-t)$ for any $t \ge 0$ and any $s \in \mathbb{R}/2\pi\mathbb{Z}$. Then $H^p(\mathbb{T}; X)$ and $H_0^p(\mathbb{T}; X)$ are both invariant subspaces under the action of $(T_t)_{t\ge 0}$ and we let d/dt denote the negative generator of $(T_t)_{t\ge 0}$ either on $L^p(\mathbb{T}; X)$, or on $H^p(\mathbb{T}; X)$, or on $H_0^p(\mathbb{T}; X)$.

Before stating our result, we note that for any X and any $1 \le p < \infty$, d/dt is a sectorial operator of type $\pi/2$ (in the sense of (2.1) below) on either $H^p(\mathbb{R}; X)$ or $H_0^p(\mathbb{T}; X)$.

THEOREM 1.2. *Given a Banach space X, the following assertions are equivalent.*

(i) *X is an AUMD Banach space.*

(ii) *For all $1 \le p < \infty$, and for all $\theta > \pi/2$, d/dt has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $H^p(\mathbb{R}; X)$.*

(iii) *There exist $1 \le p < \infty$ and $s \in \mathbb{R}^*$ such that $(d/dt)^{is}$ is bounded on $H^p(\mathbb{R}; X)$.*

(iv) *For all $1 \le p < \infty$, and for all $\theta > \pi/2$, d/dt has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on $H_0^p(\mathbb{T}; X)$.*

(v) *There exist $1 \le p < \infty$ and $s \in \mathbb{R}^*$ such that $(d/dt)^{is}$ is bounded on $H_0^p(\mathbb{T}; X)$.*

Section 2 contains the necessary background on H^∞ functional calculus and AUMD Banach spaces. Section 3 is mainly devoted to the proof of Theorem 1.1. The latter relies on remarkable recent results of Kalton and Weis [18] connecting H^∞ functional calculus and R-bounded sets of operators in the sense of [3]. Section 4 is mainly devoted to the proof of Theorem 1.2. The latter reduces to the study of certain Fourier multipliers on vector valued Hardy spaces and we include several results on this topic.

NOTE. The reader should notice that we use the same notation $f \mapsto \widehat{f}$ for all sorts of Fourier transforms.

2. Preliminaries and notation

Given a Banach space X, we let $B(X)$ denote the Banach algebra of all bounded operators on X. If A is a linear operator on X, we let $D(A)$ and $R(A)$ denote the domain and the range of A respectively. Furthermore, we denote by $\sigma(A)$ the spectrum of A and by $\rho(A)$ the resolvent set of A. For $\lambda \in \rho(A)$, we let $R(\lambda, A) = (\lambda - A)^{-1}$ denote the corresponding resolvent operator. For $\omega \in (0, \pi)$, let Σ_ω be the open sector of all $z \in \mathbb{C} \setminus \{0\}$ such that $|\operatorname{Arg}(z)| < \omega$. By definition, A is a sectorial operator of type $\omega \in (0, \pi)$ if A is closed and densely defined, A is one-to-one, A has a dense range, $\sigma(A) \subset \overline{\Sigma_\omega}$, and for any $\theta \in (\omega, \pi)$ there is a constant $C > 0$ such that

$$\forall \lambda \in \overline{\Sigma_\theta}^c, \quad \|\lambda R(\lambda, A)\| \leq C. \tag{2.1}$$

We note the classical fact that if $(T_t)_{t\geq 0}$ is a bounded C_0-semigroup on X and if $-A$ denotes its infinitesimal generator, then A is a sectorial operator of type $\pi/2$ provided that A is one-to-one and has dense range.

For $\theta \in (0, \pi)$, and for a Banach space E, we let $H^\infty(\Sigma_\theta; E)$ be the space of all bounded analytic functions $F : \Sigma_\theta \to E$. This is a Banach space for the norm

$$\|F\|_{\infty,\theta} = \sup\{\|F(z)\|_E : z \in \Sigma_\theta\}.$$

Then we let $H_0^\infty(\Sigma_\theta; E)$ be the subspace of all $F \in H^\infty(\Sigma_\theta; E)$ for which there exist two positive numbers s, $C > 0$ such that

$$\|F(z)\|_E \leq C\frac{|z|^s}{(1+|z|)^{2s}}, \qquad z \in \Sigma_\theta. \tag{2.2}$$

We will simply denote $H^\infty(\Sigma_\theta; \mathbb{C})$ and $H_0^\infty(\Sigma_\theta; \mathbb{C})$ by $H^\infty(\Sigma_\theta)$ and $H_0^\infty(\Sigma_\theta)$.

We now come to H^∞ functional calculus for sectorial operators. The definitions and basic facts below essentially go back to [23] and [6]. The reader may also consult [20] or [18] for more information. Given a sectorial operator A of type $\omega \in (0, \pi)$ on a Banach space X, we define its commutant by

$$E_A = \{T \in B(X) : TR(\lambda, A) = R(\lambda, A)T,\ \lambda \in \rho(A)\}.$$

Clearly E_A is a closed subalgebra of $B(X)$. Let $\omega < \gamma < \theta < \pi$, and let Γ_γ be the oriented contour defined by

$$\Gamma_\gamma(t) = \begin{cases} -te^{i\gamma}, & t \in \mathbb{R}_-; \\ te^{-i\gamma}, & t \in \mathbb{R}_+. \end{cases} \tag{2.3}$$

Then for any function $F \in H_0^\infty(\Sigma_\theta; E_A)$, we set

$$F(A) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} F(\lambda)R(\lambda, A)\, d\lambda. \tag{2.4}$$

Since A satisfies (2.1) and F satisfies (2.2), $F(A)$ is well defined and belongs to $B(X)$. By Cauchy's Theorem, the definition (2.4) does not depend on the choice of $\gamma \in (\omega, \theta)$. Furthermore, $H_0^\infty(\Sigma_\theta; E_A)$ is an algebra and the mapping $F \mapsto F(A)$ is an algebra homomorphism. Note that the latter is unbounded in general. Now let φ be the scalar valued function defined by $\varphi(z) = z/(1+z)^2$. Then the bounded operator $\varphi(A) = A(1+A)^{-2}$ is one-to-one and its range is equal to $D(A) \cap R(A)$, which is dense in X. Given any $F \in H^\infty(\Sigma_\theta; E_A)$, we may therefore define $F(A)$ as follows. We note that the function φF belongs to $H_0^\infty(\Sigma_\theta; E_A)$ and we set

$$F(A) = \varphi(A)^{-1}(\varphi F)(A).$$

This possibly unbounded operator has domain equal to the space of all $x \in X$ such that $\big[(\varphi F)(A)\big](x) \in R(\varphi(A))$. The latter is dense and $F(A)$ is closed. We record for further the following well-known lemma.

LEMMA 2.1. *Let $0 < \omega < \theta < \pi$ be two numbers and let A be a sectorial operator of type ω on X.*

(1) *Let $F \in H^\infty(\Sigma_\theta; E_A)$ and let $(F_n)_{n\geq 1}$ be the uniformly bounded sequence of $H_0^\infty(\Sigma_\theta; E_A)$ defined by letting $F_n(z) = \big(n/(n+z) - 1/(1+nz)\big)F(z)$ for any $z \in \Sigma_\theta$. Then $F(A)$ is bounded if and only if the sequence $(F_n(A))_{n\geq 1} \subset B(X)$ is bounded and in this case, $F(A)$ is the strong limit of $(F_n(A))_{n\geq 1}$.*

(2) *Let $F \in H^\infty(\Sigma_\theta; E_A)$ be a function with the following property: there exist two positive numbers s, $C > 0$ such that*

$$\|F(z)\|_{E_A} \leq C\frac{1}{|z|^s}, \quad z \in \Sigma_\theta. \tag{2.5}$$

Assume moreover that A is invertible. Then $F(A)$ is bounded and for any $\gamma \in (\omega, \theta)$,

$$F(A) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} F(\lambda)R(\lambda, A)\,d\lambda. \tag{2.6}$$

PROOF. Part (1) is a variant of the so-called 'convergence lemma' [23, 6]. Turning to (2), note that since $0 \in \rho(A)$, the mapping $\lambda \mapsto F(\lambda)R(\lambda, A)$ is bounded on Γ_γ. Hence (2.1) and (2.5) ensure that the integral in the right-hand side of (2.6) converges and defines an element of $B(X)$. It is easy to conclude from (1) and Lebesgue's Theorem that this element of $B(X)$ equals $F(A)$. $\square$

We finally recall two major definitions. First, let A be a sectorial operator on X of type $\omega \in (0, \pi)$, and let $\theta > \omega$. We say that A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if $F(A)$ is bounded for any $F \in H^\infty(\Sigma_\theta)$. Second, let B be another sectorial operator on X. We say that A and B commute if $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$ for any $\lambda \in \rho(A)$, $\mu \in \rho(B)$.

We now turn to some background and notation on AUMD Banach spaces for which we refer to [11] and [15, Section 4]. Equip the compact space $\mathbb{T}^{\mathbb{N}}$ with its product measure and let $(t_1, \dots, t_n, \dots)$ denote a typical element of $\mathbb{T}^{\mathbb{N}}$. For any integer $n \geq 1$, let $\mathcal{F}_n$ denote the σ-field generated by the first n variables $t_1, \dots, t_n$. Let $(g_n)_{n\geq 1}$ be an X-valued martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$, that is, each

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$g_n : \mathbb{T}^{\mathbb{N}} \to X$ is an $\mathscr{F}_n$-measurable function and letting $d_n = g_n - g_{n-1}$, we have $\mathbb{E}(d_n|\mathscr{F}_{n-1}) = 0$ for any $n \geq 1$. As usual the convention is that $g_0 = 0$ and $\mathscr{F}_0$ is the trivial σ-field. We say that $(g_n)_{n\geq 1}$ is analytic if for any $n \geq 1$, there exists a measurable function $\Phi_n : \mathbb{T}^{n-1} \to X$ such that

$$d_n(t_1, \dots, t_n) = \Phi_n(t_1, \dots, t_{n-1})e^{it_n}, \qquad t_1, \dots, t_n \in \mathbb{T}. \tag{2.7}$$

Let $1 \leq p < \infty$ be a number. By definition X is an AUMD Banach space if there is a constant K_p such that whenever $(g_n)_{n\geq 1}$ is an X-valued analytic martingale, $N \geq 1$ is an integer, and $\varepsilon_1, \dots, \varepsilon_N \in \{-1, 1\}$, we have an estimate

$$\left\| \sum_{n=1}^{N} \varepsilon_n d_n \right\|_p \leq K_p \left\| \sum_{n=1}^{N} d_n \right\|_p , \tag{2.8}$$

where the norms are computed in $L^p(\mathbb{T}^N; X)$. This property does not depend on p. Furthermore, to prove that a given Banach space is AUMD, it suffices to show (2.8) in the case when the martingale is finite (that is, d_n is eventually 0), and each Φ_n in (2.7) is an X-valued trigonometric polynomial, that is, a sum of elements of the form $x e^{iq_1t_1} \cdots e^{iq_{n-1}t_{n-1}}$, where $x \in X$ and $q_1, \dots, q_{n-1} \in \mathbb{Z}$.

