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SOME SIMILARITY TEMPERATURE PROFILES FOR THE MICROWAVE HEATING OF A HALF-SPACE

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Abstract

There is presently considerable interest in the utilisation of microwave heating in areas such as cooking, sterilising, melting, smelting, sintering and drying. In general, such problems involve Maxwell's equations coupled with the heat equation, for which all thermal, electrical and magnetic properties of the material are nonlinear. The heat source arising from microwaves is proportional to the square of the modulus of the electric field intensity, and is known to increase with increasing temperature. In an attempt to find a simple model of microwave heating, we examine here simple transient temperature profiles corresponding to a heat source with spatial exponential decay but increasing with temperature, for which we assume either a power-law dependence or an exponential dependence. The spatial exponential decay is known to apply exactly when the electrical and magnetic properties of the material are assumed constant. A number of transient temperature profiles for this model are examined which arise from the invariance of the governing heat equation under simple one-parameter transformation groups. Some closed analytical expressions are obtained, but in general the resulting ordinary differential equations need to be solved numerically, and extensive numerical results are presented. For both models, these results indicate the appearance of moving fronts.

1. Introduction

The use of microwave radiation for heating is now common in many industrial situations such as smelting, sintering and drying. Microwave heating has advantages over conventional heating in that it can quickly provide uniform heating throughout the material. Even though the standard kitchen microwave oven has been in existence for almost fifty years, it is only in recent years that this technology has been considered for industrial applications. This may in part account for the lack of scientific enquiry into the

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many experimental and theoretical aspects of microwave heating.

In general, a mathematical analysis of microwave heating involves solving both the heat equation and Maxwell's equations of electromagnetism, with all thermal, magnetic and electrical properties being dependent upon the temperature. Hill [3] examines the simplest types of exact solutions which apply when all the physical properties of the material have a power-law type dependence and which, moreover, involves the same reference temperature. This means that all physical temperature-dependent properties can be replaced by some power of the temperature, and, under appropriate restrictions, the full one-dimensional equations remain invariant under a stretching group of transformations and accordingly admit certain similarity solutions. Even so, the number of closed analytical solutions is limited, and in general the governing system of ordinary differential equations still needs to be solved numerically. In contrast, Smyth [8] obtains approximate analytic solutions following the methods of geometric optics applicable to the high-frequency limit, small thermal diffusivity and all other material properties assumed to be linearly dependent on temperature. The solutions so obtained are not uniformly valid for all time, except when the electrical conductivity is small. Pincombe and Smyth [7] extend the work of Smyth [8] to the case when the various material properties are assumed to have a power-law dependence on temperature but the electrical conductivity is still assumed small.

In an attempt to understand the problem of hot-spots which occur in microwave heating, Hill and Smyth [4] and Coleman [2] consider only the heat equation together with a source term which is nonlinearly dependent upon temperature and increases with increasing temperature. Hill and Smyth [4] demonstrate the occurrence of hot-spots for a variety of geometries using an exponential source term, while Coleman [2] employs both power law and Arrhenius dependence on temperature, and demonstrates that hot-spot type phenomena can occur for materials which do not normally exhibit thermal runaway, in the sense that the temperature can still have a local maximum. In this paper we attempt to improve the simple model utilised in both Hill and Smyth [4] and Coleman [2] by incorporating a spatial dependence on the heat source, and this model is consistent with that proposed by Coleman [1] in examining the Stefan problem for microwave heating.

In particular, we consider the special case of the one-dimensional microwave heating of the half-space $x \geq 0$ with temperature, electric and magnetic fields given respectively by

$$T = T(x, t), \qquad \mathbf{E} = E(x, t)\hat{\mathbf{j}}, \qquad \mathbf{H} = H(x, t)\hat{\mathbf{k}}, \tag{1.1}$$

where as usual $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ denote unit vectors in the (x, y, z) directions. Now the nonlinear heat equation and Maxwell's equations for a linear conducting medium of density ρ become

$$\begin{gathered}\rho c(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right)+q(T)|E|^2,\\ \frac{\partial H}{\partial x}+\frac{\partial}{\partial t}[\varepsilon(T)E]+\sigma(T)E=0,\qquad \frac{\partial E}{\partial x}+\frac{\partial}{\partial t}[\mu(T)H]=0,\end{gathered}\tag{1.2}$$

where $c(T)$, $k(T)$ and $q(T)$ denote the temperature-dependent specific heat, thermal conductivity and body heating coefficient respectively, while $\mu(T)$, $\varepsilon(T)$ and $\sigma(T)$ denote the temperature-dependent magnetic permeability, electric permittivity and electrical conductivity of the medium respectively, and $|E|^2$ denotes the square of the modulus of the complex electric intensity. In order to present a simplified account of microwave heating, we follow Coleman [1] and assume here that $|E|^2$ decays exponentially with distance; that is, we assume

$$|E|^2=E_0^2e^{-\kappa x},\tag{1.3}$$

for certain constants E_0 and κ. We make this assumption because firstly, it is a well known result in the case when the permeability, permittivity and conductivity are known to be constants, say μ_0, ε_0 and σ_0 respectively, in which case κ is given by

$$\kappa=\omega(2\mu_0\varepsilon_0)^{\frac{1}{2}}\left[\left(1+\left(\frac{\sigma_0}{\varepsilon_0\omega}\right)^2\right)^{\frac{1}{2}}-1\right]^{\frac{1}{2}},\tag{1.4}$$

where ω denotes the wave frequency (see Tralli [9] or Metaxas and Meredith [5]). Secondly, the assumption (1.3) will be locally valid within a limited region, depending on the variation in $\mu(T)$, $\varepsilon(T)$ and $\sigma(T)$. Thirdly, it is worth emphasing that the assumption (1.3) pertains to the modulus of E rather than E itself, so that as long as $E(x,t)$ takes the form

$$E(x,t)=E_0e^{-\kappa x/2}e^{i\Theta(x,t)},\tag{1.5}$$

for some real function $\Theta(x,t)$, the assumption (1.3) remains valid. Thus (1.3) represents the simplest possible spatial dependence which has some physical basis, and which enables the heating aspects of the problem to be isolated from the electrical and magnetic fields.

Metaxas and Meredith [5] present experimental evidence which indicates that the physical properties of the material have a power-law dependence on temperature. In particular, if we assume that $c(T)$, $k(T)$ and $q(T)$ have a power-law dependence, then on introducing a new temperature variable T^* defined by

$$T^*=\rho\int c(T)\,dT,\tag{1.6}$$

and rescaling x, then dropping the asterisk, it is not difficult to show that the heat equation becomes

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(T^m\frac{\partial T}{\partial x}\right)+\alpha e^{-\beta x}T^n, \tag{1.7}$$

for certain constants, m, n, α and β. Physical requirements indicate that the source term due to microwave heating decreases spatially and increases with temperature so that $n>0$, $\alpha>0$ and $\beta>0$. Similarly, assuming constant specific heat $c(T)$, and that $k(T)$ and $q(T)$ have an exponential dependence on temperature, then a similar procedure yields

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(e^{\gamma T}\frac{\partial T}{\partial x}\right)+\alpha e^{-\beta x}e^{\delta T}, \tag{1.8}$$

for certain constants γ and δ, where $\delta>0$ for a heat source which is an increasing function of temperature. Equations (1.7) and (1.8) constitute the two basic models which we consider in this paper.

