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COINITIAL GRAPHS AND WHITEHEAD AUTOMORPHISMS

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Coinitial graphs were used in [**2**; **3**; **4**] as a combinatorial tool in the Reidemeister-Schreier process in order to prove subgroup theorems for Fuchsian groups. Whitehead had previously introduced such graphs but used topological methods for his proofs [**8**; **9**]. Subsequently Rapaport [**7**] and Higgins and Lyndon [**1**] gave algebraic proofs of the results in [**9**], and McCool [**5**; **6**] has further developed these methods so that presentations of automorphism groups could be found.

In this paper it is shown that Whitehead automorphisms can be described by a "cutting and pasting" operation on coinitial graphs. Section 1 defines and gives some combinatorial properties of these operations, based on [**1**]. Section 2 gives algebraic properties, based on [**5**]. Section 3 gives a unified proof and extension of the results of [**1**], [**5**] and [**6**].

I would like to thank the referee for his many valuable suggestions, particularly in the wording of the first two pages.

1. Let X be a set of letters with a fixed involution $x \mapsto x^{-1}$, where x may be equal to x^{-1} and where for convenience we will use the term involution to include the case in which $x = x^{-1}$ for all x. Let W be a set of reduced cyclic words in X. With the pair (X, W) we associate a directed graph Γ with vertex set X and with directed edges defined as follows. With each occurrence of a letter x in an element w of the set W we associate a directed edge $e : x \mapsto y$ where y^{-1} is the letter which occurs in w immediately following the given occurrence of x. If w is of length one then x immediately follows itself in the cyclic word giving an edge $e : x \mapsto x^{-1}$. This will be a loop if $x = x^{-1}$, but apart from this Γ has no loops.

We define also a one-to-one map ϕ of the edges of Γ to themselves which takes an edge $e_1 : x \mapsto y$ associated with an occurrence of x to the edge $e_2 : y^{-1} \mapsto z$ associated with the occurrence of y^{-1} immediately following the given occurrence of x.

Definition. A *coinitial graph* is a pair (Γ, ϕ) where Γ is a directed graph whose vertex set X has an involution $x \mapsto x^{-1}$ as above, and where ϕ is a one-to-one map of the edges of Γ taking each edge ending at each vertex y to an edge beginning at y^{-1}.

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Clearly each orbit under ϕ determines a succession of letters in X, and if each orbit is finite then from a coinitial graph (Γ, ϕ) we can recover a unique pair (X, W) with which it is associated.

It should be noted that the present concept of coinitial graph is an extension of that used in [**2**; **3**; **4**].

If A and B are subsets of X, not necessarily disjoint, then by an edge between A and B we will mean an edge $e : a \mapsto b$ or an edge $e : b \mapsto a$ where $a \in A$ and $b \in B$. A' will denote the complement of A in X, and a will as usual be used to denote either the element a or the set $\{a\}$ according to the context.

We now define an operation on coinitial graphs analogous to the well known cutting and pasting of a fundamental region of a Fuchsian group. Let $a \neq a^{-1}$ be an element of X and A a subset of X containing a but not a^{-1}. The operation (A, a) on a coinitial graph (Γ, ϕ) is defined in three stages.

(i) Replace each edge e in Γ between $x \in A$ and $y^{-1} \in A'$ by edges e_1 between x and α and e_2 between α^{-1} and y^{-1}, where α and α^{-1} are new and distinct vertices. The directions of e_1 and e_2 are defined by $e_1 : x \mapsto \alpha$ and $e_2 : \alpha^{-1} \mapsto y^{-1}$ if $e : x \mapsto y^{-1}$, and by $e_2 : y^{-1} \mapsto \alpha^{-1}$ and $e_1 : \alpha \mapsto x$ if $e : y^{-1} \mapsto x$. In the first case we modify ϕ by defining $\phi(e_1) = e_2$ and replacing $\phi(e') = e$ by $\phi(e') = e_1$ and $\phi(e) = e''$ by $\phi(e_2) = e''$. In the second case ϕ is modified by defining $\phi(e_2) = e_1$ and replacing $\phi(e'') = e$ by $\phi(e'') = e_2$ and $\phi(e) = e'$ by $\phi(e_1) = e'$.

(ii) Relabel α, α^{-1} as a, a^{-1} and conversely.

(iii) Do the converse of operation (i) with the new α and α^{-1}. That is, remove the vertices α and α^{-1} and replace each pair of edges at α and α^{-1} which correspond under ϕ by a single edge with suitable direction and adjust ϕ accordingly.

We will refer to operation (i) as cutting the edges between A and A' and joining them to α and α^{-1}, and to operation (iii) as joining the edges at α and α^{-1}. Clearly after cutting the edges between A and A' we have two disjoint graphs and operation (iii) joins edges in distinct components so no loops can be created or destroyed by (A, a).

We will write the operation (A, a) on the right so that $(A, a)(B, b)$ will denote (A, a) followed by (B, b). The conditions $a \in A$ and $a^{-1} \notin A$ will be assumed whenever we write (A, a). We will use A^ϵ to denote A if $\epsilon = 1$ and to denote A', if $\epsilon = -1$, so that (A, a) is defined if and only if $a^\epsilon \in A^\epsilon$ for $\epsilon = \pm 1$. We will also, as is usual, use $A + B$ to denote $A \cup B$ when $A \cap B = \emptyset$, and $A - B$ to denote $A \cap B'$ when $A \supseteq B$. In longer formulae the operations of plus and minus for sets are performed from left to right. Let T denote the set of all operations (A, a).