The class of AUMD spaces includes UMD Banach spaces, L^1-spaces, and quotients L^1/R of an L^1-space by one of its reflexive subspaces R. Also, it is stable under taking subspaces. Conversely, $C(\Omega)$-spaces (where Ω is an infinite compact set), the quotient space $L^1(\mathbb{T})/H^1(\mathbb{T})$, and the Schatten space $S^1(H)$ of trace class operators on an infinite dimensional Hilbert space H are not AUMD.

3. Perturbation of R-sectorial operators and proof of Theorem 1.1

The main purpose of this section is the proof of Theorem 1.1. The latter relies on some recent work of Kalton and Weis [18] involving R-boundedness, and on a perturbation result (Proposition 3.2 below) of independent interest, which is the key ingredient of the proof. We shall first give the necessary background on R-boundedness. Our main reference for this notion is [3], see also [18].

Let X be a Banach space and let $\mathscr{T} \subset B(X)$ be a set of bounded operators on X. By definition, we say that $\mathscr{T}$ is R-bounded if there is a constant $C > 0$ such that for any finite families $T_1, \dots, T_n$ in $\mathscr{T}$, and $x_1, \dots, x_n$ in X, we have

$$\left\| \sum_{i=1}^{n} \varepsilon_i T_i(x_i) \right\|_{\mathrm{Rad}(X)} \leq C \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_{\mathrm{Rad}(X)} . \tag{3.1}$$

In this definition, the norms $\| \ \|_{\mathrm{Rad}(X)}$ are defined by (1.2). The least constant C satisfying (3.1) is called the R-boundedness constant of $\mathscr{T}$ and is denoted by $R(\mathscr{T})$.

Obviously any R-bounded set $\mathcal{T}$ is bounded and $\|T\| \leq \mathcal{R}(\mathcal{T})$ for any $T \in \mathcal{T}$, but the converse does not hold on non-Hilbertian Banach spaces. Given any two sets $\mathcal{T}_1, \mathcal{T}_2 \subset B(X)$, we let $\mathcal{T}_1 + \mathcal{T}_2 = \{T_1 + T_2 : T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2\}$ and $\mathcal{T}_1\mathcal{T}_2 = \{T_1T_2 : T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2\}$. In the next lemma, we record some well-known stability results concerning R-bounded sets.

LEMMA 3.1. (1) *If $\mathcal{T} \subset B(X)$ is R-bounded, then its closure $\overline{\mathcal{T}}$ is R-bounded and $R(\overline{\mathcal{T}}) = R(\mathcal{T})$.*

(2) *If $\mathcal{T}_1, \mathcal{T}_2 \subset B(X)$ are R-bounded, then $\mathcal{T}_1 + \mathcal{T}_2$ is R-bounded and $R(\mathcal{T}_1 + \mathcal{T}_2) \leq R(\mathcal{T}_1) + R(\mathcal{T}_2)$.*

(3) *If $\mathcal{T}_1, \mathcal{T}_2 \subset B(X)$ are R-bounded, then $\mathcal{T}_1\mathcal{T}_2$ is R-bounded and $R(\mathcal{T}_1\mathcal{T}_2) \leq R(\mathcal{T}_1)R(\mathcal{T}_2)$.*

(4) *If $\mathcal{T} \subset B(X)$ is R-bounded, then its absolute convex hull* $\mathrm{aco}(\mathcal{T})$ *is R-bounded and $R(\mathrm{aco}(\mathcal{T})) \leq 2R(\mathcal{T})$.*

(5) *Let $\mathcal{T} \subset B(X)$ be R-bounded and let $C > 0$ be a constant. Then the set $\left\{\int_0^\infty f(t)T(t)\,dt \,\middle|\, T : \mathbb{R}_+^* \to \mathcal{T} \text{ and } f : \mathbb{R}_+^* \to \mathbb{C} \text{ are continuous, and } \int_0^\infty |f(t)|\,dt \leq C\right\}$ is R-bounded and its R-boundedness constant is less than or equal to $2CR(\mathcal{T})$.*

PROOF. The first three assertions are more or less obvious. The assertion (4) is [3, Lemma 3.2]. To prove (5), it therefore suffices to check that if $T : \mathbb{R}_+^* \to \mathcal{T}$ and $f : \mathbb{R}_+^* \to \mathbb{C}$ are two continuous functions and if $\int_0^\infty |f(t)|\,dt \leq C$, then

$$\int_0^\infty f(t)T(t)\,dt \in C\,\overline{\mathrm{aco}}(\mathcal{T}).$$

For this it suffices to show that for any $0 < \alpha < \beta < \infty$,

$$\int_\alpha^\beta f(t)T(t)\,dt \in C\,\overline{\mathrm{aco}}(\mathcal{T}).$$

The latter property clearly follows from an approximation of the integral by Riemann sums. $\square$

PROPOSITION 3.2. *Let A be a sectorial operator on a Banach space X, and let $\mu \in (0, \pi)$ be such that the set*

$$\left\{\lambda R(\lambda, A) : \lambda \notin \overline{\Sigma_\mu}\right\} \tag{3.2}$$

is R-bounded. Let $\nu, \varphi \in (0, \pi)$ be two numbers such that $\mu + \nu < \pi$ and $\max\{\mu, \nu\} < \varphi$. Suppose that $F \in H^\infty(\Sigma_\varphi)$ and that $F(A)$ is bounded. Then the set

$$\{F(A + z) : z \in \Sigma_\nu\}$$

is R-bounded.

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Note that all the operators considered in the previous statement make sense. Indeed, our assumption on (3.2) implies that A is sectorial of type $< \mu$. (According to the terminology of [18], A is actually an R-sectorial operator of R-sectorial type $< \mu$.) Since $\mu + \nu < \pi$, $A + z$ is therefore a sectorial operator of type $\max\{\mu, \nu\}$ for any $z \in \Sigma_\nu$, which allows us to define $F(A + z)$ for any $F \in H^\infty(\Sigma_\varphi)$.

Replacing R-boundedness by boundedness, our statement corresponds to the following perturbation result established by Uiterdijk in his Ph.D. thesis [28]: if A is a sectorial operator of type $< \mu$, if $\mu + \nu < \pi$ and $\max\{\mu, \nu\} < \varphi$ and if $F \in H^\infty(\Sigma_\varphi)$ is a function such that $F(A)$ is bounded, then $F(A + z)$ is bounded for any $z \in \Sigma_\nu$ and the resulting family of operators is bounded. Strictly speaking, Uiterdijk proved that result only for $\nu = 0$ but it is possible to extend his proof to the general case. It turns out that his arguments also yield our Proposition 3.2, up to some estimates on the R-boundedness of certain sets of operators, as explained in the proof below.

PROOF OF PROPOSITION 3.2. We let A be as in Proposition 3.2 and we let $F \in H^\infty(\Sigma_\varphi)$ be such that $F(A)$ is bounded. Following Uiterdijk's idea in [28, 2.3], we decompose $F(A+z)$ for any $z \in \Sigma_\nu$ as follows. Writing $I_X = z(A+z)^{-1} + A(A+z)^{-1}$, we have $F(A + z) = z(A + z)^{-1}F(A + z) + A(A + z)^{-1}F(A + z)$ hence

$$\begin{aligned} F(A + z) = z(A + z)^{-1}F(A + z) + A(A + z)^{-1}F(A) \\ + A(A + z)^{-1}(F(A + z) - F(A)). \end{aligned}$$

Since $\nu < \pi - \mu$, the R-boundedness of (3.2) implies that $\{z(A + z)^{-1} : z \in \Sigma_\nu\}$ is R-bounded. Applying Lemma 3.1 (1), we deduce that $\{A(A + z)^{-1}F(A) : z \in \Sigma_\nu\}$ is R-bounded and that it suffices to show that

$$\{z(A + z)^{-1}F(A + z) : z \in \Sigma_\nu\} \quad \text{is } R\text{-bounded;} \tag{3.3}$$

and

$$\{A(A + z)^{-1}\big(F(A + z) - F(A)\big) : z \in \Sigma_\nu\} \quad \text{is } R\text{-bounded.} \tag{3.4}$$

We first prove (3.3). Let $\mu' > \mu$ and $\nu' > \nu$ be such that $\mu' + \nu' < \pi$ and let $\varphi > \max\{\mu', \nu'\}$. Then there exists a positive number $r > 0$ (depending on ν and ν') such that

$$\forall z \in \Sigma_\nu, \quad z - 2r|z| \in \Sigma_{\nu'}. \tag{3.5}$$

Fix some $z \in \Sigma_\nu$. For any $\lambda \in \Sigma_{\mu'}$, the complex number $\lambda + z - r|z|$ belongs to $\Sigma_{\max\{\mu', \nu'\}}$, hence to Σ_φ. Indeed, $z - r|z| \in \Sigma_{\nu'}$ by (3.5) and $\mu' + \nu' < \pi$. We may therefore define $h_z \in H^\infty(\Sigma_{\mu'})$ by letting

$$h_z(\lambda) = \frac{z}{\lambda + z - r|z|}F(\lambda + z - r|z|), \quad \lambda \in \Sigma_{\mu'}.$$