The next three sections deal with (1.7), while the final three sections of the paper deal with (1.8). In the following section, we present a similarity solution of (1.7) valid for $m\neq 0$ and $n\neq m+1$. Solutions appropriate to the special cases $n=m+1$ and $m=0$ are discussed separately in Sections 3 and 4 respectively. Similarly in Section 5 we present a similarity solution of (1.8) assuming $\gamma\neq 0$ and $\gamma\neq\delta$, and then solutions applying to the special cases $\gamma=\delta$ and $\gamma=0$ are presented in Sections 6 and 7 respectively.

Finally in this section we, comment briefly on various results for time independent solutions of these equations. For steady solutions of (1.7) and (1.8), with $T=T(x)$ we may readily deduce,

$$\begin{aligned}\frac{d^2y}{dx^2}+\alpha(m+1)e^{-\beta x}y^{n/(m+1)}&=0, \qquad y=T^{m+1},\\ \frac{d^2z}{dx^2}+\alpha\gamma e^{-\beta x}z^{\delta/\gamma}&=0, \qquad z=e^{\gamma T},\end{aligned} \tag{1.9}$$

from which it is apparent that the special cases $n=m+1$ and $\gamma=\delta$ give rise to linear equations and in both cases the transformation $\xi=e^{-\beta x/2}$ produces equations which have Bessel function solutions; thus

$$\begin{aligned}\frac{d^2y}{d\xi^2}+\frac{1}{\xi}\frac{dy}{d\xi}+\frac{4\alpha(m+1)}{\beta^2}y&=0,\\ \frac{d^2z}{d\xi^2}+\frac{1}{\xi}\frac{dz}{d\xi}+\frac{4\alpha\gamma}{\beta^2}z&=0.\end{aligned} \tag{1.10}$$

These equations have solutions of the form $C_1J_0(k\xi)+C_2Y_0(k\xi)$, where C_1 and C_2 denote arbitrary constants and $k=2[\alpha(m+1)]^{1/2}/\beta$ or

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$k = 2(\alpha\gamma)^{1/2}/\beta$. If $n \neq m+1$ or $\gamma \neq \delta$, then possibly both of (1.9) admit simple solutions of the form Ae^{Bx}, where A and B are constants such that

$$\begin{aligned} B &= \frac{\beta(m+1)}{(n-m-1)}, \qquad B^2 + \alpha(m+1)A^{(n-m-1)/(m+1)} = 0, \\ B &= \frac{\beta\gamma}{(\delta-\gamma)}, \qquad B^2 + \alpha\gamma A^{(\delta-\gamma)/\gamma} = 0, \end{aligned} \tag{1.11}$$

which certainly have solutions with $A < 0$ when $(n-m-1)/(m+1)$ or $(\delta-\gamma)/\gamma$ is an odd integer and assuming $\alpha(m+1)$ and $\alpha\gamma$ are both positive. However, in general it appears to be difficult to obtain further simple analytical expressions for solutions of (1.9) (see Murphy [6], page 387). A careful examination of each of the above special steady solutions indicates that there are no steady-state solutions of the microwave heating problem such that $T \geq 0$ at all times and T tends to zero as x tends to infinity. In subsequent sections, we examine a number of transient solutions of (1.7) and (1.8) which arise from the invariance of these equations under simple one-parameter transformation groups. Since we are not at liberty to impose arbitrary boundary and initial conditions for such solutions, we illustrate these solutions by assuming that both $T(x, t)$ and $\frac{\partial T}{\partial x}(x, t)$ are prescribed on the boundary $x = 0$ at some time $t = t_1$, and we then display the temperature profile at time t_1. These numerical results indicate the presence of moving fronts.

2. Power law thermal conductivity and heat source

In this section we examine a similarity solution of (1.7) which applies for $m \neq 0$ and $n \neq m+1$. The appropriate version of the solution for these special cases is detailed in the subsequent two sections. It is a simple matter to show that (1.7) remains invariant under the one-parameter group of transformations

$$x_1 = x + a\varepsilon, \qquad t_1 = e^{\varepsilon} t, \qquad T_1 = e^{b\varepsilon} T, \tag{2.1}$$

provided the constants a and b are given by

$$a = \frac{(m+1-n)}{\beta m}, \qquad b = -\frac{1}{m}, \tag{2.2}$$

in which case two invariants of (2.1) are $Te^{x/ma}$ and $te^{-x/a}$ and therefore the functional form of the solution corresponding to (2.1) is given by

$$T(x, t) = e^{-x/ma}\phi(\xi), \qquad \xi = te^{-x/a}, \tag{2.3}$$

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where ϕ denotes some function which is determined by substitution of (2.3) into the partial differential equation (1.7). From (2.2) and (2.3) we observe that this solution is only meaningful for $m \neq 0$ and $n \neq m+1$.

In addition, from (2.3) we note that the solution must satisfy an initial condition of the form

$$T(x, 0) = e^{-x/ma}\phi(0), \tag{2.4}$$

and therefore we can if necessary accommodate the usual zero initial condition $T(x, 0) = 0$ by simply taking $\phi(0) = 0$. Further, at $x = 0$ the solution (2.3) would satisfy any one of the time-dependent boundary conditions

$$\begin{gathered} T(0, t) = \phi(t) \qquad \frac{\partial T}{\partial x}(0, t) = -\frac{[mt\phi'(t) + \phi(t)]}{ma}, \\ \frac{\partial T}{\partial x}(0, t) + \frac{T(0, t)}{ma} = -\frac{t\phi'(t)}{a}, \end{gathered} \tag{2.5}$$

where here primes denote differentiation with respect to t, and the function $\phi(t)$ is not arbitrary but is that determined by solving the ordinary differential equation (2.6). If both a and m are positive, the solution and its partial derivatives all tend to zero as x tends to infinity. If, however, a is negative, zero temperature at infinity might be achieved by the condition $\phi(\infty) = 0$.

On substituting (2.3) into (1.7), we may eventually deduce the second-order ordinary differential equation

$$\frac{d\phi}{d\xi} = \frac{1}{ma^2}\left\{\frac{d}{d\xi}\left[m\xi^2\phi^m\frac{d\phi}{d\xi} + \left(\frac{m+2}{m+1}\right)\xi\phi^{m+1}\right] + \frac{\phi^{m+1}}{m(m+1)} + \alpha ma^2\phi^n\right\}, \tag{2.6}$$

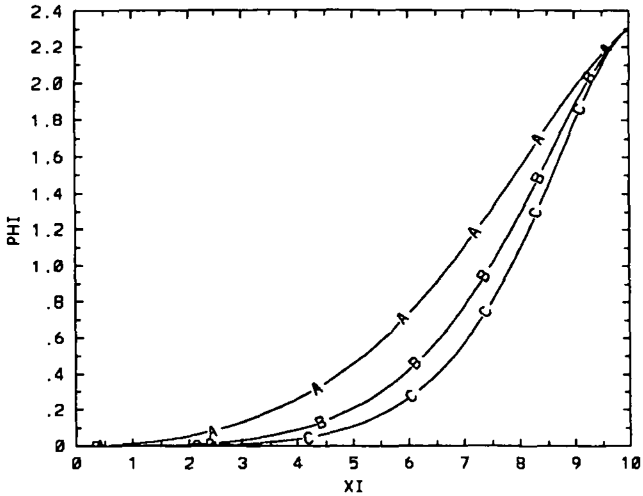
which appears not to admit any simple first integrals (unless $\alpha(m+1) < 0$ and $n = m+1$, which is not possible here) and accordingly must be solved numerically. We note, however, that in terms of $\psi = \phi^{m+1}$ $(\phi = \psi^{\frac{1}{(m+1)}})$, (2.6) takes on the alternative compact form,

$$a^2\psi^{\frac{-m}{(m+1)}}\left\{\frac{d\psi}{d\xi} - \alpha\psi^{\frac{n+m}{(m+1)}}\right\} = \left(\xi\frac{d}{d\xi} + \frac{(m+1)}{m}\right)\left(\xi\frac{d\psi}{d\xi} + \frac{(m+1)}{m}\psi\right). \tag{2.7}$$