Suppose now that $|\Gamma|$, the number of edges of Γ, is finite. Let σ be any finite succession of operations in T, then $\Gamma\sigma$ also has a finite number of edges and we write $\Delta_\Gamma\sigma$, or simply $\Delta\sigma$, for $|\Gamma\sigma| - |\Gamma|$. Let A and B be any two subsets of X, then we define $A._\Gamma B$, or $A.B$ when Γ is understood, to be the number of edges in Γ between A and B.

We have immediately that

$$\begin{aligned} &a.a = 0 \quad \text{if } a \neq a^{-1},\\ &A.B = B.A \geqq 0,\\ &(A + B).C = A.C + B.C,\\ &(B - A).C = B.C - A.C,\\ &a.X = a^{-1}.X \end{aligned}$$

Also operation (i) above replaces each edge between A and A' by two edges thereby increasing the number of edges by $A.A'$, while operation (iii) replaces pairs of edges at α and α^{-1}, which were previously the edges at a and a^{-1}, by single edges thereby decreasing the number by $a.X$, hence

$$\Delta(A, a) = A.A' - a.X = \Delta(A', a^{-1}).$$

Let $P_{\epsilon,\zeta}$ denote $A^\epsilon \cap B^\zeta$. Then $A^\epsilon = \sum_\zeta P_{\epsilon,\zeta}$, $B^\zeta = \sum_\epsilon P_{\epsilon,\zeta}$ and $X = \sum_{\epsilon,\zeta} P_{\epsilon,\zeta}$, where summation is over ϵ, $\zeta = \pm 1$. Hence

$$\begin{aligned} A.A' = A^\epsilon.A^{-\epsilon} &= (P_{\epsilon,\zeta} + P_{\epsilon,-\zeta}).(P_{-\epsilon,\zeta} + P_{-\epsilon,-\zeta})\\ &\geqq P_{\epsilon,-\zeta}.(P_{-\epsilon,\zeta} + P_{-\epsilon,-\zeta}) + P_{-\epsilon,\zeta}.P_{\epsilon,\zeta} \end{aligned}$$

and

$$\begin{aligned} B.B' = B^\zeta.B^{-\zeta} &= (P_{\epsilon,\zeta} + P_{-\epsilon,\zeta}).(P_{\epsilon,-\zeta} + P_{-\epsilon,-\zeta})\\ &\geqq P_{\epsilon,-\zeta}.P_{\epsilon,\zeta} + P_{-\epsilon,\zeta}.(P_{\epsilon,-\zeta} + P_{-\epsilon,-\zeta,}). \end{aligned}$$

Thus

$$\begin{aligned} (1)\quad A.A' + B.B' &\geqq P_{\epsilon,-\zeta}.(P_{\epsilon,\zeta} + P_{-\epsilon,\zeta} + P_{-\epsilon,-\zeta})\\ &+ P_{-\epsilon,\zeta}.(P_{\epsilon,\zeta} + P_{\epsilon,-\zeta} + P_{-\epsilon,-\zeta}) = P_{\epsilon,-\zeta}.P'_{\epsilon,-\zeta} + P_{-\epsilon,\zeta}.P'_{-\epsilon,\zeta}. \end{aligned}$$

By symmetry

$$A.A' + B.B' \geqq P_{-\epsilon,-\zeta}.P_{-\epsilon,-\zeta}' + P_{\epsilon,\zeta}.P_{\epsilon,\zeta}'$$

and so

$$(2)\quad 2(A.A' + B.B') \geqq \textstyle\sum_{\epsilon,\zeta} P_{\epsilon,\zeta}.P_{\epsilon,\zeta}'.$$

LEMMA 1. *If $A \cap B = \emptyset$ and $a^{-1} \notin B$ then for all finite* Γ

$$|\Gamma(A, a)(B, b)| - |\Gamma(A, a)| = |\Gamma(B, b)| - |\Gamma|.$$

Proof. Consider the effect of (A, a) on edges between B and B' in Γ.

Since $B \subseteq A'$, the only edges between B and B' which are affected by operation (i) are those between B and A and these are replaced by an equal number of edges between B and α^{-1} and between α and A. Therefore the number of edges between B and $B' + \alpha + \alpha^{-1}$ is the same as between B and B' in Γ.

Since a and a^{-1} are in B' operation (ii) does not change the label of any vertex of B, hence the number of edges we are concerned with does not alter.

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Since the new α and α^{-1} are not in B, reversing the argument for operation (i) shows that the number of edges we are considering remains unaltered.

We have shown therefore that the number of edges between B and B' is the same in $\Gamma(A, a)$ as in Γ. Applying the same argument with b for B and using the equations above gives

$$|\Gamma(A, a)(B, b)| - |\Gamma(A, a)| = B.B' - b.X = |\Gamma(B, b)| - |\Gamma|.$$

LEMMA 2. *Let* Γ *and* Γ_i, $i \in I$, *be finite coinitial graphs, where* I *is a set inductive under a relation* $>$. *Suppose that, for each* $i \in I$, $|\Gamma_i|$ *is minimal under operations in* T *which do not increase* $|\Gamma_j|$ *for* $j < i$. *Suppose also that* (A, a) *and* (B, b) *satisfy*

$$\Delta_i(A, a) = \Delta_i(B, b) = 0, \quad \textit{for all } i \in I, \textit{ where } \Delta_i = \Delta_{\Gamma_i}, \textit{ and}$$
$$\Delta(A, a) \leqq 0, \; \Delta(B, b) \leqq 0,$$

with at least one of these inequalities strict. Then for some $p = a^{\pm 1}, b^{\pm 1}$

$$\Delta_i(P, p) = 0, \quad \textit{for all } i \in I, \textit{ and}$$
$$\Delta(P, p) < 0,$$

where P *denotes the set* $A^\epsilon \cap B^\zeta$ *containing* p.