Let $\Gamma = \Gamma_\gamma$ be defined by (2.3), for some $\gamma \in (\mu, \mu')$. Clearly $\lambda h_z(\lambda)$ is bounded away from 0 on $\Sigma_{\mu'}$ and $A + r|z|$ is an invertible sectorial operator of type μ. Hence by Lemma 2.1 (2), $h_z(A + r|z|)$ is bounded and

$$h_z(A + r|z|) = \frac{1}{2\pi i} \int_\Gamma h(\lambda) R(\lambda, A + r|z|)\, d\lambda\,.$$

However, $h_z(A+r|z|) = z(A+z)^{-1}F(A+z)$ hence we have the integral representation

$$z(A+z)^{-1}F(A+z) = \frac{1}{2\pi i} \int_\Gamma \frac{zF(\lambda + z - r|z|)}{\lambda + z - r|z|} R(\lambda - r|z|, A)\, d\lambda\,. \tag{3.6}$$

Let $I(z)$ be the integral in the right-hand side of (3.6). Letting $\Gamma_+ = \Gamma \cap \{\mathrm{Im}(\lambda) > 0\}$ and $\Gamma_- = \Gamma \cap \{\mathrm{Im}(\lambda) < 0\}$, we write $I(z) = I_+(z) + I_-(z)$, where $I_+(z) = \frac{1}{2\pi i}\int_{\Gamma_+} \cdots$ and $I_-(z) = \frac{1}{2\pi i}\int_{\Gamma_-} \cdots$. Then

$$\begin{aligned} I_+(z) &= -\frac{e^{i\gamma}}{2\pi i} \int_0^\infty \frac{zF(te^{i\gamma} + z - r|z|)}{te^{i\gamma} + z - r|z|} R(te^{i\gamma} - r|z|, A)\, dt \\ &= -\frac{e^{i\gamma}}{2\pi i} \int_0^\infty \frac{zF(te^{i\gamma} + z - r|z|)}{(te^{i\gamma} + z - r|z|)(te^{i\gamma} - r|z|)} \big((te^{i\gamma} - r|z|) R(te^{i\gamma} - r|z|, A)\big)\, dt\,. \end{aligned}$$

We will prove below that

$$\sup_{z \in \Sigma_\nu} \int_0^\infty \left| \frac{zF(te^{i\gamma} + z - r|z|)}{(te^{i\gamma} + z - r|z|)(te^{i\gamma} - r|z|)} \right| dt < \infty. \tag{3.7}$$

Now recall that we chose $\gamma > \mu$. Thus for any $t > 0$, $te^{i\gamma} \notin \overline{\Sigma_\mu}$ hence $te^{i\gamma} - r|z| \notin \overline{\Sigma_\mu}$. Hence the continuous function $T(t) = (te^{i\gamma} - r|z|)R(te^{i\gamma} - r|z|, A)$ is valued in the R-bounded set (3.2). Thus according to Lemma 3.1 (5), and (3.7), we obtain that $\{I_+(z) : z \in \Sigma_\nu\}$ is R-bounded. Similarly, the set $\{I_-(z) : z \in \Sigma_\nu\}$ is R-bounded, and so the first required result (3.3) follows using (3.6) and Lemma 3.1 (1).

We now prove the crucial estimate (3.7). Write any $z \in \Sigma_\nu$ as $z = |z|e^{i\theta}$, with $|\theta| < \nu$. Then the integral in (3.7) is

$$\leq \|F\|_{\infty,\varphi} \int_0^\infty \frac{|z|}{\big|te^{i\gamma} + |z|(e^{i\theta} - r)\big|\big|te^{i\gamma} - r|z|\big|}\, dt\,.$$

Changing t into $|z|t$, the latter is equal to

$$\|F\|_{\infty,\varphi} \int_0^\infty \frac{1}{\big|te^{i\gamma} + e^{i\theta} - r\big|\big|te^{i\gamma} - r\big|}\, dt\,. \tag{3.8}$$

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Now observe that $e^{i\theta} \in \Sigma_\nu$, hence $e^{i\theta} - 2r \in \Sigma_{\nu'}$ by (3.5). Hence $2r - e^{i\theta}$ belongs to $\overline{\Sigma_{\pi-\nu'}}^c$, hence to $\overline{\Sigma_{\mu'}}^c$. Consequently, we have

$$\begin{aligned}\left|te^{i\gamma} + e^{i\theta} - r\right| &= \left|(te^{i\gamma} + r) - (2r - e^{i\theta})\right| \\ &\geq \operatorname{dist}\left(te^{i\gamma} + r, \overline{\Sigma_{\mu'}}^c\right) \geq K|te^{i\gamma} + r|,\end{aligned}$$

where $K > 0$ is a constant not depending on $t > 0$. Therefore the integral in (3.8) is

$$\leq \frac{\|F\|_{\infty,\varphi}}{K} \int_0^\infty \frac{1}{|te^{i\gamma} + r||te^{i\gamma} - r|}\, dt\,.$$

This shows (3.7).

We now turn to the proof of (3.4). Let $\Gamma = \Gamma_\gamma$ be defined by (2.3), for some γ satisfying $\max\{\mu, \nu\} < \gamma < \varphi$. We will need the following integral representation, which is essentially due to [28]. For any $z \in \Sigma_\nu$,

$$(3.9) \qquad \begin{aligned}&A(A+z)^{-1}\big(F(A+z) - F(A)\big) \\ &\quad = \frac{1}{2\pi i}\int_\Gamma F(\lambda) z\big(AR(\lambda, A)(A+z)^{-1}\big) R(\lambda - z, A)\, d\lambda\,.\end{aligned}$$

To prove this, first note from the boundedness of the operator $A(A+z)^{-1}$ and the sectoriality of A that there exists a constant $K > 0$ (depending on z) such that

$$(3.10) \quad \left\|\big(AR(\lambda, A)(z+A)^{-1}\big)R(\lambda - z, A)\right\| \leq K \min\left\{1, \frac{1}{|\lambda||\lambda - z|}\right\}, \qquad \lambda \in \Gamma.$$

Hence the right-hand side of (3.9) makes sense. Assume that $F \in H_0^\infty(\Sigma_\varphi)$. Then applying (2.4), we have

$$F(A+z) = \frac{1}{2\pi i}\int_\Gamma F(\lambda) R(\lambda - z, A)\, d\lambda \quad \text{and} \quad F(A) = \frac{1}{2\pi i}\int_\Gamma F(\lambda) R(\lambda, A)\, d\lambda,$$

hence (3.9) follows by applying the identity

$$R(\lambda - z, A) - R(\lambda, A) = zR(\lambda - z, A)R(\lambda, A).$$

Now for an arbitrary $F \in H^\infty(\Sigma_\varphi)$, we let $(F_n)_{n\geq 1}$ be the sequence of $H_0^\infty(\Sigma_\varphi)$ defined in Lemma 2.1 (1) so that $A(A+z)^{-1}\big(F(A+z) - F(A)\big)$ is the strong limit of $A(A+z)^{-1}\big(F_n(A+z) - F_n(A)\big)$. Thanks to (3.10), we may apply Lebesgue's Theorem to deduce that since (3.9) holds for each F_n, it holds as well for F.

We let $\Gamma_1^z = \Gamma \cap \{|\lambda| \leq |z|\}$ and $\Gamma_2^z = \Gamma \cap \{|\lambda| > |z|\}$. According to (3.9), we have that for any $z \in \Sigma_\nu$

$$(3.11) \quad A(A+z)^{-1}\big(F(A+z) - F(A)\big) = z(A+z)^{-1}B_1(z) + A(A+z)^{-1}B_2(z),$$

with

$$B_1(z) = \frac{1}{2\pi i}\int_{\Gamma_1^z} F(\lambda) A R(\lambda, A) R(\lambda - z, A)\, d\lambda\,;$$

$$B_2(z) = \frac{1}{2\pi i}\int_{\Gamma_2^z} F(\lambda) z R(\lambda, A) R(\lambda - z, A)\, d\lambda\,.$$

Then we decompose again as $B_1(z) = B_1^+(z) + B_1^-(z)$ and $B_2(z) = B_2^+(z) + B_2^-(z)$, where $B_i^+(z)$ corresponds to the integration along $\Gamma_i^z \cap \{\mathrm{Im}(\lambda) > 0\}$ and $B_i^-(z)$ corresponds to the integration along $\Gamma_i^z \cap \{\mathrm{Im}(\lambda) < 0\}$. Let us show that

$$\{B_i^+(z) : z \in \Sigma_\nu\} \quad \text{is } R\text{-bounded}, \qquad i = 1, 2. \tag{3.12}$$

For any $z \in \Sigma_\nu$, we have

$$\begin{aligned} B_1^+(z) &= -\frac{e^{i\gamma}}{2\pi i}\int_0^{|z|} F(te^{i\gamma}) A R(te^{i\gamma}, A) R(te^{i\gamma} - z, A)\, dt \\ &= -\frac{e^{i\gamma}}{2\pi i}\int_0^{|z|} \frac{F(te^{i\gamma})}{te^{i\gamma} - z}\big(A R(te^{i\gamma}, A)(te^{i\gamma} - z) R(te^{i\gamma} - z, A)\big)\, dt\,. \end{aligned}$$

By assumption, the set $\{\sigma R(\sigma, A) : \sigma \notin \overline{\Sigma_\mu}\}$ is R-bounded hence the set $\{A R(\sigma, A) : \sigma \notin \overline{\Sigma_\mu}\}$ is R-bounded as well. Hence applying Lemma 3.1 (2) we find that

$$\big\{\sigma R(\sigma, A) \omega R(\omega, A) : \sigma, \omega \notin \overline{\Sigma_\mu}\big\} \tag{3.13}$$

is R-bounded. Now for any $0 < t < |z|$, the operator $A R(te^{i\gamma}, A)(te^{i\gamma} - z) R(te^{i\gamma} - z, A)$ belongs to the set (3.13). Hence by Lemma 3.1 (5), the set $\{B_1^+(z) : z \in \Sigma_\nu\}$ is R-bounded provided that

$$\sup_{z\in\Sigma_\nu}\int_0^{|z|} \frac{\big|F(te^{i\gamma})\big|}{\big|te^{i\gamma} - z\big|}\, dt < \infty\,. \tag{3.14}$$