In order to illustrate the behaviour of solutions of (2.6), we write the equation as a pair of first-order ordinary differential equations,

$$\frac{d\phi}{d\xi} = \omega, \qquad \frac{d\omega}{d\xi} = \left[\frac{a^2}{\phi^m\xi^2} - \frac{(3m+2)}{m\xi} - \frac{m\omega}{\phi}\right]\omega - \frac{(m+1)\phi}{m^2\xi^2} - \frac{\alpha a^2\phi^{n-m}}{\xi^2}, \tag{2.8}$$

and assume that both $T(x, t)$ and $\frac{\partial T}{\partial x}(x, t)$ are prescribed on the boundary $x = 0$ at some fixed time $t = t_1$, so that from $(2.5)_1$ and $(2.5)_2$ we may deduce $\phi(t_1)$ and $\omega(t_1)$, which can be used as starting conditions in a



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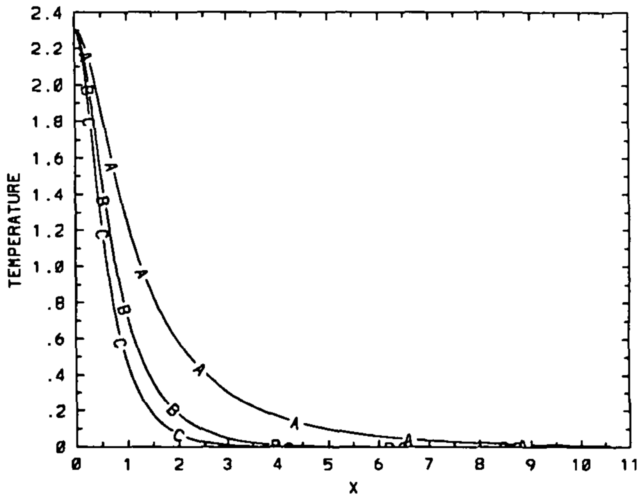
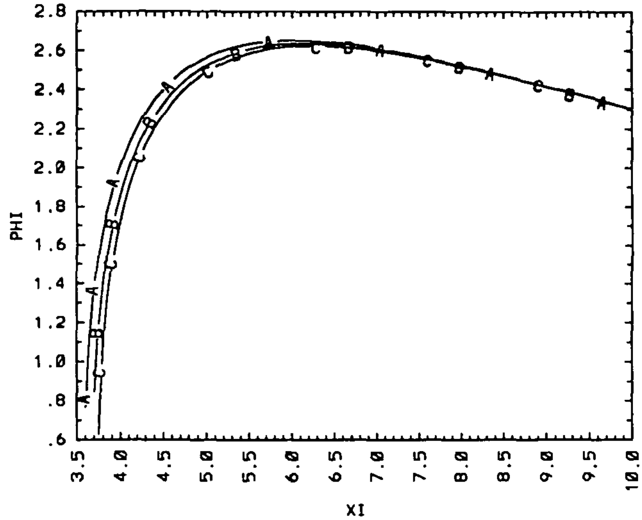


(b)

FIGURE 1. Variation of $\phi(\xi)$ and $T(x, t)$ for (2.3) for the case a positive and m negative $(m = -1, n = 2)$.



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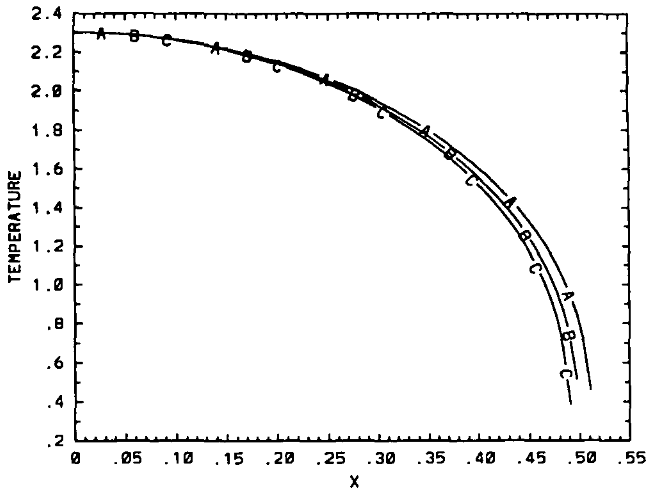

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FIGURE 2. Variation of $\phi(\xi)$ and $T(x, t)$ for (2.3) for the case a and m positive $(m = n = 2)$.

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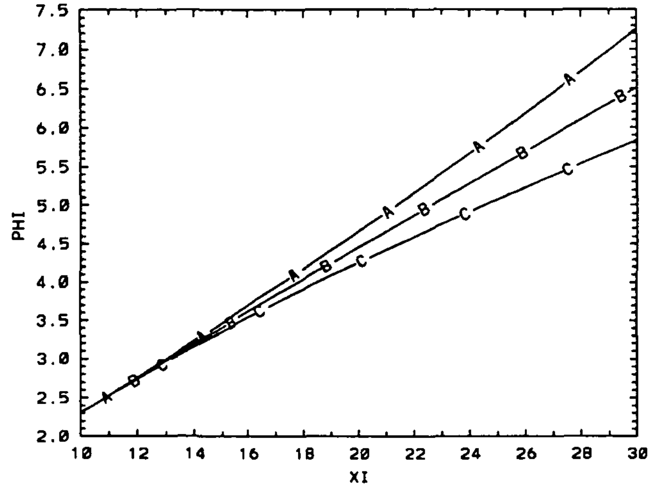

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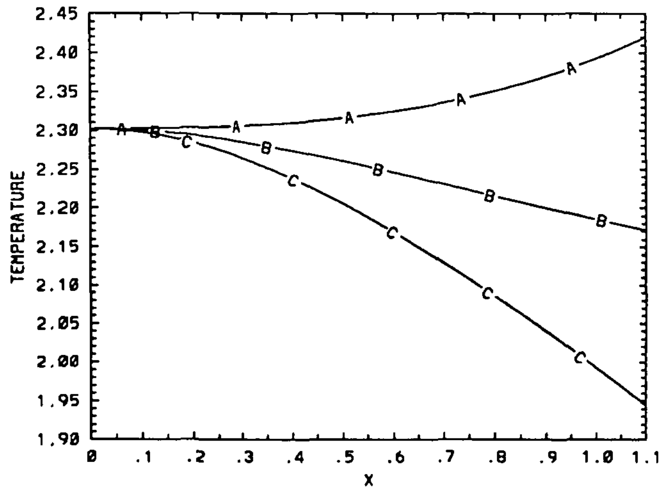

(b)

FIGURE 3. Variation of $\phi(\xi)$ and $T(x, t)$ for (2.3) for the case a and m negative $(m = n = -1)$.