Moreover if $(Y, y) = (A^{-\epsilon}, a^{-\epsilon})$ *or* $(B^{-\zeta}, b^{-\zeta})$ *then*

$$\Delta_i\sigma = 0, \; \textit{for all } i \in I, \textit{ and}$$
$$\Delta\sigma < 0,$$

where

$$\sigma = \begin{cases} (Y, y)(P, p) & \textit{if } y^{-1} \notin P, \\ (P, p)(Y, y) & \textit{if } p^{-1} \notin Y, \\ \textit{either } (P, y^{-1}) \textit{ or } (Y, p^{-1}) & \textit{otherwise.} \end{cases}$$

Proof. For $x = a^{\pm 1}, b^{\pm 1}$ let $P(x)$ denote the set $A^\epsilon \cap B^\zeta$ containing x. If the four sets $P(a)$, $P(a^{-1})$, $P(b)$ and $P(b^{-1})$ are all distinct then

$$\begin{aligned} \sum_x \Delta(P(x), x) &= \sum_{\epsilon,\zeta} P_{\epsilon,\zeta}.P_{\epsilon,\zeta}' - 2a.X - 2b.X \\ &\leqq 2A.A' + 2B.B' - 2a.X - 2b.X, \quad \text{by (2)}, \\ &< 0 \quad \text{by hypothesis.} \end{aligned}$$

Hence $\Delta(P, p) < 0$ for some $p = a^{\pm 1}, b^{\pm 1}$ where $P = P(p)$. Similarly

$$\sum_x \Delta_i(P(x), x) \leqq 0 \quad \text{for all } i \in I.$$

Hence by the given conditional minimality and using induction on I we have

$$\Delta_i(P, p) = 0 \quad \text{for all } i \in I.$$

If there are three or less distinct sets $P(x)$ for $x = a^{\pm 1}, b^{\pm 1}$ then for some ϵ and ζ, $A^\epsilon \cap B^\zeta$ does not contain any of a, a^{-1}, b or b^{-1}. In particular it does not

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contain a^ϵ or b^ζ, so

$$P(a^\epsilon) = A^\epsilon \cap B^{-\zeta} = P_{\epsilon,-\zeta} \quad \text{and} \quad P(b^\zeta) = A^{-\epsilon} \cap B^\zeta = P_{-\epsilon,\zeta}.$$

Hence

$$\begin{aligned}\Delta(P(a^\epsilon), a^\epsilon) + \Delta(P(b^\zeta), b^\zeta) &= P_{\epsilon,-\zeta}.P_{\epsilon,-\zeta}' - a^\epsilon.X + P_{-\epsilon,\zeta}.P_{-\epsilon,\zeta}' \\ &\qquad - b^\zeta.X \\ &\leqq A.A' - a.X + B.B' - b.X, \quad \text{by (1)} \\ &< 0 \quad \text{by hypothesis.}\end{aligned}$$

Similarly

$$\Delta_i(P(a^\epsilon), a^\epsilon) + \Delta_i(P(b^\zeta), b^\zeta) \leqq 0 \quad \text{for all } i \in I.$$

As above it follows that $\Delta(P, p) < 0$ and $\Delta_i(P, p) = 0$ for $(P, p) = (P(a^\epsilon), a^\epsilon)$ or $(P(b^\zeta), b^\zeta)$ and for all $i \in I$. This completes the proof of the first part.

By a change of notation take $P = A^\epsilon \cap B^\zeta$. Then for $Y = A^{-\epsilon}$, or $Y = B^{-\zeta}$, $P \cap Y = \emptyset$ so if $y^{-1} \notin P$ we have

$$\begin{aligned}\Delta\sigma &= |\Gamma(Y, y)(P, p)| - |\Gamma(Y, y)| + |\Gamma(Y, y)| - |\Gamma| \\ &= |\Gamma(P, p)| - |\Gamma| + |\Gamma(Y, y)| - |\Gamma| \quad \text{by Lemma 1} \\ &= \Delta(P, p) + \Delta(Y, y) < 0, \quad \text{since } \Delta(Y, y) \leqq 0 \text{ by assumption.}\end{aligned}$$

Similarly, $\Delta_i\sigma = 0$, since $\Delta_i(Y, y) = 0$.

The proof for $p^{-1} \notin Y$ is similar.

If $y^{-1} \in P$ and $p^{-1} \in Y$, then (P, y^{-1}) and (Y, p^{-1}) are defined and

$$\begin{aligned}\Delta(Y, p^{-1}) + \Delta(P, y^{-1}) &= Y.Y' - p.X + P.P' - y.X \\ &= \Delta(Y, y) + \Delta(P, p) \leqq 0.\end{aligned}$$

Similarly, $\Delta_i(Y, p^{-1}) + \Delta_i(P, y^{-1}) = 0$. As before this gives $\Delta\sigma < 0$ and $\Delta_i\sigma = 0$ either for $\sigma = (Y, p^{-1})$ or for $\sigma = (P, y^{-1})$.