To check (3.14), we let $z = |z|e^{i\theta}$ be an arbitrary element of Σ_ν with $|\theta| < \nu$. Then changing t into $|z|t$, the integral in (3.14) is

$$\leq \|F\|_{\infty,\varphi}\int_0^1 \frac{dt}{|te^{i\gamma} - e^{i\theta}|}\,.$$

Hence it remains to observe that the latter integral is less than or equal to the inverse of the distance between the two disjoint compact sets $\{te^{i\gamma} : t \in [0, 1]\}$ and $\{e^{i\theta} : \theta \in [-\nu, \nu]\}$. This concludes the proof of the R-boundedness of $\{B_1^+(z) : z \in \Sigma_\nu\}$. Similarly we write

$$\begin{aligned} B_2^+(z) &= -\frac{e^{i\gamma}}{2\pi i}\int_{|z|}^{\infty} F(te^{i\gamma}) z R(te^{i\gamma}, A) R(te^{i\gamma} - z, A)\, dt \\ &= -\frac{e^{i\gamma}}{2\pi i}\int_{|z|}^{\infty} \frac{zF(te^{i\gamma})}{(te^{i\gamma})(te^{i\gamma} - z)}\big(te^{i\gamma} R(te^{i\gamma}, A)(te^{i\gamma} - z) R(te^{i\gamma} - z, A)\big)\, dt\,. \end{aligned}$$

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The proof that $\{B_2^+(z) : z \in \Sigma_\nu\}$ is R-bounded reduces to showing that

$$\sup_{z\in\Sigma_\nu} \int_{|z|}^{\infty} \frac{|z||F(te^{i\gamma})|}{|te^{i\gamma}||te^{i\gamma} - z|}\, dt < \infty. \tag{3.15}$$

We let $z = |z|e^{i\theta}$ with $|\theta| < \nu$ and changing t into $|z|t$, we find that the integral in (3.15) is

$$\leq \|F\|_{\infty,\varphi} \int_1^{\infty} \frac{dt}{t|te^{i\gamma} - e^{i\theta}|}.$$

There is a constant $K > 0$ such that $|te^{i\gamma} - e^{i\theta}| \geq Kt$ for any $t \geq 1$ and any $\theta \in [-\nu, \nu]$. Consequently the latter integral is less than or equal to $K^{-1}\|F\|_{\infty,\varphi} \int_1^\infty dt/t^2$. This completes the proof of (3.12).

We now conclude the proof by applying Lemma 3.1. Arguing as above we find that the sets $\{B_i^-(z) : z \in \Sigma_\nu\}$ are R-bounded hence the sets $\{B_i(z) : z \in \Sigma_\nu\}$ are R-bounded. Since $-\Sigma_\nu = \overline{\Sigma_{\pi-\nu}}^c \subset \overline{\Sigma_\mu}^c$, our assumption implies that $\{z(A+z)^{-1} : z \in \Sigma_\nu\}$ is R-bounded. We therefore deduce that $\{z(A+z)^{-1}B_1(z) : z \in \Sigma_\nu\}$ is R-bounded. Likewise $\{A(A+z)^{-1} : z \in \Sigma_\nu\}$ hence $\{A(A+z)^{-1}B_2(z) : z \in \Sigma_\nu\}$ is R-bounded. Now the result (3.4) follows from the decomposition (3.11). □

PROOF OF THEOREM 1.1. Suppose that A, B satisfy the assumptions of Theorem 1.1 on a Banach space X with property (Δ). We already know that $A + B$ is closed by [18, Corollary 6.4], hence is sectorial of type $\max\{\theta_1, \theta_2\}$. We let $\theta > \max\{\theta_1, \theta_2\}$ and $F \in H^\infty(\Sigma_\theta)$. Then we choose $\mu > \theta_1$ and $\nu > \theta_2$ such that $\mu + \nu < \pi$ and $\theta > \max\{\mu, \nu\}$. Since A has a bounded $H^\infty(\Sigma_{\theta_1})$ functional calculus and X satisfies (Δ), the set $\{\lambda R(\lambda, A) : z \notin \overline{\Sigma_\mu}\}$ is R-bounded by [18, Theorem 5.3, (3)]. Hence the set

$$\{F(A+z) : z \in \Sigma_\nu\} \tag{3.16}$$

is R-bounded by Proposition 3.2 applied with $\varphi = \theta$. Since A and B commute, the function $F(A + \cdot) : z \mapsto F(A+z)$ takes values in the commutant algebra E_B, and hence belongs to $H^\infty(\Sigma_\theta; E_B)$. Now the R-boundedness of (3.16) implies that $(F(A+\cdot))(B)$ is bounded by [18, Theorem 4.4]. Hence it remains to check that

$$(F(A+\cdot))(B) = F(A+B), \tag{3.17}$$

which shows that $F(A+B)$ is a bounded operator.

To check this equality, we use the function $\varphi(z) = z/(1+z)^2$ considered in Section 2. We let ω_1 and ω_2 denote the respective types of A and B. We let $\Gamma_1, \Gamma_2, \Gamma_3$ be three contours as defined by (2.3) corresponding to three numbers $\gamma_1, \gamma_2, \gamma_3$ such that $\omega_1 < \gamma_1 < \theta_1$, $\omega_2 < \gamma_2 < \theta_2$, and $\max\{\theta_1, \theta_2\} < \gamma_3 < \theta$. It follows from the first part of the proof of [19, Theorem 4.1] that for any $\lambda \notin \overline{\Sigma_{\max\{\theta_1,\theta_2\}}}$, we have

(3.18)
$$\varphi(A)\varphi(B)R(\lambda, A+B) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{\varphi(z')\varphi(z)}{\lambda - (z'+z)} R(z', A)R(z, B)\,dz\,dz'.$$

Hence if $F \in H_0^\infty(\Sigma_\theta)$, we have

$$\begin{aligned}
&F(A+B)\varphi(A)\varphi(B)\\
&\quad = \frac{1}{2\pi i}\int_{\Gamma_3} F(\lambda)\varphi(A)\varphi(B)R(\lambda, A+B)\,d\lambda \quad \text{by (2.4)}\\
&\quad = \left(\frac{1}{2\pi i}\right)^3 \int_{\Gamma_3}\int_{\Gamma_1}\int_{\Gamma_2} \frac{F(\lambda)\varphi(z')\varphi(z)}{\lambda - (z'+z)} R(z', A)R(z, B)\,d\lambda\,dz\,dz' \qquad \text{by (3.18)}\\
&\quad = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} F(z'+z)\varphi(z')\varphi(z)R(z', A)R(z, B)\,dz\,dz'\\
&\qquad \text{by Cauchy's Theorem}\\
&\quad = \frac{1}{2\pi i}\int_{\Gamma_2} F(A+z)\varphi(A)\varphi(z)R(z, B)\,dz = \big(F(A+\cdot)\big)(B)\varphi(A)\varphi(B).
\end{aligned}$$

Since $\varphi(A)\varphi(B)$ has a dense range this shows (3.17) in the case when $F \in H_0^\infty(\Sigma_\theta)$. The general case now follows from Lemma 2.1 (1). $\square$

We note that Theorem 1.1 does not remain true if sums are replaced by products, even on UMD Banach spaces. Namely, let A, B be two commuting sectorial operators on a UMD Banach space, and assume that A has a bounded $H^\infty(\Sigma_{\theta_1})$ functional calculus, B has a bounded $H^\infty(\Sigma_{\theta_2})$ functional calculus, and $\theta_1 + \theta_2 < \pi$. Then the operator AB, with domain equal to the space of all $x \in D(B)$ such that $B(x) \in D(A)$, is closable and it is proved in [26] that its closure $\overline{AB}$ is a sectorial operator of type $\theta_1 + \theta_2$ which admits bounded imaginary powers. However $\overline{AB}$ does not admit a bounded H^∞ functional calculus in general. Indeed given $1 \le p < \infty$, let S^p denote the Schatten space of all compact operators $T : \ell^2 \to \ell^2$ such that $|T|^p$ has a finite trace, equipped with the norm $\|T\|_p = (\mathrm{tr}(|T|^p))^{1/p}$. Then S^p is a UMD Banach space if $1 < p < \infty$ and the example given in the proof of [19, Theorem 3.9] to show that S^p fails the so-called joint calculus property shows as well that if $p \neq 2$, there exist commuting operators A, B on S^p so that $\overline{AB}$ does not admit a bounded H^∞ functional calculus although for any $\theta_1, \theta_2 > 0$, A (respectively B) has a bounded $H^\infty(\Sigma_{\theta_1})$ (respectively $H^\infty(\Sigma_{\theta_2})$) functional calculus.

We do not know any Banach space satisfying (P1) without satisfying (P2) or satisfying (P2) without satisfying (P1). The example below [18, Corollary 6.4] showing

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that if X is a Banach space such that $\mathrm{Rad}^2(X)$ satisfies (P1), then X has property (Δ), can be easily adapted to show that if $\mathrm{Rad}^2(X)$ satisfies (P2), then X has property (Δ). On the other hand we notice that the Banach space S^∞ of compact operators on ℓ^2 does not satisfy (P2) by [28, Chapter 7]. The same argument shows that S^1 does not satisfy (P2). We conclude this section by a remark and an open question.

REMARK 3.3. Up to now, we know two classes of Banach spaces satisfying (P1) and (P2), namely Banach spaces with property (Δ) and Banach spaces with the so-called property (A) introduced in [19]. It is natural to consider the following property, which is both weaker than (Δ) and (A). Let us say that a Banach space X satisfies $(W\Delta)$ (for weak (Δ)) if there is a constant $C > 0$ such that for any finite families $(x_{ij})_{1\leq i,j\leq n}$ in X and $(x_{ij}^*)_{1\leq i,j\leq n}$ in X^* we have

$$\left|\sum_{1\leq i\leq j\leq n} \langle x_{ij}, x_{ij}^*\rangle\right| \leq C \left\|\sum_{i,j=1}^n \varepsilon_i\varepsilon_j' x_{ij}\right\|_{\mathrm{Rad}^2(X)} \left\|\sum_{i,j=1}^n \varepsilon_i\varepsilon_j' x_{ij}^*\right\|_{\mathrm{Rad}^2(X^*)}.$$

Any Banach space with property $(W\Delta)$ satisfies (P1). Indeed the argument in the proof of [18, Theorem 5.3, (3)] can be easily adapted to show that if A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on X, then A is WR-sectorial (in the sense of [18]) with respect to any $\omega > \theta$. The result therefore follows from [18, Theorem 4.5]. However we do not know whether any Banach space with property $(W\Delta)$ satisfies (P2). In particular we do not know whether our Proposition 3.2 remains true if 'R-bounded' is replaced by 'WR-bounded'.