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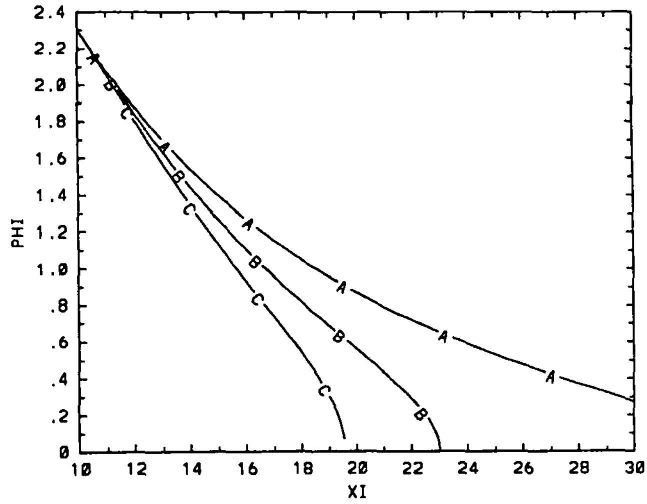

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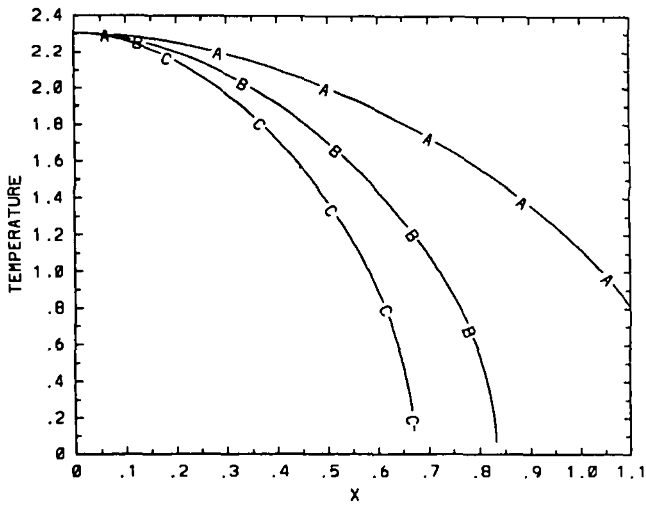

(b)

FIGURE 4. Variation of $\phi(\xi)$ and $T(x, t)$ for (2.3) for the case a negative and m positive $(m = 1, n = 3)$.

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Runge-Kutta scheme to determine $\phi(\xi)$ and hence the complete temperature distribution at time $t = t_1$. In the numerical results shown in Figures 1-4 we adopt throughout the values

$$\beta = 1, \qquad t_1 = 10, \qquad T(0, t_1) = 2.3, \qquad \frac{\partial T}{\partial x}(0, t_1) = 0, \tag{2.9}$$

and the A, B and C shown on the curves refers to the three values of α considered, namely $\alpha = 1/2(A)$, $\alpha = 1(B)$ and $\alpha = 3/2(C)$. We consider the two cases $a > 0$ and $a < 0$ separately.

The case $a > 0$ arises if either $m < 0$ and $n > m + 1$ or if $m > 0$ and $n < m + 1$. Further, in this case, ξ defined by $(2.3)_2$ maps the x interval $(0, \infty)$ into the ξ interval $(0, t)$. The two sub-cases $m < 0$ and $m > 0$ give qualitatively different behaviour for $\phi(\xi)$. If $m < 0$ then both $\phi(\xi)$ and $\phi'(\xi)$ approach zero as ξ tends to zero (that is, as x tends to infinity) while for $m > 0$ there is a value $\xi = \xi_0$ $(0 < \xi_0 < t_1)$ such that $\phi(\xi_0) = 0$ and $\lim_{\xi \to \xi_0} \phi'(\xi_0) = \infty$ and with $\phi(\xi)$ not defined by (2.8) for $\xi < \xi_0$. The latter situation corresponds to a moving front, and for a zero initial condition it is appropriate to take $\phi(\xi) = 0$ for $\xi < \xi_0$. Moreover, this qualitative behaviour of solutions coincides with the well-known behaviour of solutions of the nonlinear diffusion equation. Typical curves corresponding to these two sub-cases are shown in Figures 1 and 2 (see pp. 296–297).

The case $a < 0$ arises if either $m < 0$ and $n < m + 1$ or if $m > 0$ and $n > m + 1$ and in this case ξ defined by $(2.3)_2$ maps the x interval $(0, \infty)$ into the ξ interval (t, ∞). If $m < 0$, the solutions for $\phi(\xi)$ are monotonically increasing with ξ and give rise to a variety of temperature distributions as indicated in Figure 3(b). For $m > 0$ the solutions for $\phi(\xi)$ initially have the appearance of an exponential but then move steeply to zero at $\xi = \xi_0$ with $\phi(\xi) = 0$ for $\xi > \xi_0$ and again a moving front is exhibited. Typical curves corresponding to $m < 0$ and $m > 0$ are shown in Figures 3 and 4 (see pp. 298–299) respectively.

3. Power-law dependence with $n = m + 1$

If $n = m + 1$ then the one-parameter group (2.1) becomes

$$x_1 = x, \qquad t_1 = e^{\varepsilon} t, \qquad T_1 = e^{-\varepsilon/m} T, \tag{3.1}$$

so that in this case, two invariants are x and $Tt^{1/m}$, and accordingly the functional form of the solution is

$$T(x, t) = t^{-1/m} \phi(x), \tag{3.2}$$

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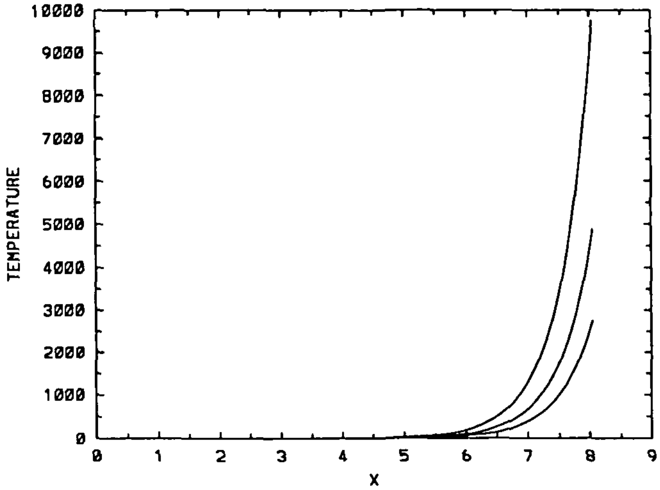

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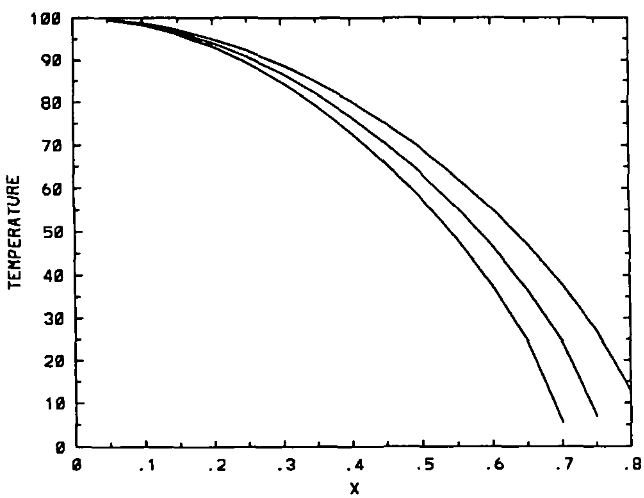

(b)

FIGURE 5. Temperature variation for (3.2) for the cases of m negative $(m = -1/2)$ and m positive $(m = 1/2)$.