If $p = y^{-1}$ take $\sigma = (P, y^{-1})$.

When I is empty this reduces to the lemma of [**1**].

2. We now consider the transformation on the set W corresponding to the operation (A, a) on the coinitial graph of W.

Let there be given any $w \in W$ of length greater than one and an occurrence of x^{-1} in w, say $\ldots yx^{-1}z \ldots$. Suppose $y \neq z^{-1}$, and let W' be the set of reduced cyclic words obtained by deleting the occurrence of x^{-1} from w. It follows from the definition that the coinitial graph (Γ', ϕ') of W' is obtained from (Γ, ϕ) by deleting the corresponding pair of distinct edges

$$e_1 : y \mapsto x \quad \text{and} \quad e_2 : x^{-1} \mapsto z^{-1}$$

from Γ, and replacing them by a new edge $e : y \mapsto z^{-1}$ which is not a loop since by assumption $y \neq z^{-1}$. ϕ' is obtained by adjusting ϕ in the obvious way, that is by deleting e_1 and e_2 from the domain and range of ϕ and substituting the new edge e for them, ϕ' being otherwise the same as ϕ. This deletion operation for coinitial graphs was used in [**2**], [**3**] and [**4**] to obtain subgroup theorems.

Comparing this with stage (iii) in the operation (A, a) and comparing its converse with stage (i) we have that the transformations on the set W corresponding to the three stages of (A, a) are:

(i)′ For every occurrence of $\ldots yz \ldots$ or $\ldots z^{-1}y^{-1} \ldots$ with $y \in A$ and $z^{-1} \notin A$, insert α^{-1} or α respectively,
(ii)′ Substitute a, a^{-1} for α, α^{-1} and conversely,
(iii)′ Delete all occurrences of α and α^{-1}.

It is now easily checked using (i)′ (ii)′ and (iii)′ that the following lemma holds.

LEMMA 3. *The operation (A, a) is equivalent to the transformation of W given by mapping each $x \in X$ as follows:*

$$a \leftrightarrow a^{-1},$$
$$x \mapsto axa^{-1} \quad \textit{if } x \in A \textit{ and } x^{-1} \in A,$$
$$x \mapsto xa^{-1} \quad \textit{if } a \neq x \in A \textit{ and } x^{-1} \notin A,$$
$$x \mapsto ax \quad \textit{if } a^{-1} \neq x \notin A \textit{ and } x^{-1} \in A,$$
$$x \mapsto x \quad \textit{if } x \notin A \textit{ and } x^{-1} \notin A,$$

and deleting any $a^{-1}a$ in the resulting cyclic words.

Note that this transformation differs from the Whitehead automorphism only in taking a, a^{-1} to a^{-1}, a. If A is the subset $\{a\}$, then we have that (a, a) is equivalent to the transformation $a \leftrightarrow a^{-1}$.

We now introduce the notation $(a \leftrightarrow b)$, if $a \neq a^{-1} \neq b^{\pm 1}$ and $b \neq b^{-1}$, for interchanging the labels of the vertices a and b and those of a^{-1} and b^{-1}, and interchanging similarly all occurrences of a, a^{-1}, b, b^{-1} in the reduced words. Clearly $(a^{-1} \leftrightarrow b^{-1})$ denotes the same operation as $(a \leftrightarrow b)$.

LEMMA 4. *If $b \neq a$ the following relations hold.*
R1. $(A, a) = (A, a)^{-1}$, $(x \leftrightarrow y) = (x \leftrightarrow y)^{-1}$,
R2. $(A, a) = (A', a^{-1})$,
R3. $(a, a) = (a^{-1}, a^{-1})$,
R4. $(A, a)\pi = \pi(A\pi, a\pi)$, $\pi = (x \leftrightarrow y)$ *or* (x, x),
R5. $(A, a)(A, b) = (A, b)(b \leftrightarrow a)$.
If $A \cap B = \emptyset$ the following also hold.
R6. $(A, a)(B, a^{-1}) = (B + A - a^{-1}, a)(a^{-1}, a^{-1})$,
R7. $(A, a)(B, b) = (B, b)(A, a)$ *whenever* $a^{-1} \notin B$, $b^{-1} \notin A$,
R8. $(A, a)(B, b) = (B, b)(B + A - b^{-1}, a)$ *whenever* $a^{-1} \notin B$, $b^{-1} \in A$.

Proof. Since we have proved the equivalence of the operations (A, a) on coinitial graphs and on reduced cyclic words, and since this clearly extends to $(a \leftrightarrow b)$ we can prove the relations by showing that they hold either for all coinitial graphs or for all reduced cyclic words. Consider first all coinitial graphs.