4. Fourier multipliers on vector valued Hardy spaces and proof of Theorem 1.2

In this section, we shall first explain how to define Fourier multipliers on spaces $H^p(\mathbb{R};X)$ or $H_0^p(\mathbb{T};X)$ and their link with the H^∞ functional calculus of the derivation operators. We shall then study the relationships between multipliers on $H^p(\mathbb{R};X)$ and multipliers on $H_0^p(\mathbb{T};X)$ and establish the proof of Theorem 1.2.

Let $1 \leq p < \infty$ be a number and let X be an arbitrary Banach space. Let $\mathscr{P}$ denote the space of all complex trigonometric polynomials, that is, the linear span of the functions $e_k(t) = e^{ikt}$, $k \in \mathbb{Z}$. Then we let $\mathscr{P}^A$ (respectively $\mathscr{P}_0^A$) denote the subspace of $\mathscr{P}$ spanned by $\{e_k : k \geq 0\}$ (respectively $\{e_k : k \geq 1\}$). Using for example Fejér's approximation, we see that $\mathscr{P}^A \otimes X$ and $\mathscr{P}_0^A \otimes X$ are dense subspaces of $H^p(\mathbb{T};X)$ and $H_0^p(\mathbb{T};X)$ respectively. Let $(m_k)_{k\geq 1}$ be a bounded sequence of complex numbers. We say that $(m_k)_{k\geq 1}$ is a bounded Fourier multiplier on $H_0^p(\mathbb{T};X)$ if there exists a constant $C > 0$ such that for any $f = \sum_{k\geq 1} \widehat{f}(k) \otimes e_k \in \mathscr{P}_0^A \otimes X$,

$$(4.1) \qquad \left\| \sum_{k\geq 1} m(k)\widehat{f}(k) \otimes e_k \right\|_p \leq C \left\| \sum_{k\geq 1} \widehat{f}(k) \otimes e_k \right\|_p .$$

In that case there is a (necessarily unique) bounded operator on $H_0^p(\mathbb{T}; X)$ mapping $x \otimes e_k$ to $m(k)x \otimes e_k$ for any $x \in X$ and any $k \geq 1$. Its norm is the least constant C satisfying (4.1) and is called the norm of the Fourier multiplier $(m_k)_{k\geq 1}$ on $H_0^p(\mathbb{T}; X)$. Of course, similar definitions can be given on $H^p(\mathbb{T}; X)$.

We now proceed to Fourier multipliers on $H^p(\mathbb{R}; X)$. We first note that the tensor product $H^p(\mathbb{R}) \otimes X$ is a dense subspace of $H^p(\mathbb{R}; X)$. Indeed the arguments in [17, Chapter 8] showing that $H^p(\mathbb{R})$ and $H^p(\mathbb{T})$ are isometric extend almost verbatim to the vector-valued case giving an isometric isomorphism $J_{p,X} : H^p(\mathbb{R}; X) \to H^p(\mathbb{T}; X)$. Moreover $J_{p,X}$ maps $H^p(\mathbb{R}) \otimes X$ onto $H^p(\mathbb{T}) \otimes X$. Since the latter is dense in $H^p(\mathbb{T}; X)$ (by Fejér's approximation), we deduce the density of $H^p(\mathbb{R}) \otimes X$ in $H^p(\mathbb{R}; X)$. Consequently, $(H^2(\mathbb{R}) \cap H^p(\mathbb{R})) \otimes X$ is a dense subspace of $H^p(\mathbb{R}; X)$.

Let $m : \mathbb{R}_+^* \to \mathbb{C}$ be a bounded measurable function, and let $\tau_m : H^2(\mathbb{R}) \to H^2(\mathbb{R})$ be the associated Fourier multiplier defined by letting $\widehat{\tau_m(f)} = m\widehat{f}$ on $\mathbb{R}_+^*$ for any $f \in H^2(\mathbb{R})$. We say that m is a bounded Fourier multiplier on $H^p(\mathbb{R}; X)$ if τ_m maps $H^2(\mathbb{R}) \cap H^p(\mathbb{R})$ into itself and if there exists a constant $C > 0$ such that for any $\sum_k f_k \otimes x_k$ in $\big(H^2(\mathbb{R}) \cap H^p(\mathbb{R})\big) \otimes X$,

$$(4.2) \qquad \left\| \sum_k \tau_m(f_k) \otimes x_k \right\|_p \leq C \left\| \sum_k f_k \otimes x_k \right\|_p .$$

In other words, m is a bounded Fourier multiplier on $H^p(\mathbb{R}; X)$ if $\tau_m \otimes I_X$ is bounded with respect to the $H^p(\mathbb{R}; X)$-norm. In that case, $\tau_m \otimes I_X$ uniquely extends to a bounded operator on $H^p(\mathbb{R}; X)$ whose norm is the least constant C satisfying (4.2). This norm will be called the norm of the Fourier multiplier m on $H^p(\mathbb{R}; X)$.

LEMMA 4.1. *Let $1 \leq p < \infty$ and consider a function $F \in H^\infty(\Sigma_\theta)$ for some $\theta > \pi/2$.*

(1) *Let $A = d/dt$ on the space $H^p(\mathbb{R}; X)$. Then $F(A)$ is bounded if and only if the function $m : \mathbb{R}_+^* \to \mathbb{C}$ defined by $m(\xi) = F(i\xi)$ is a bounded Fourier multiplier on $H^p(\mathbb{R}; X)$.*

(2) *Let $A = d/dt$ on the space $H_0^p(\mathbb{T}; X)$. Then $F(A)$ is bounded if and only if the sequence $(F(ik))_{k\geq 1}$ is a bounded Fourier multiplier on $H_0^p(\mathbb{T}; X)$.*

PROOF. This result is elementary. Using the definitions above, one can reduce to the scalar case. Then it suffices to apply the formula $\widehat{f'}(\xi) = i\xi\widehat{f}(\xi)$ (in case (1)) or $\widehat{f'}(k) = ik\widehat{f}(k)$ (in case (2)) for suitable functions. $\square$

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LEMMA 4.2. *Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous 2π-periodic function, and let $\varphi \in L^1(\mathbb{R})$ with $\int \varphi(t)\,dt = 1$. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \lim_{\eta\to 0}\eta\int f(t)\varphi(\eta t)\,dt. \tag{4.3}$$

PROOF. By equicontinuity, we may reduce to the case when $f \in \mathcal{P}$ hence by linearity, we may assume that $f = e_k$ for some $k \in \mathbb{Z}$. Since we have

$$\eta\int e_k(t)\varphi(\eta t)\,dt = \widehat{\varphi}\left(\frac{-k}{\eta}\right),$$

the result follows at once. □

The next result extends the well-known fact that if m is a bounded and continuous function on $\mathbb{R}$, then m is a bounded Fourier multiplier on $L^p(\mathbb{R})$ if and only if the sequences $(m(\varepsilon k))_{k\in\mathbb{Z}}$ are uniformly bounded Fourier multipliers on $L^p(\mathbb{T})$. In the latter result, the 'if' part is due to Stein and Weiss [27, VII, Theorem 3.18] whereas the 'only if' part goes back to de Leeuw [21]. Note that de Leeuw's Theorem can be regarded as a consequence of the Coifman-Weiss transference principle (see [4, Chapter 3]). The above mentioned equivalence extends to vector-valued L^p-spaces with identical proofs. To deal with vector-valued H^p-spaces, we will need some substantial modifications that are indicated below. Note for example that the Coifman-Weiss transference principle is no longer available for Hardy spaces.

PROPOSITION 4.3. *Let $1 \le p < \infty$ be a number, let $m : \mathbb{R}_+^* \to \mathbb{C}$ be a bounded and continuous function, and let $C \ge 0$ be a constant. Then the following two assertions are equivalent*:

(i) *m is a bounded Fourier multiplier on $H^p(\mathbb{R}; X)$ whose norm is less than or equal to C*;

(ii) *for any $\varepsilon \in (0, 1]$, the sequence $(m(\varepsilon k))_{k\ge 1}$ is a bounded Fourier multiplier on $H_0^p(\mathbb{T}; X)$ whose norm is less than or equal to C.*

PROOF. We first show that (i) implies (ii), by adapting the proof of [21, Theorem 2.3]. In proving (ii), we may clearly assume that $\varepsilon = 1$. Indeed if m satisfies (i), then the function $\xi \mapsto m(\varepsilon\xi)$ satisfies it as well for any $\varepsilon > 0$. We let $1 < q \le \infty$ be the conjugate number of p, that is, $p^{-1} + q^{-1} = 1$. Suppose that

$$P(t) = \sum_{k\ge 1}\widehat{P}(k)e^{ikt} \quad \text{and} \quad Q(t) = \sum_{n}\widehat{Q}(n)e^{int}. \tag{4.4}$$

are two vector-valued trigonometric polynomials with $P \in \mathcal{P}_0^A \otimes X$ and $Q \in \mathcal{P} \otimes X^*$. Let us consider two (scalar-valued) functions $\gamma_1 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and $\gamma_2 \in L^1(\mathbb{R}) \cap$

$L^q(\mathbb{R})$ which satisfy

$$\|\gamma_1\|_p = \|\gamma_2\|_q = 1 \quad \text{and} \quad \operatorname{Supp}(\widehat{\gamma_1}) \subset [-1, 1]. \tag{4.5}$$

Then for any $\eta \in (0, 1)$, let us define $P_\eta \in L^1(\mathbb{R}; X) \cap L^p(\mathbb{R}; X)$ and $Q_\eta \in L^1(\mathbb{R}; X^*) \cap L^q(\mathbb{R}; X^*)$ by letting $P_\eta(t) = P(t)\gamma_1(\eta t)$ and $Q_\eta(t) = Q(t)\gamma_2(\eta t)$ for any $t \in \mathbb{R}$. Given $k \geq 1$, $\eta \in (0, 1)$, and $\xi \in \mathbb{R}$, the Fourier transform of $t \mapsto e^{ikt}\gamma_1(\eta t)$ at the point ξ equals $(1/\eta)\widehat{\gamma_1}\big((\xi - k)/\eta\big)$. Hence we have

$$\widehat{P_\eta}(\xi) = \frac{1}{\eta} \sum_{k\geq 1} \widehat{P}(k)\widehat{\gamma_1}\left(\frac{\xi - k}{\eta}\right), \quad \xi \in \mathbb{R}. \tag{4.6}$$

For any $\xi \leq 0$ and $k \geq 1$, we have $\xi - k \leq -1$. Hence $(\xi - k)/\eta \leq -1$ and hence $\widehat{\gamma_1}\big((\xi - k)/\eta\big) = 0$. It therefore follows from (4.6) that P_η actually belongs to $H^1(\mathbb{R}; X) \cap L^p(\mathbb{R}; X) \subset H^p(\mathbb{R}; X)$.