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which is simply a separable solution and it is not difficult to show that $n = m+1$ is an essential condition for the existence of solutions of the form $T(x, t) = f(x)g(t)$. On substituting (3.2) into (1.7), we can readily deduce

$$\frac{d}{dx}\left(\phi^m \frac{d\phi}{dx}\right) + \alpha e^{-\beta x}\phi^{m+1} = -\frac{\phi}{m}, \tag{3.3}$$

and the substitution $y = \phi^{m+1}$ yields

$$\frac{d^2y}{dx^2} + \alpha(m+1)e^{-\beta x}y = -\frac{(m+1)}{m}y^{1/(m+1)}. \tag{3.4}$$

By rearranging this equation as a pair of first-order ordinary differential equations, as described in the previous section, we can readily generate a numerical solution for $y(x)$ for given values of $y(0)$ and $\frac{dy}{dx}(0)$. The temperature profiles at time t_1 shown in Figure 5(a) (see p. 301) apply for $m = -1/2$, with the values

$$\beta = 1, \qquad t_1 = 10, \qquad T(0, t_1) = 0, \qquad \frac{\partial T}{\partial x}(0, t_1) = 0, \tag{3.5}$$

while those shown in Figure 5(b) apply for $m = 1/2$, with

$$\beta = 1, \qquad t_1 = 10, \qquad T(0, t_1) = 100, \qquad \frac{\partial T}{\partial x}(0, t_1) = 0, \tag{3.6}$$

and again A, B and C denote the temperature profile appropriate to the three values of α: $1/2$, 1 and $3/2$ respectively. Since the corresponding curves for both $y(x)$ and $\phi(x)$ are similar to those shown in the figures, the former curves are not presented.

4. Power-law dependence with m zero

When $m = 0$, (1.7) remains invariant under the one-parameter group of transformations

$$x_1 = x + a\varepsilon, \qquad t_1 = t + b\varepsilon, \qquad T_1 = e^{\varepsilon}T, \tag{4.1}$$

where $a = (n-1)/\beta$ and b is arbitrary. Two invariants of this group are $at - bx$ and $Te^{\kappa x}$, so that in this case the functional form of the solution becomes

$$T(x, t) = e^{-\kappa x}\phi(x - \lambda t), \tag{4.2}$$

where for $n \neq 1$, κ and λ are given by

$$\kappa = \frac{\beta}{(1-n)}, \qquad \lambda = \frac{(n-1)}{\beta b}. \tag{4.3}$$

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On substitution (4.2) into (1.7) with m zero, we may readily deduce

$$\phi'' + (\lambda - 2\kappa)\phi' + \kappa^2\phi + \alpha\phi^n = 0, \tag{4.4}$$

where primes denote differentiation with respect to $\xi = x - \lambda t$. This equation can be integrated in a number of special cases.

Evidently, if $\lambda = 2\kappa$ we have immediately

$$\phi'^2 + \kappa^2\phi^2 + \frac{2\alpha}{(n+1)}\phi^{n+1} = C, \tag{4.5}$$

where C denotes the constant of integration. A further integration gives

$$\xi = \xi_0 \pm \int_{\phi_0}^{\phi} \left[C - \kappa^2\phi^2 - \frac{2\alpha\phi^{n+1}}{(n+1)}\right]^{-1/2} d\phi, \tag{4.6}$$

where ξ_0 denotes a further arbitrary constant and ϕ_0 denotes $\phi(\xi_0)$. Special cases of this integral can be evaluated in terms of elliptic functions. However, for purposes of illustrating the temperature profile it is easy to evaluate (4.6) with the aid of Simpson's rule. Now since

$$\frac{d\phi}{d\xi} = \pm\left[C - \kappa^2\phi^2 - \frac{2\alpha\phi^{n+1}}{(n+1)}\right]^{1/2}, \tag{4.7}$$

we see that the constant C is related to the maximum value of $\phi(\xi)$ by

$$C = \kappa^2\phi_{\max}^2 + \frac{2\alpha\phi_{\max}^{n+1}}{(n+1)}. \tag{4.8}$$

If the positive sign is taken in (4.6), $\phi(\xi) < \phi_{\max}$ in the interval $\xi_0 \le \xi < \infty$ and $\phi(\xi)$ tends to $\phi_{\max}$ as ξ tends to infinity. If the negative sign is adopted then $\phi(\xi) < \phi_{\max}$ in the interval $-\infty < \xi \le \xi_0$ and $\phi(\xi)$ tends to $\phi_{\max}$ as ξ tends to minus infinity. If for given ξ_0 we adopt the positive sign for $\xi > \xi_0$ and the negative sign for $\xi < \xi_0$ then the resulting temperature profile has a cusp at $\xi = \xi_0$ and the function $\phi(\xi)$ is symmetrical about this point. The physical conditions applying to the microwave heating problem are such that the negative sign is the appropriate choice with ξ_0 positive. Since if ξ_0 is negative, the temperature at $x = 0$ will be initially positive but will fall to zero before rising to a maximum value. For microwave heating we would expect a monotonic temperature increase at $x = 0$ and this implies ξ_0 positive. Further, we would expect an initial temperature profile which is nonzero at $x = 0$ but which falls to zero at some point $x = \xi_0$ within the material being heated. To obtain such an initial temperature profile, we take the negative sign to define ϕ on $-\infty < \xi \le \xi_0$, and set $\phi(\xi) \equiv 0$ for $\xi_0 < \xi < \infty$. Even though $\phi(\xi)$ is defined for $\xi < 0$, it makes sense for T to be defined by (4.2) only for $x \ge 0$ and $T \equiv 0$ for $x < 0$. In this case

$$T(x, t) \to e^{-\kappa x}\phi_{\max}, \tag{4.9}$$

as $t \to \infty$ so that in particular $T(0, t) \to \phi_{\max}$ as $t \to \infty$. In order to illustrate this solution we set $\phi_{\max} = \xi_0 = 1$ along with the values

$$\alpha = 0.1\,, \qquad \beta = 0.2\,, \qquad n = 1/2\,, \tag{4.10}$$

and the resulting variation in $\phi(\xi)$ is shown in Figure 6(a) (see p. 305), while the corresponding temperature profiles are shown in Figure 6(b) at three times $t = 0(A)$, $t = 1(B)$, and $t = 2(C)$.