In the operation (A, a), (i) and (iii) are inverses and (ii) is its own inverse, and by definition $(x \leftrightarrow y)$ is its own inverse, so R1 holds. Operations (i), (ii) and (iii) are the same for (A, a) and (A', a^{-1}), so R2 holds. Now (a, a) and (a^{-1}, a^{-1}) both merely interchange a and a^{-1}, so R3 holds. The operation (A, a) followed by a change of label may also be achieved by first relabelling the vertices and then carrying out the operation on the same edges and with the same vertex, i.e., with the new labelling, $(A\pi, a\pi)$, so R4 holds. In applying $(A, a)(A, b)$ operation (i) of (A, b) cuts precisely those edges which have been joined by operation (iii) of (A, a). Therefore $(A, a)(A, b)$ is equivalent to

(i) cut all edges between A and A' and join to α and α^{-1},
(ii) apply $(\alpha \leftrightarrow a)$ then apply $(\alpha \leftrightarrow b)$,
(iii) join the edges at α and α^{-1}.

Now (ii) above is the same as applying $(\alpha \leftrightarrow b)$ and then applying $(a \leftrightarrow b)$; moreover the edges at α and α^{-1} may be joined before the last relabelling and so R5 holds.

The proofs of R6, R7 and R8 are more succinct using reduced words than coinitial graphs, so we consider the action on letters.

Now $a(A, a)(B', a) = a$ and similarly for a^{-1}. If $x \neq a^{\pm 1}$ then

$$x(A, a)(B', a) = \begin{cases} .xa & \text{if } x \in A \cap B, \\ .xa^{-1} & \text{if } x \in A' \cap B', \\ .x & \text{otherwise}, \end{cases}$$

where the dot is determined by x^{-1}. Hence, using $(B', a) = (B, a^{-1})$,

$$(A, a)(B, a^{-1}) = (C, a)(D, a^{-1})$$

whenever $A \cap B = C \cap D$ and $A' \cap B' = C' \cap D'$. In particular R6 holds.

If $A \cap B = \emptyset$ and $a^{-1} \notin B$ then

$$\begin{aligned} a(A, a)(B, b) &= a^{-1} = a(B, b)(A, a) \\ &= a(B, b)(B + A - b^{-1}, a), \end{aligned}$$

and similarly for a^{-1}. Moreover if $x \neq a^{\pm 1}, b^{\pm 1}$ then

$$x(A, a)(B, b) = \begin{cases} .xa^{-1} & \text{if } x \in A, \\ .xb^{-1} & \text{if } x \in B, \\ .x & \text{otherwise}. \end{cases}$$

Now if also $b^{-1} \notin A$ then by symmetry

$$\begin{aligned} b(A, a)(B, b) &= b^{-1} = b(B, b)(A, a) \quad \text{and} \\ x(A, a)(B, b) &= x(B, b)(A, a) \quad \text{for all } x \neq a^{\pm 1}, b^{\pm 1}, \end{aligned}$$

so R7 holds.

Whereas if $b^{-1} \in A$ then

$$b(A, a)(B, b) = ab^{-1} = b(B, b)(B + A - b^{-1}, a)$$

and for $x \neq a^{\pm 1}, b^{\pm 1}$

$$x(B, b)(B + A - b^{-1}, a) = \begin{cases} .xa^{-1} & \text{if } x \in A, \\ .xa^{-1}ab^{-1} & \text{if } x \in B, \\ .x & \text{otherwise,} \end{cases}$$

which proves R8.

3. We now combine the algebraic results in Section 2 with the combinatorial results from Section 1. Suppose that the conditions of Lemma 2 hold and that (P, p) and σ are given by the lemma. Let $\sigma_0 = (A, a)$ and $\sigma_3 = (B, b)$. If $p = a^\epsilon$ let σ_1 be (P, p) and σ_2 be σ with $(Y, y) = (B^{-\zeta}, b^{-\zeta})$. If $p = b^\zeta$ let σ_2 be (P, p) and σ_1 be σ with $(Y, y) = (A^{-\epsilon}, a^{-\epsilon})$.

LEMMA 5. *With $\sigma_0, \ldots, \sigma_3$ defined as above, the relations* R1–R8 *give*

$$\sigma_t = \sigma_{t-1}(Q_t, c_t)\pi_t \quad \textit{for some } Q_t,\ t = 1, 2, 3,$$

where $c_t = a^{-\epsilon}$ or $b^{-\zeta}$ and $\pi_t = (a^\epsilon \leftrightarrow b^\zeta)$, (c_t, c_t) or the identity, and with the convention that (Q_t, c_t) may be the identity.

Proof. The relations R1–R3 will be used freely throughout, the other relations will be explicitly mentioned.
Suppose $p = a^\epsilon$; then, with $(Y, y) = (B^{-\zeta}, b^{-\zeta})$,

$$\sigma_1 = (P, a^\epsilon) = (A^{-\epsilon}, a^{-\epsilon})(A^{-\epsilon} + P - a^{-\epsilon}, a^{-\epsilon})(a^\epsilon, a^\epsilon),$$

by R6, which is of the required form.