Applying the assumption (i), let $T : H^p(\mathbb{R}; X) \to H^p(\mathbb{R}; X)$ be the bounded Fourier multiplier operator induced by m, which we may apply to P_η. We will show that

$$\lim_{\eta\to 0} \eta\langle T(P_\eta),\, Q_\eta\rangle = \langle\gamma_1, \gamma_2\rangle \left\langle \sum_{k\geq 1} m(k)\,\widehat{P}(k) \otimes e_k,\, Q\right\rangle, \tag{4.7}$$

where the brackets in the left-hand side stand for the duality between $L^p(\mathbb{R}; X)$ and $L^q(\mathbb{R}; X^*)$, the first brackets in the right-hand side stand for the duality between $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, and the second brackets in the right-hand side stand for the duality between $L^p(\mathbb{T}; X)$ and $L^q(\mathbb{T}; X^*)$.

Once this is established, one can conclude as follows. On the one hand, we note that

$$\eta \left\| P_\eta \right\|_p^p = \eta \int \|P(t)\|_X^p \, |\gamma_1(\eta t)|^p \, dt \to \frac{1}{2\pi} \int_{-\pi}^{\pi} \|P(t)\|_X^p \, dt = \|P\|_p^p$$

by Lemma 4.2. Similarly, $\eta \left\| Q_\eta \right\|_q^q \to \|Q\|_q^q$ if $q < \infty$, and in the case when $q = \infty$, we have an estimate $\left\| Q_\eta \right\|_\infty \leq \|Q\|_\infty$. Hence, we finally have that

$$\limsup_{\eta\to 0} \eta \left\| P_\eta \right\|_p \left\| Q_\eta \right\|_q \leq \|P\|_p \, \|Q\|_q \,,$$

and hence

$$\limsup_{\eta\to 0} \left|\eta\langle T(P_\eta),\, Q_\eta\rangle\right| \leq C \, \|P\|_p \, \|Q\|_q \,. \tag{4.8}$$

On the other hand, recall that the natural embedding $L^p(\mathbb{T}; X) \hookrightarrow L^q(\mathbb{T}; X^*)^*$ is an isometry (see for example [8, IV.1]). We easily deduce that

$$\left\| \sum_{k\geq 1} m(k)\,\widehat{P}(k) \otimes e_k \right\|_p = \sup \left\{ \left| \left\langle \sum_{k\geq 1} m(k)\,\widehat{P}(k) \otimes e_k,\, Q \right\rangle \right| \right\},$$

where the supremum runs over all $Q \in \mathscr{P} \otimes X^*$ with $\|Q\|_q \leq 1$. Combining with (4.8) and (4.7), we obtain that $|\langle \gamma_1, \gamma_2 \rangle| \left\| \sum_{k \geq 1} m(k) \widehat{P}(k) \otimes e_k \right\|_p \leq C \|P\|_p$. Since the supremum of $|\langle \gamma_1, \gamma_2 \rangle|$ over all γ_1 and γ_2 satisfying (4.5) equals 1, we finally obtain that $(m(k))_{k \geq 1}$ is a bounded Fourier multiplier on $H_0^p(\mathbb{T}; X)$ with norm $\leq C$.

It therefore remains to prove (4.7). Applying (4.6) we see that

$$\widehat{T(P_\eta)}(\xi) = \frac{1}{\eta} \sum_{k \geq 1} \widehat{P}(k) m(\xi) \widehat{\gamma_1}\left(\frac{\xi - k}{\eta}\right), \quad \xi > 0.$$

Furthermore arguing as in the proof of (4.6), we find that

$$\widehat{Q_\eta}(\xi) = \frac{1}{\eta} \sum_{n} \widehat{Q}(n) \widehat{\gamma_2}\left(\frac{\xi - n}{\eta}\right), \quad \xi \in \mathbb{R}.$$

Hence we have for any $\eta \in (0, 1)$

$$\begin{aligned}
\langle T(P_\eta), Q_\eta \rangle &= \frac{1}{2\pi} \left\langle \widehat{T(P_\eta)}, \widehat{Q_\eta}(-\cdot) \right\rangle \\
&= \frac{1}{2\pi \eta^2} \int \left\langle \sum_{k \geq 1} \widehat{P}(k) m(\xi) \widehat{\gamma_1}\left(\frac{\xi - k}{\eta}\right), \sum_{n} \widehat{Q}(n) \widehat{\gamma_2}\left(\frac{-\xi - n}{\eta}\right) \right\rangle d\xi \\
&= \frac{1}{2\pi \eta^2} \sum_{k,n} \left\langle \widehat{P}(k), \widehat{Q}(n) \right\rangle \int m(\xi) \widehat{\gamma_1}\left(\frac{\xi - k}{\eta}\right) \widehat{\gamma_2}\left(\frac{-\xi - n}{\eta}\right) d\xi.
\end{aligned}$$

Let us denote by $I_{k,n}(\eta)$ the integral on the right-hand side of this equality. By the change of variable $s = (\xi + n)/\eta$, we see that

$$I_{k,n}(\eta) = \eta \int m(\eta s - n) \widehat{\gamma_1}\left(s - \frac{n + k}{\eta}\right) \widehat{\gamma_2}(-s)\, ds.$$

In the case when $n = -k$, this integral equals $\eta \int m(k + \eta s) \widehat{\gamma_1}(s) \widehat{\gamma_2}(-s)\, ds$ hence the continuity of m at the point k yields

$$\lim_{\eta \to 0} \frac{1}{\eta} I_{k,-k}(\eta) = m(k) \int \widehat{\gamma_1}(s) \widehat{\gamma_2}(-s)\, ds = 2\pi m(k) \langle \gamma_1, \gamma_2 \rangle.$$

Assume now that $n + k \neq 0$. Since $\mathrm{Supp}(\widehat{\gamma_1}) \subset [-1, 1]$, we have

$$\begin{aligned}
\left| \frac{1}{\eta} I_{k,n}(\eta) \right| &\leq \|m\|_\infty \int \left| \widehat{\gamma_1}\left(s - \frac{n + k}{\eta}\right) \right| \left| \widehat{\gamma_2}(-s) \right| ds \\
&\leq \|m\|_\infty \, \|\widehat{\gamma_1}\|_1 \sup \left\{ \left| \widehat{\gamma_2}(-s) \right| : -1 + \frac{n + k}{\eta} \leq s \leq 1 + \frac{n + k}{\eta} \right\} \\
&\to 0 \quad \text{when} \quad \eta \to 0 \quad \text{since} \quad \lim_{s \to \infty} \widehat{\gamma_2}(s) = 0.
\end{aligned}$$

Combining these estimates, we finally obtain that

$$\begin{aligned}\lim_{\eta\to 0}\eta\langle T(P_\eta),\, Q_\eta\rangle &= \langle\gamma_1,\gamma_2\rangle\sum_{k\geq 1} m(k)\big\langle\widehat{P}(k),\, \widehat{Q}(-k)\big\rangle\\ &= \langle\gamma_1,\gamma_2\rangle\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\langle\sum_{k\geq 1} m(k)\widehat{P}(k)e^{ikt},\sum_n \widehat{Q}(n)e^{int}\right\rangle dt\\ &= \langle\gamma_1,\gamma_2\rangle\left\langle\sum_{k\geq 1} m(k)\widehat{P}(k)\otimes e_k,\, Q\right\rangle,\end{aligned}$$

which completes the proof of (4.7).

We now prove that (ii) implies (i), by adapting the proof of [27, VII, Theorem 3.18] to our analytic setting. In particular we will use the Poisson summation principle. We let $\mathscr{U}$ be the space of all C^∞ functions $f : \mathbb{R}\to\mathbb{C}$ belonging to $H^1(\mathbb{R})$ such that $\lim_{|t|\to\infty}|t^2 f(t)| = 0$. Then according to [12, Chapter II, Corollary 3.3], $\mathscr{U}$ is a dense subspace of $H^p(\mathbb{R})$ for any $1\leq p<\infty$. Therefore, $\mathscr{U}\otimes X$ is dense in $H^p(\mathbb{R};X)$. Hence to prove (i), it suffices to prove (4.2) on $\mathscr{U}\otimes X$.