Other integrals of (4.4) can be obtained by the successive substitutions

$$\phi(\xi) = e^{\mu\xi}\psi(\xi)\,, \qquad \eta = e^{\tau\xi}\,, \tag{4.11}$$

so that (4.4) becomes

$$\tau^2\eta^2\frac{d^2\psi}{d\eta^2} + \tau\eta(\lambda - 2\kappa + \tau + 2\mu)\frac{d\psi}{d\eta} + [\mu^2 + \mu(\lambda - 2\kappa) + \kappa^2]\psi + \alpha\eta^{(n-1)\mu/\tau}\psi^n = 0\,. \tag{4.12}$$

Thus, if we choose μ and τ such that

$$\lambda - 2\kappa + \tau + 2\mu = 0\,, \qquad \mu^2 + \mu(\lambda - 2\kappa) + \kappa^2 = 0\,, \qquad (n-1)\mu = 2\tau\,, \tag{4.13}$$

then (4.12) becomes simply

$$\frac{d^2\psi}{d\eta^2} + \frac{\alpha\psi^n}{\tau^2} = 0\,, \tag{4.14}$$

which integrates once to give

$$\left(\frac{d\psi}{d\eta}\right)^2 + \frac{2\alpha\psi^{n+1}}{(n+1)\tau^2} = C\,, \tag{4.15}$$

and a further integration yields

$$\eta = \eta_0 \pm \int_0^{\psi}\left[C - \frac{2\alpha\psi^{n+1}}{(n+1)\tau^2}\right]^{-1/2} d\psi\,, \tag{4.16}$$

where C and η_0 denote constants of integration. From $(4.13)_1$ and $(4.13)_3$, we find that μ and τ are given by

$$\mu = \frac{2(2\kappa - \lambda)}{(n+3)}\,, \qquad \tau = \frac{(n-1)(2\kappa - \lambda)}{(n+3)}\,, \tag{4.17}$$

so that from $(4.13)_2$, this integration procedure is possible for (4.4) provided the constants κ and λ are such that

$$\lambda = \left\{2 \pm \frac{(n+3)}{[2(n+1)]^{1/2}}\right\}\kappa\,. \tag{4.18}$$

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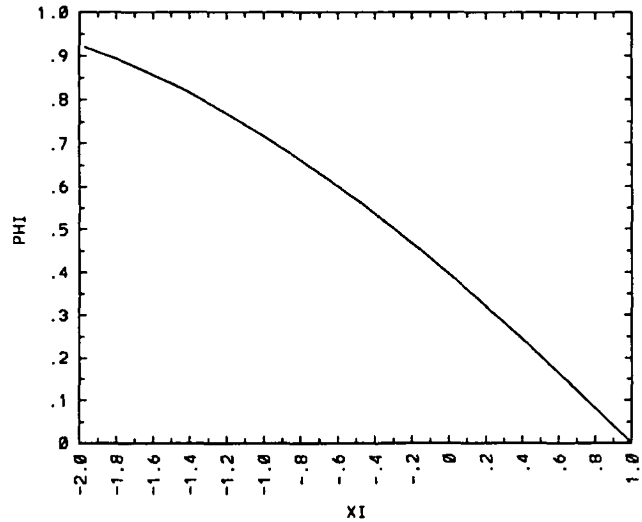
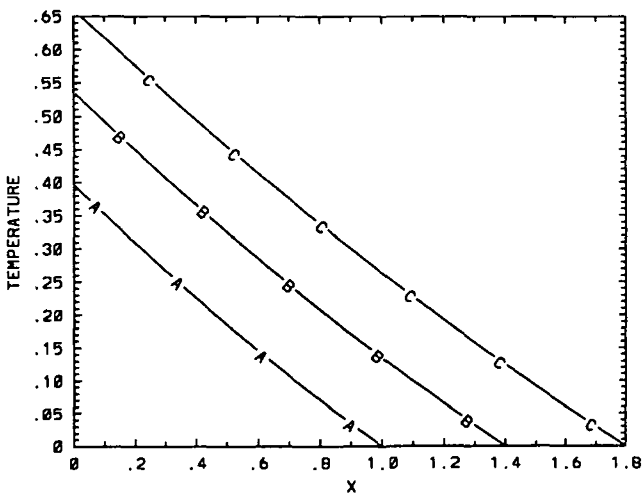


(a)



(b)

FIGURE 6. Variation of $\phi(\xi)$ and $T(x, t)$ for (4.2) at times $t = 0(A)$, $t = 1(B)$ and $t = 2(C)$ and with $\alpha = -0.1$, $\beta = 0.2$ and $n = 1/2$.

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Thus, in this case, for all n except $n \leq -1$ there is a real value of λ for which (4.4) can be formally integrated. From (4.3) and (4.18) we see that the possible values of the constant b are determined from the equation

$$\frac{1}{b} = -\left(\frac{\beta}{n-1}\right)^2 \left\{2 \pm \frac{(n+3)}{[2(n+1)]^{1/2}}\right\}. \tag{4.19}$$

Again for purposes of illustration, Figure 7(c) (see pp. 307–308) gives two temperature profiles corresponding to $t = 0(A)$ and $t = 10(B)$ for $n = 2$, and these are obtained by direct numerical integration of (4.16) employing the values $\alpha = \beta = 1$. The procedure adopted follows that used for (4.6) apart from the additional transformation (4.11). Since we require ξ in the range $-\infty < \xi \leq \xi_0$ for some positive ξ_0, this means that we require η in the range $-\infty < \eta \leq e^{\tau\xi_0}$ for $\tau > 0$ and $e^{\tau\xi_0} \leq \eta < \infty$ for $\tau < 0$. From (4.2) we see that $\lambda > 0$ represents heating while $\lambda < 0$ represents cooling, so for heating the appropriate λ values are

$$\begin{aligned} \lambda &= \left\{2 - \frac{(n+3)}{[2(n+1)]^{1/2}}\right\}\kappa, \qquad n > 1, \\ &= \left\{2 + \frac{(n+3)}{[2(n+1)]^{1/2}}\right\}\kappa, \qquad n < 1, \end{aligned} \tag{4.20}$$

and with this choice of λ, we find that $\tau < 0$ for $n > 1$ while $\tau > 0$ for $n < 1$. Thus for $-1 < n < 1$, we take λ given by $(4.20)_2$ and integrate over $-\infty < \eta \leq e^{\tau\xi_0}$ where $\tau > 0$ while for $n > 1$, we take λ given by $(4.20)_1$ and integrate over $e^{\tau\xi_0} \leq \eta < \infty$ where $\tau < 0$. From $(4.3)_1$ and these values of λ we see that the wave speed λ is discontinuous at $n = 1$ since we have

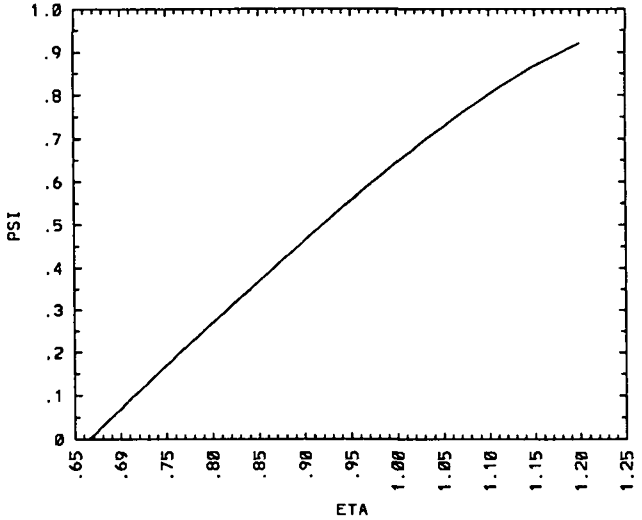
$$\lim_{n\to 1-} \lambda = \infty, \qquad \lim_{n\to 1+} \lambda = 0, \tag{4.21}$$

and moreover for $n > 1$, the wave speed λ increases slowly (for $\beta = 0.2$, $\lambda = 0.004$ for $n = 2$ while $\lambda = 0.008$ for $n = 5$). For $n = 2$ and $\alpha = \beta = 1$, Figures 7(a) and 7(b) show $\psi(\eta)$ as determined by(4.16) and the corresponding $\phi(\xi)$ as obtained from $(4.11)_1$. In utilising (4.16) we have taken the plus sign, $\eta_0 = 1$, $\psi_{\max} = 1$ and where the constant C is given by