$$\sigma_2 = \begin{cases} (Y, y)(P, p) = (P, p)(Y, y) \text{ or } (P, p)(P + Y - p^{-1}, y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{by R7 or R8 if } y^{-1} \notin P, \\ (P, p)(Y, y) = (P, p)(Y, y) \quad \text{if } p^{-1} \notin Y, \\ (P, y^{-1}) \qquad = (P, p)(P, y^{-1})(y^{-1} \leftrightarrow p) \quad \text{by R5 if } p \neq y^{-1} \in P, \\ \qquad\qquad\quad\ = (P, p) \quad \text{if } p = y^{-1}, \\ (Y, p^{-1}) \qquad = (P, p)(Y + P - p^{-1}, p)(p^{-1}, p^{-1}) \quad \text{by R6,} \end{cases}$$

which in every case is of the required form.
Moreover

$$\sigma_2 = \begin{cases} (Y, y)(P, p) = (Y, y)(P, p) \quad \text{if } y^{-1} \notin P, \\ (P, p)(Y, y) = (Y, y)(P, p) \text{ or } (Y, y)(Y + P - y^{-1}, p) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{by R7 or R8 if } p^{-1} \notin Y, \\ (P, y^{-1}) \qquad = (Y, y)(P + Y - y^{-1}, y)(y^{-1}, y^{-1}) \quad \text{by R6,} \\ (Y, p^{-1}) \qquad = (Y, y)(Y, p^{-1})(y \leftrightarrow p^{-1}) \quad \text{by R5 if } y \neq p^{-1}, \\ \qquad\qquad\quad\ = (Y, y) \quad \text{if } y = p^{-1}, \end{cases}$$

which, since $\sigma_3 = (Y, y)$, may be transformed to the required expression for σ_3 using R4 if necessary.

This proves the lemma when $p = a^\epsilon$. If $p = b^\zeta$ the proof is the same with (A, a) and (B, b) interchanged.

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When I is empty we get Lemmas 1 and 2 of [**5**].

Let T now denote the set of transformations (A, a) and $(a \leftrightarrow b)$, and Σ the group of transformations generated by T. Lemma 2 still holds with the slight change of notation.

LEMMA 6. *Let I be a finite partially ordered set and let Γ and Γ_i for $i \in I$, be finite coinitial graphs. Suppose that, for each $i \in I$, $|\Gamma_i|$ is minimal under $\tau \in$ T that do not increase $|\Gamma_j|$ for any $j < i$. Let σ be any element of Σ such that $|\Gamma_i\sigma| = |\Gamma_i|$ for $i \in I$, and $|\Gamma\sigma|$ is minimal under all $\sigma \in \Sigma$ that do not increase $|\Gamma_i|$ for $i \in I$. Then any expression for σ as a product of τ in T can be changed by repeated use of the relations in Lemma 5 to $\sigma = \tau_1 \ldots \tau_n$ such that*

$$|\Gamma_i\tau_1 \ldots \tau_r| = |\Gamma_i| \text{ for } r = 1, 2, \ldots, n, \text{ and all } i \in I, \text{ and}$$

$$|\Gamma| > |\Gamma\tau_1| > \ldots > |\Gamma\tau_1 \ldots \tau_s| = \ldots = |\Gamma\sigma| \quad \text{for some } s = 0, 1, \ldots, n.$$

Proof. Add an element i_0 to I with $i_0 > i$ for all $i \in I$, and put $\Gamma = \Gamma_{i_0}$. We use induction on the relation $>$. Let k be a minimal element of I for which the conclusion does not hold for some σ satisfying the hypotheses.

Now by the induction we can assume that the expression for σ has been changed by the relations to $\sigma = \tau_1 \ldots \tau_n$ with $|\Gamma_i\tau_1 \ldots \tau_r| = |\Gamma_i|$ for $r = 1, \ldots, n$, and all $i < k$. Put $|\Gamma_k\tau_1 \ldots \tau_r| = l_r$ for $r = 0, \ldots, n$, and consider the sequence $l_0, l_1, \ldots, l_n$. If $k \neq i_0$ then $k \in I$ and by the hypothesis $l_r \geqq l_0 = l_n$, so we have $l_{r-1} \leqq l_r > l_{r+1}$ for some r. If $k = i_0$, then by hypothesis $l_r \geqq l_n$ and again $l_{r-1} \leqq l_r > l_{r+1}$ for some r. In either case choose r to give a maximum l_r satisfying these inequalities.

If τ_r is not a permutation then we assert that the conditions of Lemma 2 hold for $\Gamma = \Gamma_k\tau_1 \ldots \tau_r$ and $\Gamma_j\tau_1 \ldots \tau_r$, $j < k$, with $\tau_r = (A, a)$ and $\tau_{r+1} = (B, b)$. For if $\{|\Gamma_j\tau_1 \ldots \tau_r|, j < k\}$ does not satisfy the required minimality condition for Lemma 2 then there exists some $j' < k$ and some $\tau \in$ T such that $|\Gamma_j\tau_1 \ldots \tau_r\tau| = |\Gamma_j\tau_1 \ldots \tau_r|$ all $j < j'$, and $|\Gamma_{j'}\tau_1 \ldots \tau_r\tau| < |\Gamma_{j'}\tau_1 \ldots \tau_r|$. But then by the induction hypothesis we may apply the present lemma to j' and a suitable σ to show the existence of τ_1' such that $|\Gamma_j\tau_1'| = |\Gamma_j|$ and $|\Gamma_{j'}\tau_1'| < |\Gamma_{j'}|$. Since $j' \in I$ this contradicts the hypotheses given for this lemma. The other hypotheses of Lemma 2 clearly hold. We may therefore apply Lemma 2 and Lemma 5 to find σ_1, σ_2 such that $\Delta\sigma_t = |\Gamma_k\tau_1 \ldots \tau_r\sigma_t| - |\Gamma_k\tau_1 \ldots \tau_r| < 0$, $|\Gamma_j\tau_1 \ldots \tau_r\sigma_t| = |\Gamma_j\tau_1 \ldots \tau_r|$, $j < k$, and $\sigma_t = \sigma_{t-1}(Q_t, c_t)\pi_t$, $t = 1, 2, 3$, where $\sigma_0 = (A, a) = \tau_r$ and $\sigma_3 = (B, b) = \tau_{r+1}$. Therefore