Let $T=\tau_m\otimes I_X$ be the Fourier multiplier operator corresponding to m defined on $\mathscr{U}\otimes X$ and let f be an arbitrary element of $\mathscr{U}\otimes X$. Since any function in $\mathscr{U}$ is C^∞ hence C^2, we have $\lim_{|\xi|\to\infty}|\xi|^2\,\big\|\widehat{f}(\xi)\big\| = 0$ hence $\lim_{|\xi|\to\infty}|\xi|^2|m(\xi)|\,\big\|\widehat{f}(\xi)\big\| = 0$. By Fourier's inversion formula we deduce that for any $t\in\mathbb{R}$,

$$Tf(t) = \frac{1}{2\pi}\int_0^\infty m(\xi)\widehat{f}(\xi)e^{it\xi}\,d\xi\,.$$

Since m is continuous, we deduce by means of Riemann sums that

$$Tf(t) = \lim_{\varepsilon\to 0}\left(\frac{\varepsilon}{2\pi}\sum_{k=1}^{\infty} m(\varepsilon k)\widehat{f}(\varepsilon k)e^{it\varepsilon k}\right),\qquad t\in\mathbb{R}. \tag{4.9}$$

For any $\varepsilon\in(0,1)$, let F_ε be defined by

$$F_\varepsilon(t) = \frac{2\pi}{\varepsilon}\sum_{j\in\mathbb{Z}} f\left(\frac{t+2\pi j}{\varepsilon}\right),\qquad t\in\mathbb{R}.$$

Since $f\in\mathscr{U}\otimes X$, F_ε is a well-defined continuous function on $\mathbb{R}$. Moreover F_ε is 2π-periodic. Then regarding it as an element of $C(\mathbb{T};X)$, we see that

$$\forall\, k\in\mathbb{Z},\qquad \widehat{F_\varepsilon}(k) = \widehat{f}(\varepsilon k). \tag{4.10}$$

Indeed, this follows from the standard proof of the Poisson summation formula. This shows in particular that $F_\varepsilon\in H_0^p(\mathbb{T};X)$. We now claim that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{p-1}}{(2\pi)^p}\int_{-\pi}^{\pi}\|F_\varepsilon(t)\|^p\,dt = \|f\|_p^p\,. \tag{4.11}$$

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Indeed, by our assumption that $f \in \mathscr{U} \otimes X$, there is a constant $K > 0$ such that $\|f(s)\| \leq K|s|^{-2}$ for any s satisfying $|s| \geq \pi$. Let $t \in [-\pi, \pi]$. For any non-zero integer j and any $\varepsilon \in (0, 1)$, we have $\left|(t + 2\pi j)/\varepsilon\right| \geq \pi$ and hence

$$\left\|f\left(\frac{t+2\pi j}{\varepsilon}\right)\right\| \leq \frac{K\varepsilon^2}{(t+2\pi j)^2} \leq \frac{K\varepsilon^2}{\pi^2(2|j|-1)^2}.$$

We deduce that for some absolute constant $K' > 0$, we have

$$\left\|F_\varepsilon(t) - \frac{2\pi}{\varepsilon} f\left(\frac{t}{\varepsilon}\right)\right\| \leq K'\varepsilon, \quad t \in [-\pi, \pi],\ \varepsilon \in (0, 1).$$

Integrating this inequality and passing to the limit, we obtain that

$$\lim_{\varepsilon\to 0} \varepsilon^{p-1} \int_{-\pi}^{\pi} \left\|F_\varepsilon(t) - \frac{2\pi}{\varepsilon} f\left(\frac{t}{\varepsilon}\right)\right\|^p dt = 0.$$

Our claim (4.11) now follows from the fact that

$$\begin{aligned} \varepsilon^{p-1} \int_{-\pi}^{\pi} \left\|\frac{2\pi}{\varepsilon} f\left(\frac{t}{\varepsilon}\right)\right\|^p dt &= (2\pi)^p \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \|f(s)\|^p\, ds \\ &\to (2\pi)^p \|f\|_p^p \quad \text{when } \varepsilon \to 0. \end{aligned}$$

For any $\varepsilon \in (0, 1)$, we let $T_\varepsilon : H_0^p(\mathbb{T}; X) \to H_0^p(\mathbb{T}; X)$ be the bounded Fourier multiplier operator induced by the sequence $(m(\varepsilon k))_{k\geq 1}$. Then we let $\eta : \mathbb{R} \to \mathbb{R}_+$ be a compactly supported continuous function such that $\eta(0) = 1$ and

$$\sum_{j\in\mathbb{Z}} \eta(t + 2\pi j)^p = 1, \quad t \in \mathbb{R}.$$

(See [27, VII, Lemma 3.21].) Arguing as in [27, page 266], we find that

$$\varepsilon^p \int \|T_\varepsilon F_\varepsilon(\varepsilon t)\eta(\varepsilon t)\|^p\, dt = \varepsilon^{p-1} \int_{-\pi}^{\pi} \|T_\varepsilon F_\varepsilon(y)\|^p\, dy.$$

Hence applying our assumption, we have

$$\varepsilon^p \int \|T_\varepsilon F_\varepsilon(\varepsilon t)\eta(\varepsilon t)\|^p\, dt \leq C^p \varepsilon^{p-1} \int_{-\pi}^{\pi} \|F_\varepsilon(y)\|^p\, dy. \tag{4.12}$$

Combining (4.10) and (4.9), we infer that

$$Tf(t) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{2\pi} T_\varepsilon F_\varepsilon(\varepsilon t)\eta(\varepsilon t), \quad t \in \mathbb{R}.$$

It therefore follows from Fatou's Lemma, (4.12), and (4.11) that

$$\int \|Tf(t)\|^p\, dt \leq C^p \int \|f(t)\|^p\, dt.$$

This concludes the proof that m satisfies (i). □

In proving Theorem 1.2, we will need the following elementary result whose proof is left to the reader.

LEMMA 4.4. *Let $A \subset \mathbb{Z}$ be a finite set of integers, let $s, \gamma > 0$ be two positive numbers, and let $\varepsilon \in \{-1, 1\}$ be a sign. Then there exists an integer $k \geq 1$ such that for any $a \in A$, $a + k \geq 1$ and $|(a+k)^{is} - \varepsilon| < \gamma$.*

PROOF OF THEOREM 1.2. Clearly (ii) implies (iii) and (iv) implies (v). Moreover, (ii) implies (iv) and (iii) implies (v) by Lemma 4.1 and Proposition 4.3. Thus we only have to show that (i) implies (ii) and (v) implies (i).

Assume (i), that is, X is an AUMD Banach space, and let $(m_k)_{k\geq 1}$ be a bounded sequence of complex numbers. Then according to Blower's extension of Mikhlin's Theorem in [1], $(m_k)_{k\geq 1}$ is a bounded Fourier multiplier on $H_0^1(\mathbb{T}; X)$ provided that

$$\begin{aligned} C_1 &= \sup_{k\geq 1} k|m_{k+1} - m_k| < \infty \qquad \text{and} \\ C_2 &= \sup_{k\geq 1} k^2|m_{k+2} - 2m_{k+1} + m_k| < \infty. \end{aligned} \tag{4.13}$$

Moreover, letting $C_0 = \sup_{k\geq 1} |m_k|$, the norm of the Fourier multiplier $(m_k)_{k\geq 1}$ on $H_0^1(\mathbb{T}; X)$ only depends on C_0, C_1, C_2 and on the 'AUMD constant' K_1 appearing in (2.8) for $p = 1$. Furthermore it is easy to check that for any $1 < p < \infty$, Blower's Theorem extends to Fourier multipliers on $H_0^p(\mathbb{T}; X)$ with the same proof. Of course in this case, K_p replaces K_1 in the estimate of the norm of the Fourier multiplier $(m_k)_{k\geq 1}$ on $H_0^p(\mathbb{T}; X)$.

We now prove (ii). Let $1 \leq p < \infty$ and $\theta > \pi/2$ be two numbers, and let $F \in H^\infty(\Sigma_\theta)$. Then according to Lemma 4.1, Proposition 4.3, and the preceding discussion, it suffices to show that for any $\varepsilon \in (0, 1)$, the complex numbers $m_k = F(i\varepsilon k)$ satisfy (4.13) and that the resulting constants C_1, C_2 are uniformly bounded with respect to $\varepsilon \in (0, 1)$. To prove this, we first note the well-known fact that for any $z \in i\mathbb{R}$, $|zF'(z)| \leq K_\theta \|F\|_{\infty,\theta}$ and $|z^2 F''(z)| \leq K_\theta \|F\|_{\infty,\theta}$, for some constant K_θ only depending on θ. Indeed, this follows from Cauchy's Theorem and probably goes back to [5]. Now for any $k \geq 1$ and any $\varepsilon \in (0, 1)$, we have

$$m_{k+1} - m_k = F(i\varepsilon(k+1)) - F(i\varepsilon k) = i\varepsilon \int_0^1 F'(i\varepsilon(k+t))\,dt$$

hence

$$\begin{aligned} k|m_{k+1} - m_k| &\leq k\varepsilon \int_0^1 |F'(i\varepsilon(k+t))|\,dt \\ &\leq k\varepsilon K_\theta \|F\|_{\infty,\theta} \int_0^1 \frac{dt}{\varepsilon(k+t)} \leq K_\theta \|F\|_{\infty,\theta}. \end{aligned}$$

Similarly, we have

$$m_{k+2}-2m_{k+1}+m_k = \big(F(i\varepsilon(k+2))-F(i\varepsilon(k+1))\big)-\big(F(i\varepsilon(k+1))-F(i\varepsilon k)\big)$$
$$= (i\varepsilon)^2\int_0^1\int_0^1 F''(i\varepsilon(k+t+s))\,dt\,ds$$

hence $k^2|m_{k+2}-2m_{k+1}+m_k| \leq K_\theta\|F\|_{\infty,\theta}$. This completes the proof of (ii).

We now assume (v) and prove (i). We will use Bourgain's transference technique introduced in [2]. We note that our proof is close to the Guerre-Delabrière characterization of UMD spaces [14]. Let p and s be given by (v). For any integer $k \geq 1$, $(ik)^{is} = k^{is}e^{-s\pi/2}$ hence by Lemma 4.1, the sequence $(k^{is})_{k\geq 1}$ is a bounded Fourier multiplier on $H_0^p(\mathbb{T};X)$. We denote by $T : H_0^p(\mathbb{T};X) \to H_0^p(\mathbb{T};X)$ the resulting bounded operator. Let $N \geq 1$ be an integer, and for any $1 \leq n \leq N$, let $\Phi_n : \mathbb{T}^{n-1} \to X$ be an X-valued trigonometric polynomial ($\Phi_1 \in X$ being a constant), and let $d_n : \mathbb{T}^n \to X$ be defined by (2.7). Given $\varepsilon_1, \ldots, \varepsilon_N \in \{-1, 1\}$, we aim to prove that

$$\left\|\sum_{n=1}^N \varepsilon_n d_n\right\|_p \leq 2^{1-1/p}\,\|T\|\,\left\|\sum_{n=1}^N d_n\right\|_p, \tag{4.14}$$

where the norms are computed in $L^p(\mathbb{T}^N;X)$. This will show that X is an AUMD Banach space.

