A COMBINATORIAL INTERPRETATION OF RAMANUJAN'S CONTINUED FRACTION

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The purpose of the present note is to give a combinatorial interpretation of the coefficients of expansion of the Ramanujan continued fraction ([1], p. 295)

$$\frac{x}{1+}\ \frac{xt}{1+}\ \frac{xt^2}{1+}\ \frac{xt^3}{1+}\ \cdots \quad .$$

The result is expressed by formula (12) below.

The enumeration of distinct score vectors of a tournament leads to the following problem: (Erdős and Moser, see Moon [2], p. 68). Given $n \geq 1$, $k \geq 0$, determine the number of distinct sequences of positive integers

(1) $$a_1 \leq a_2 \leq \ldots \leq a_n = n$$

satisfying

(2) $$a_i \geq i \quad \text{for} \quad 1 \leq i < n$$

and

(3) $$a_1 + \ldots + a_{n-1} = \binom{n}{2} + k.$$

Denote by $A(n, k)$ this number and set

(4) $$A(0, 0) = 1.$$

Clearly

(5) $$A(n, 0) = A(n, \binom{n}{2}) = 1, \quad A(n, k) = 0 \text{ for } k > \binom{n}{2}.$$

Let $B(n, k)$, $n \geq 1$, $k \geq 0$ be the number of sequences of integer

(1') $$b_1 \leq b_2 \leq \ldots \leq b_n = n$$

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with

(2') $$b_i > i \quad \text{for} \quad 1 \le i < n$$

and

(3') $$b_1 + \ldots + b_{n-1} = \binom{n}{2} + k.$$

Note that

(5') $$B(n, k) = 0 \quad \text{for} \quad 0 \le k < n - 1 \text{ and for } k > \binom{n}{2},$$
$$B(n, n-1) = B(n, \binom{n}{2}) = 1,$$

also

(4') $$B(1, 0) = 1.$$

Since $b_{n-1} = n$ from (1') and (2'), $d_i = b_i - 1$ $(1 \le i < n)$ satisfies the conditions (1), (2) and (3) with $n - 1$, $k - n + 1$ instead of n, k, and conversely. Hence

(6) $$B(n, k) = A(n-1, k - n + 1) \; (k \ge n - 1 \ge 0),$$

and trivially

(7) $$A(n, k) = \sum_{r=0}^{n} \sum_{\substack{n_1 + \ldots + n_r = n \\ k_1 + \ldots + k_r = k}} B(n_i, k_i).$$

Set

(8) $$F(\mathbf{x}, t) = \sum_{\substack{n \ge 0 \\ k \ge 0}} A(n, k)\, \mathbf{x}^n t^k,$$

(9) $$f(\mathbf{x}, t) = \sum_{0 \le n-1 \le k} B(n, k)\, \mathbf{x}^n t^k.$$

Then by (6) and (7)

$$F(\mathbf{x}, t) = \sum_{r=0}^{\infty} (f(\mathbf{x}, t))^r = \frac{1}{1 - f(\mathbf{x}, t)},$$

$$f(x,t) = \sum_{k \geq n-1 \geq 0} A(n-1, k-n+1)\, x^n t^k$$

$$= x \sum_{k \geq n-1 \geq 0} A(n-1, k-n+1)\, (xt)^{n-1}\, t^{k-n+1}$$

$$= x\, F(xt, t),$$

giving

(10) $$F(x,t) = \frac{1}{1-x\,F(xt,t)} = \frac{1}{1-}\,\frac{x}{1-}\,\frac{xt}{1-}\,\frac{xt^2}{1-}\,\cdots$$

$$= \sum A(n,k)\, x^n\, {}^k$$

and

(11) $$f(x,t) = \frac{x}{1-}\,\frac{xt}{1-}\,\frac{xt^2}{1-}\,\cdots\quad .$$

In particular we have for the Ramanujan fraction

(12) $$\frac{x}{1+}\,\frac{xt}{1+}\,\frac{xt^2}{1+} = \sum_{0 \leq n-1 \leq k} (-1)^{n-1}\, B(n,k)\, x^n t^k \quad ,$$

where $B(n,k)$ is the number of solutions of (1'), (2') and (3').

Observe that from (10)

$$F(x,1) = \frac{1}{1-x\,F(x,1)} = \frac{1}{2x}\,(1-(1-4x)^{1/2});$$

hence by (8)

$$\sum_{k \geq 0} A(n,k) = \frac{1}{n+1}\binom{2n}{n}$$

which is indeed the total number of integer solutions of

$$a_1 \leq a_2 \leq \ldots \leq a_n = n,$$

$$a_i \geq i \quad \text{for} \quad 1 \leq i < n.$$

Similarly

$$F(x,-1) = \frac{1}{1-x\,F(-x,-1)} = \frac{1}{1-\dfrac{x}{1+x\,F(x,-1)}}$$

$$= \frac{1+x\,F(x,-1)}{1-x+x\,F(x,-1)} = \frac{1}{2x}\,(-1+2x+(1+4x^2)^{1/2});$$

hence, for $n > 0$,

$$\sum_k A(2n, 2k) - \sum_k A(2n, 2k+1) = 0,$$

and

$$\sum_k A(2n+1, 2k) - \sum_k A(2n+1, 2k+1) = \frac{(-1)^n}{n+1}\binom{2n}{n}.$$

Generally one can calculate the explicit value of $F(x,\zeta)$ for any root of unity ζ. For example if $\zeta^3 = 1$, $\zeta \neq 1$ then

$$F(x,\zeta) = \frac{\zeta(1-(1-4x^3)^{1/2})}{2x(1-x)} + \frac{1}{1-x}.$$

It would be interesting to find an asymptotic expression for $B(n, k)$ when k is large and n varies from 1 to k. It seems likely that for fixed k, $B(n, k)$ increases monotonically to a maximum as n varies from 1 to some $\mu(k) < k$ and then decreases monotonically as n varies from $\mu(k)$ to k. However not even the approximate value of $\mu(k)$ is known.

REFERENCES

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2. J.W. Moon, Topics on tournaments. (Holt, Rinehart and Winston, New York, 1968).

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