$$\tau_{r+1} = \tau_r \prod_1^3 (Q_t, c_t)\pi_t$$

and substituting this for τ_{r+1} in the expression for σ and cancelling $\tau_r\tau_r$ we obtain a sequence in which, in place of the value l_r, we have $l_r + \Delta\sigma_1$ twice and $l_r + \Delta\sigma_2$ twice and l_{r+1} once. Moreover for the new sequence, since $|\Gamma_j\tau_1 \ldots \tau_r\sigma_t| = |\Gamma_j\tau_1 \ldots \tau_r|$, we still have $|\Gamma_j\tau_1 \ldots \tau_{r'}| = |\Gamma_j|$ all r', all

$j < k$. If τ_r is a permutation then by R2 we can move it one place to the right to obtain a sequence in which the value l_r is replaced by l_{r+1}.

Now by the usual induction on the value l_r and the number of times it is attained we have, after a finite number of steps, that the conclusion holds after all for this k and σ. Since I is finite the lemma is proved.

If I is empty we get the original result of Whitehead [**9**].

COROLLARY. *Let I be an inductive set and let $|\Gamma_i|$, $i \in I$, be minimal under all $\tau \in \mathrm{T}$ that do not increase $|\Gamma_j|$ for any $j < i$. Then $|\Gamma_i|$ is minimal under all $\sigma \in \Sigma$ that do not increase $|\Gamma_j|$ for any $j < i$.*

Proof. Suppose not. Then there is some minimal $i_0 \in I$ and some $\sigma \in \Sigma$ such that $|\Gamma_{i_0}\sigma| < |\Gamma_{i_0}|$ and $|\Gamma_j\sigma| = |\Gamma_j|$, for $j < i_0$. Applying the lemma to Γ_i for $i < i_0$ with $\Gamma = \Gamma_{i_0}$, we have $\sigma = \tau_1 \ldots \tau_n$, where $|\Gamma_i\tau_1| = |\Gamma_i|$ and $|\Gamma\tau_1| < |\Gamma|$ contradicting the hypothesis.

Now each σ in Σ acts on ordered m-tuples of finite coinitial graphs by $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m)\sigma = (\Gamma_1\sigma, \Gamma_2\sigma, \ldots, \Gamma_m\sigma)$. The following theorem was proved by McCool [**5**; **6**] for $m = 0$ and 1, and the general case can be proved similarly.

THEOREM *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ be coinitial graphs on a finite set X, then* Stab $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$ *is finitely presented.*

Proof. We may suppose that each $|\Gamma_i|$ is minimal under σ that do not increase $|\Gamma_j|$ for $j < i$. For in general if σ_i is chosen inductively for $i = 1, 2, \ldots, m$, so that $|\Gamma_i\sigma_1 \ldots \sigma_i|$ is minimal under all $\sigma \in \Sigma$ not increasing $|\Gamma_j\sigma_1 \ldots \sigma_{i-1}|$ for $j < i$, and if $\sigma = \sigma_1\sigma_2 \ldots \sigma_m$ then $|\Gamma_i\sigma|$ satisfies the stated minimality condition and Stab $(\Gamma_1, \Gamma_2, \ldots, \Gamma_m) = \sigma$ Stab $(\Gamma_1\sigma, \Gamma_2\sigma, \ldots, \Gamma_m\sigma)\sigma^{-1}$.

Following the method of [**6**] we construct a 2-complex K as follows. We take as vertices v all m-tuples $(\Gamma_1\sigma, \Gamma_2\sigma, \ldots, \Gamma_m\sigma)$ such that $|\Gamma_i\sigma| = |\Gamma_i|$, where for $m = 0$ we take one vertex. For each vertex v in K and for each $\tau \in \mathrm{T}$ such that $v\tau$ is also in K we construct one edge, labelled τ, between v and $v\tau$. Since $\tau^2 = 1$ for all $\tau \in \mathrm{T}$, there is at most one edge labelled τ at each vertex. If $v\tau_1\tau_2 \ldots \tau_r$ is in K for $r = 0, 1, \ldots n$ and $v\tau_1\tau_2 \ldots \tau_n = v$, then we say that $\tau_1, \tau_2, \ldots, \tau_n$ is a loop in K at v.

By Lemma 6, K is connected and if $v_0 = (\Gamma_1, \Gamma_2, \ldots, \Gamma_m)$ and $\sigma \in$ Stab v_0 then $\sigma = \tau_1\tau_2 \ldots \tau_n$, $\tau_r \in \mathrm{T}$ where $|\Gamma_i\tau_1 \ldots \tau_r| = |\Gamma_i|$, $r = 1, \ldots, n$; that is, $\tau_1, \tau_2, \ldots, \tau_n$ is a loop in K at v_0. Conversely every loop at v_0 in K gives an element of Stab v_0. Since X is finite, T is finite and the number of vertices and edges of K is finite. Thus Stab v_0 is finitely generated.