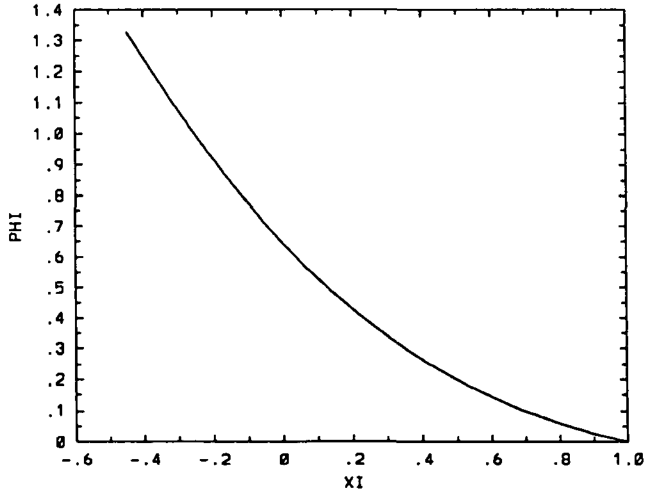
$$C = \frac{2\alpha\psi_{\max}^{n+1}}{(n+1)\tau^2} = \frac{2\alpha}{(n+1)\tau^2}. \tag{4.22}$$

In the special case of (4.1) corresponding to b zero we may deduce a simple separable solution as follows. In this case the functional form of the solution is simply

$$T(x, t) = e^{-\kappa x}\phi(t), \tag{4.23}$$

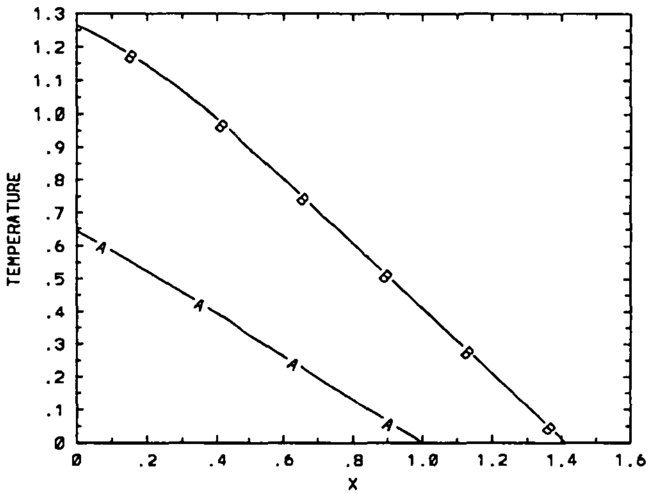


(a)



(b)

FIGURE 7. Variation of $\phi(\xi)$, $\psi(\eta)$ and $T(x, t)$ for (4.2) at times $t = 0(A)$ and $t = 10(B)$, and with $\alpha = \beta = 1$ and $n = 2$.



(c)

FIGURE 7. Continued.

where for $n \neq 1$, κ is still defined by $(4.3)_1$, and from (1.7) we may readily deduce the Bernoulli equation

$$\phi' = \kappa^2 \phi + \alpha \phi^n , \tag{4.24}$$

where here the prime denotes differentiation with respect to time. In the usual way we obtain

$$\phi(t) = \left\{ C e^{-(n-1)\kappa^2 t} - \alpha/\kappa^2 \right\}^{1/(1-n)} , \tag{4.25}$$

where C denotes the constant of integration. For $n < 1$ we may choose C such that initially the temperature is zero and the solution becomes

$$T(x, t) = e^{-\kappa x} \left\{ \frac{\alpha}{\kappa^2} (e^{(1-n)\kappa^2 t} - 1) \right\}^{1/(1-n)} . \tag{4.26}$$

If $n > 1$ then for $\beta > 0$, $\kappa < 0$ and $T(x, t)$ tends to infinity with x. In addition if $C > \alpha/\kappa^2$ then "blow-up" occurs after a finite time t_c given by

$$t_c = \frac{1}{(n-1)\kappa^2} \log \left(\frac{C\kappa^2}{\alpha} \right) . \tag{4.27}$$

Finally in this section, we note that the linear case $n = 1$ admits separable solutions of the form

$$T(x, t) = e^{-\Lambda t} y(x) , \tag{4.28}$$

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where Λ (assumed positive) denotes the separation constant and with the usual substitution $\xi = e^{-\beta x/2}$, y satisfies

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + \frac{4}{\beta^2}\left(\alpha + \frac{\Lambda}{\xi^2}\right)y = 0, \tag{4.29}$$

which has solutions of the form

$$y(\xi) = C_1 J_\nu(2\alpha^{1/2}\xi/\beta) + C_2 J_{-\nu}(2\alpha^{1/2}\xi/\beta), \tag{4.30}$$

where C_1 and C_2 denote arbitrary constants which are possibly complex and for positive Λ the index $\nu = 2i\Lambda^{1/2}/\beta$ is pure imaginary.

5. Exponential thermal conductivity and heat source

In this section we examine a similarity solution of (1.8) which applies for $\gamma \neq 0$ and $\delta \neq \gamma$. Solutions corresponding to these special cases are examined in the following sections. Equation (1.8) remains invariant under the one-parameter group of transformations

$$x_1 = x + a\varepsilon, \qquad t_1 = e^{\varepsilon}t, \qquad T_1 = T + b\varepsilon, \tag{5.1}$$

provided the constants a and b are given by

$$a = \frac{(\gamma - \delta)}{\beta\gamma}, \qquad b = -\frac{1}{\gamma}. \tag{5.2}$$

Two invariants of this group are $T - bx/a$ and $te^{-x/a}$, so that the functional form of the solution corresponding to (5.1) is

$$T(x, t) = \frac{bx}{a} + \phi(\xi), \qquad \xi = te^{-x/a}, \tag{5.3}$$

where as usual $\phi(\xi)$ is determined by substitution of (5.3) into (1.8). From these equations we observe that this solution is only sensible provided both a and γ are nonzero.

From (5.3) it is apparent that the above solution necessarily has an initial condition of the form

$$T(x, 0) = \frac{bx}{a} + \phi(0), \tag{5.4}$$

and would satisfy at $x = 0$ one of the following time dependent boundary conditions,

$$T(0, t) = \phi(t), \qquad \frac{\partial T}{\partial x}(0, t) = \frac{b}{a} - \frac{t}{a}\phi'(t), \tag{5.5}$$

where here primes denote differentiation with respect to time and as previously mentioned $\phi(t)$ is that function produced by solving the ordinary differential equation (5.6).