For $1 \leq n \leq N$, let $\Lambda(\Phi_n) \subset \mathbb{Z}^{n-1}$ be the spectrum of Φ_n, that is, the support of the Fourier transform of Φ_n. Then $\Lambda(\Phi_n)$ is a finite set and

$$\Phi_n(t_1,\ldots,t_{n-1}) = \sum_{q\in\Lambda(\Phi_n)} \widehat{\Phi}_n(q)e^{iq_1t_1}\cdots e^{iq_{n-1}t_{n-1}}, \qquad t_1,\ldots,t_{n-1}\in\mathbb{T}. \tag{4.15}$$

We let $\delta > 0$ be an arbitrary positive number. Then we define (by induction) a sequence $k_1, \ldots, k_N$ of positive integers as follows. We let $C_n = \sum_{q\in\Lambda(\Phi_n)}\big\|\widehat{\Phi}_n(q)\big\|$ for $1 \leq n \leq N$. We first choose $k_1 \geq 1$ such that

$$|k_1^{is}-\varepsilon_1| \leq \frac{\delta}{NC_1}.$$

Then we assume that $2 \leq n \leq N$ and that $k_1, \ldots, k_{n-1}$ have been chosen. We let

$$A_n = \left\{\sum_{j=1}^{n-1} q_jk_j \,:\, q = (q_1,\ldots,q_{n-1}) \in \Lambda(\Phi_n)\right\}.$$

Then A_n is a finite subset of $\mathbb{Z}$ and so applying Lemma 4.4 with $A = A_n$ and $\varepsilon = \varepsilon_n$, we choose $k_n \geq 1$ such that

$$\text{if } \sum_{j=1}^{n-1} q_jk_j \in A_n, \quad \text{then } k_n + \sum_{j=1}^{n-1} q_jk_j \geq 1 \tag{4.16}$$

and

$$\left| \left(k_n + \sum_{j=1}^{n-1} q_j k_j \right)^{is} - \varepsilon_n \right| \leq \frac{\delta}{N C_n}.$$

Thus for any $1 \leq n \leq N$, we have the following estimate

$$\sum_{q \in \Lambda(\Phi_n)} \left| \left(k_n + \sum_{j=1}^{n-1} q_j k_j \right)^{is} - \varepsilon_n \right| \|\widehat{\Phi}_n(q)\| \leq \frac{\delta}{n}. \tag{4.17}$$

We fix $t_1, \ldots, t_N \in \mathbb{T}$ and, for any $1 \leq n \leq N$, we introduce $\Delta_n : \mathbb{T} \to X$ by letting $\Delta_n(t) = d_n(t_1 + k_1 t, \ldots, t_n + k_n t)$ for any $t \in \mathbb{T}$. Then we have

$$\begin{aligned}
\Delta_n(t) &= \Phi_n(t_1 + k_1 t, \ldots, t_{n-1} + k_{n-1} t) e^{i(t_n + k_n t)} \qquad \text{by (2.7)} \\
&= \sum_{q \in \Lambda(\Phi_n)} \widehat{\Phi}_n(q)\, e^{iq_1(t_1 + k_1 t)} \cdots e^{iq_{n-1}(t_{n-1} + k_{n-1} t)} e^{i(t_n + k_n t)} \qquad \text{by (4.15)} \\
&= \sum_{q \in \Lambda(\Phi_n)} \widehat{\Phi}_n(q)\, e^{iq_1 t_1} \cdots e^{iq_{n-1} t_{n-1}} e^{it_n} e^{i\left(k_n + \sum_{j=1}^{n-1} q_j k_j\right) t}.
\end{aligned}$$

Looking at (4.16), we see that Δ_n is an analytic polynomial without constant term (that is, $\Delta_n \in \mathscr{P}_0^A \otimes X$) and applying the Fourier multiplier operator T, we obtain that

$$T\Delta_n(t) = \sum_{q \in \Lambda(\Phi_n)} \widehat{\Phi}_n(q) e^{iq_1 t_1} \cdots e^{iq_{n-1} t_{n-1}} e^{it_n} \left(k_n + \sum_{j=1}^{n-1} q_j k_j \right)^{is} e^{i\left(k_n + \sum_{j=1}^{n-1} q_j k_j\right) t}, \quad t \in \mathbb{T}.$$

It therefore follows from (4.17) that for any $1 \leq n \leq N$ and any $t \in \mathbb{T}$,

$$\| T\Delta_n(t) - \varepsilon_n \Delta_n(t) \| \leq \delta / N$$

whence $\left\| T\left(\sum_{n=1}^N \Delta_n\right)(t) - \sum_{n=1}^N \varepsilon_n \Delta_n(t) \right\| \leq \delta$, $t \in \mathbb{T}$. Integrating on $\mathbb{T}$ yields $\left\| T\left(\sum_{n=1}^N \Delta_n\right) - \sum_{n=1}^N \varepsilon_n \Delta_n \right\|_p \leq \delta$, hence $\left\| \sum_{n=1}^N \varepsilon_n \Delta_n \right\|_p \leq \delta + \|T\| \left\| \sum_{n=1}^N \Delta_n \right\|_p$, whence

$$\left\| \sum_{n=1}^N \varepsilon_n \Delta_n \right\|_p^p \leq 2^{p-1} \left(\delta^p + \|T\|^p \left\| \sum_{n=1}^N \Delta_n \right\|_p^p \right).$$

More explicitly,

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left\| \sum_{n=1}^N \varepsilon_n d_n(t_1 + k_1 t, \ldots, t_n + k_n t) \right\|^p \frac{dt}{2\pi} \\
&\qquad \leq 2^{p-1} \left(\delta^p + \|T\|^p \int_{-\pi}^{\pi} \left\| \sum_{n=1}^N d_n(t_1 + k_1 t, \ldots, t_n + k_n t) \right\|^p \frac{dt}{2\pi} \right).
\end{aligned}$$

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Now by integrating the latter inequality with respect to $(t_1, \dots, t_N) \in \mathbb{T}^N$ and applying Fubini's Theorem, we deduce that

$$\left\| \sum_{n=1}^{N} \varepsilon_n d_n \right\|_p^p \leq 2^{p-1} \left(\delta^p + \|T\|^p \left\| \sum_{n=1}^{N} d_n \right\|_p^p \right).$$

Since $\delta > 0$ is arbitrary, we finally obtain (4.14), which completes the proof. □

REMARK 4.5. Let X be any Banach space and consider the sectorial operator d/dt either on $H^p(\mathbb{R}; X)$ or on $H_0^p(\mathbb{T}; X)$. Then it is easy to see that for any $\omega > 0$, the operator $-id/dt$ is sectorial of type ω. Moreover it is clear from the proof of Theorem 1.2 that in the case when X is AUMD, the operator $-id/dt$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > 0$.

REMARK 4.6. It was first established by Lust-Piquard and independently by Drury (see [22]) that there exist bounded Fourier multipliers on $H_0^1(\mathbb{T})$ which are not bounded on $H_0^1(\mathbb{T}; S^1)$. Since S^1 is not AUMD [15], Theorem 1.2 yields several explicit examples. In particular we recover Drury's example, namely for any $s \in \mathbb{R}^*$, the sequence $(k^{is})_{k\geq 1}$ is not a bounded Fourier multiplier on $H_0^1(\mathbb{T}; S^1)$. We note that the latter fact can also be easily deduced from [24, Theorem 6.2]. More generally, if $(m_k)_{k\geq 1}$ is a bounded non-converging sequence and if $\lim_{k\to\infty}(m_{k+1} - m_k) = 0$, then $(m_k)_{k\geq 1}$ cannot be a bounded Fourier multiplier on $H_0^1(\mathbb{T}; S^1)$. Indeed, let $(m_{k_p})_{p\geq 1}$ and $(m_{l_q})_{q\geq 1}$ be two converging subsequences of $(m_k)_{k\geq 1}$, with distinct limits β_1 and β_2. Assume that $(m_k)_{k\geq 1}$ is a bounded Fourier multiplier on $H_0^1(\mathbb{T}; S^1)$. Then according to [24, Theorem 6.2], there exist a Hilbert space H and two bounded sequences $(x_k)_{k\geq 1}$ and $(y_l)_{l\geq 1}$ in H such that $m_{k+l} = \langle x_k, y_l \rangle$ for any $k, l \geq 1$. In particular, we may write $m_{k_p+l_q} = \langle x_{k_p}, y_{l_q} \rangle$ for any $p, q \geq 1$. For any $q \geq 1$, $\lim_{p\to\infty}(m_{k_p+l_q} - m_{k_p}) = 0$ hence $\lim_{p\to\infty} m_{k_p+l_q} = \beta_1$. Similarly, $\lim_{q\to\infty} m_{k_p+l_q} = \beta_2$ for any $p \geq 1$. Now if x and y are weak cluster points of the bounded sequences $(x_{k_p})_{p\geq 1}$ and $(y_{l_q})_{q\geq 1}$, we see that for any $q \geq 1$, $\beta_1 = \lim_{p\to\infty}\langle x_{k_p}, y_{l_q} \rangle = \langle x, y_{l_q} \rangle$ hence $\beta_1 = \langle x, y \rangle$. Likewise, $\beta_2 = \langle x, y \rangle$, which gives a contradiction.

REMARK 4.7. Let X be a Banach space, and let $H_{at}^1(\mathbb{R}; X)$ be the X-valued atomic H^1-space. Then the translation semigroup is well-defined and isometric on $H_{at}^1(\mathbb{R}; X)$. We may then define d/dt on $H_{at}^1(\mathbb{R}; X)$ as its negative generator, and consider the question whether it admits a bounded H^∞ functional calculus. Using Calderon-Zygmund Theory for vector-valued L^p-spaces, one can show that d/dt has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \pi/2$ if and only if d/dt has bounded imaginary powers if and only if X is UMD. We omit the details.

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