The subset of T consisting of all transformations (a, a) and $(a \leftrightarrow b)$ generates the extended symmetric group Ω on all $a \neq a^{-1} \in X$. We add sufficient relations R9 to give a presentation of this group. These relations clearly hold in Σ. We now add 2-cells to K as follows. If $\tau\tau' \ldots \tau^{(\nu)} = 1$ is a relation R2–R9 then whenever $\tau, \tau', \ldots, \tau^{(\nu)}$ is a loop at some vertex v of K we add a 2-cell with this loop as boundary. Since the number of relations and of vertices is

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finite we now have a finite complex, whose fundamental group $\pi_1(K, v_0)$ is therefore finitely presented.

Since every 2-cell corresponds to a relation in Σ, homotopic loops at v_0 give the same element of Stab v_0. We therefore have a homomorphism from $\pi_1(K, v_0)$ to Stab v_0, which we wish to prove is an isomorphism. Suppose then that τ_1, $\tau_2, \ldots, \tau_n$ is a loop in K such that $\tau_1 \ldots \tau_n = 1$. Let Γ be any coinitial graph minimal under $\sigma \in \Sigma$ not increasing $|\Gamma_i|$, $i = 1, \ldots, m$. Then by Lemma 6 we may change the product $\tau_1 \ldots \tau_n$ to one in which $|\Gamma\tau_1 \ldots \tau_r| = |\Gamma|$. Moreover this is achieved by substitutions of the form

$$\tau_1 \ldots \tau_r\sigma_t = \tau_1 \ldots \tau_r\sigma_{t-1}(Q_t, c_t)\pi_t, \quad t = 3, 2, 1,$$

where σ_t is defined as in Lemma 5 and where Lemma 2 applies to $v_r = v_0\tau_1 \ldots \tau_r$. Hence we have $|\Gamma_i\tau_1 \ldots \tau_r(P, p)| = |\Gamma_i\tau_1 \ldots \tau_r| = |\Gamma_i|$ etc., so that the words σ_t and $\sigma_{t-1}(Q_t, c_t)\pi_t$ correspond to paths in K from v_r to $v_r\sigma_t$. Moreover they form the boundary of a 2-cell in K which is the union of 2-cells corresponding to the relations used in Lemma 5. Therefore the original loop $\tau_1, \tau_2, \ldots, \tau_n$ is homotopic to one for which $|\Gamma\tau_1 \ldots \tau_r| = |\Gamma|$. It remains to show by choosing a suitable Γ that this loop is homotopic to the identity. This is similar to the method used [**6**].

If X contains an element $e = e^{-1}$, take Γ to be the coinitial graph of the reduced cyclic words $\{ex : x \neq x^{-1} \in X\}$. Then Γ is minimal and if $|\Gamma\tau_1 \ldots \tau_r| = |\Gamma|$ for all r, then $\tau_r = (a, a)$, $(X - a, a^{-1})$ or $(a \leftrightarrow b)$ so 2-cells corresponding to R2 and R9 will give $\tau_1, \ldots, \tau_n$ homotopic to the identity.

If X contains no element $e = e^{-1}$, take Γ to be the coinitial graph of the reduced cyclic words $\{x : x \in X\}$, and Γ' the coinitial graph of $\{xy : y \neq x^{-1}\}$. Γ is minimal and each xy, $y \neq x^{-1}$, is minimal under $\tau \in \mathrm{T}$ not increasing Γ, so, a fortiori, Γ' is minimal under such τ. Applying the argument above for Γ and then for Γ' we obtain an expression for which $|\Gamma\tau_1 \ldots \tau_r| = |\Gamma|$ and $|\Gamma'\tau_1 \ldots \tau_r| = |\Gamma'|$. But again this is only possible if $\tau_r = (a, a)$, $(X - a, a^{-1})$ or $(a \leftrightarrow b)$, and the result follows as before.

We now consider automorphisms of free groups. Let F be a free group of finite rank and let X consist of a set $\{x\}$ of free generators and their inverses and one other element e. Then X has an involution $x \mapsto x^{-1}$, $e \mapsto e$. The reduced cyclic words having exactly one occurrence of e are in $1 - 1$ correspondence with the elements of F, under the map $ew \mapsto w \in F$. The transformations (A, a) take this set of reduced cyclic words into itself and correspond to automorphisms of F. In fact if $e \notin A$ then (A, a) corresponds to the transformation on the generators of F given after Lemma 2, whereas if $e \in A$ then it corresponds to the same transformation followed by the inner automorphism $x \mapsto a^{-1}xa$. The transformations (x, x) and $(x + y, x)(x, x)$ correspond to the elementary Nielsen transformations. Therefore we have an epimorphism from Σ to Aut F which is easily seen to be an isomorphism.

If $W_1, W_2, \ldots, W_m$ are sets of reduced words or cyclic reduced words in the free generators of F, let $\Gamma_1, \ldots, \Gamma_m$ be the corresponding coinitial graphs, where as before a reduced word w is equated with the cyclic reduced word ew. Then Stab $(\Gamma_1, \ldots, \Gamma_m)$ is isomorphic to the subgroup of all α in Aut F such that $W_i\alpha = W_i$, $i = 1, \ldots, m$. Adding the relations $(A, a) = (A - e, a)$, whenever $e \in A$, factors out inner automorphisms. In this way presentations of some subgroups of Aut $F/I(F)$ can theoretically be given, including the subgroups given by Zieschang [**10**] which induce automorphsims of Fuchsian groups.

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