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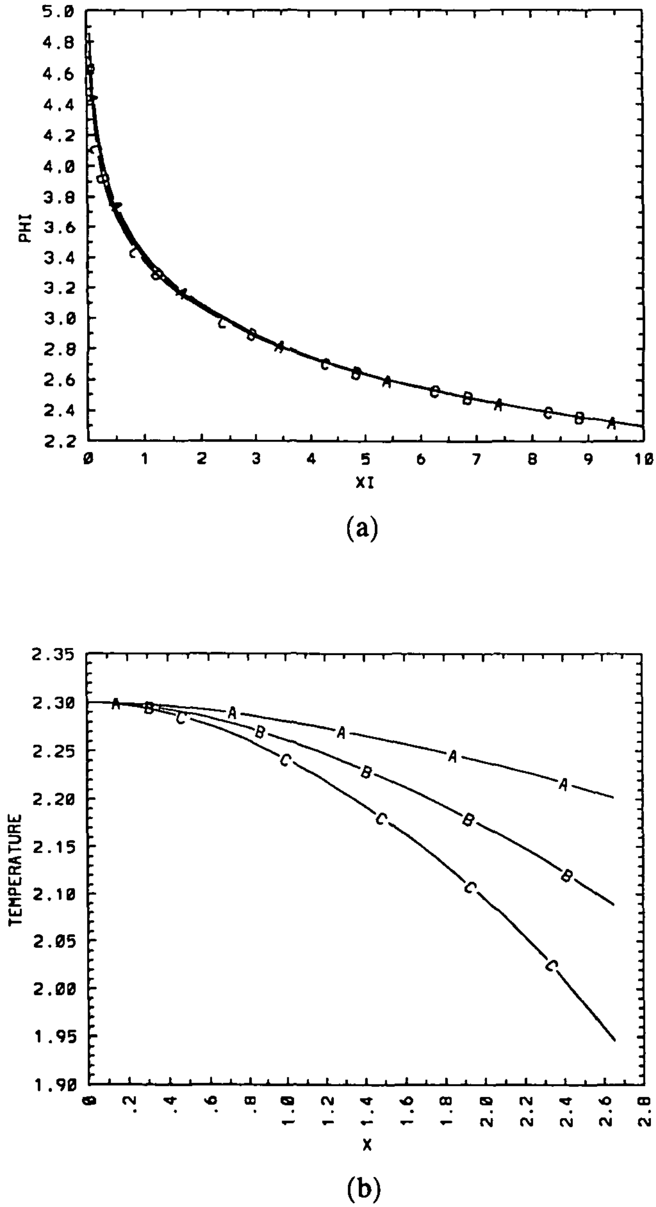
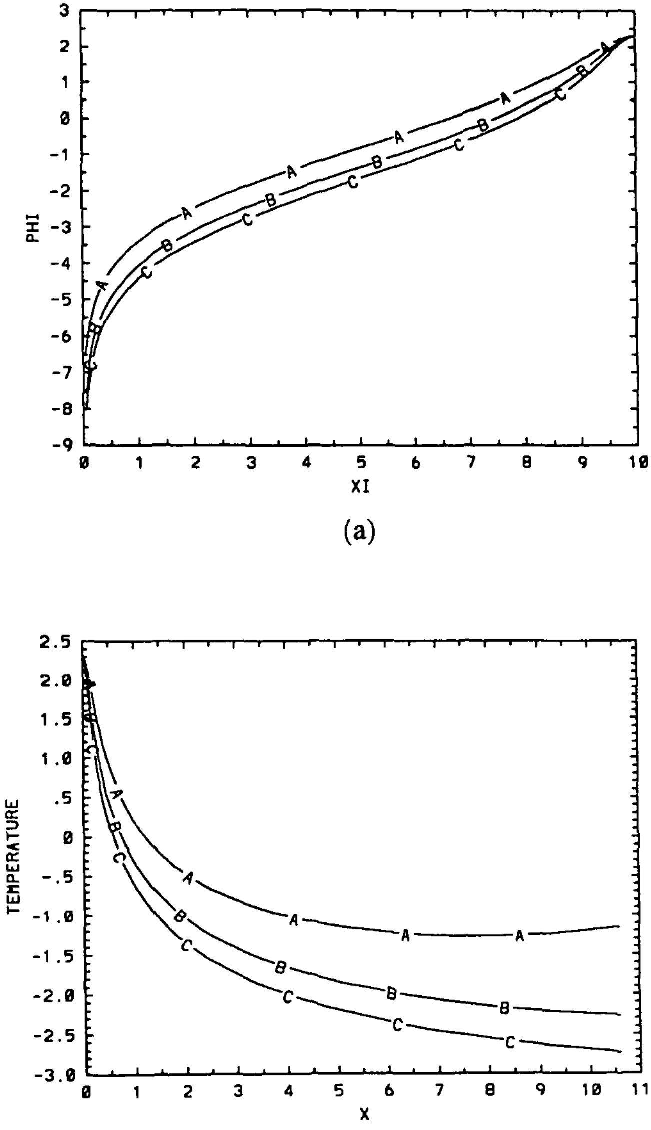


FIGURE 8. Variation of $\phi(\xi)$ and $T(x, t)$ for (5.3) for the case $\gamma > \delta$ and $\gamma > 0$ $(\gamma = 2, \delta = 1)$.



(a)

(b)

FIGURE 9. Variation of $\phi(\xi)$ and $T(x, t)$ for (5.3) for the case $\gamma < \delta$ and $\gamma < 0$ $(\gamma = -1, \delta = 1)$.

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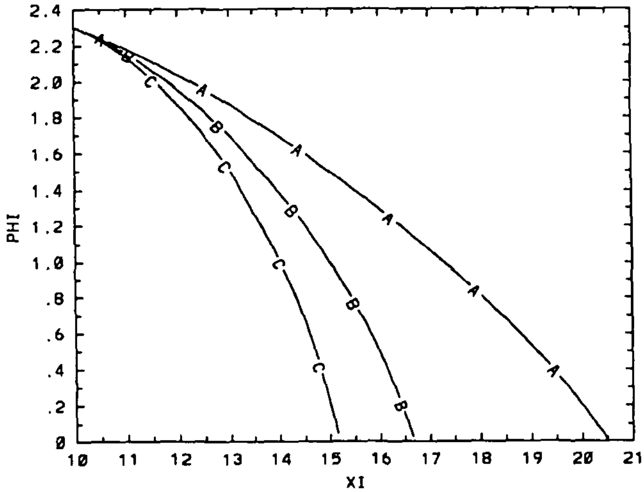

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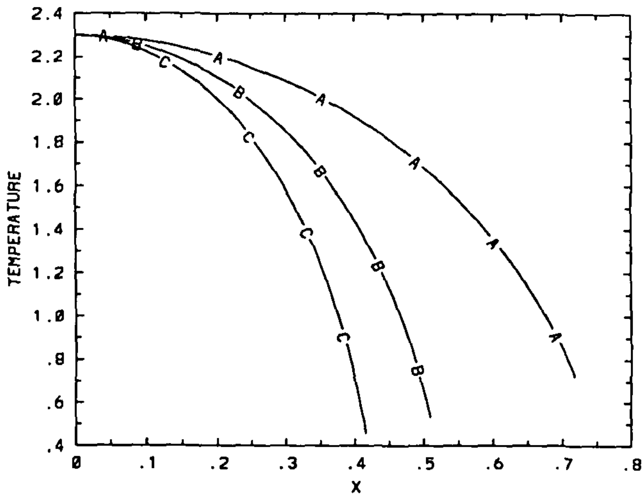

(b)

FIGURE 10. Variation of $\phi(\xi)$ and $T(x, t)$ for (5.3) for the case $\gamma < \delta$ and $2 > 0$ $(\gamma = 1, \delta = 2)$.

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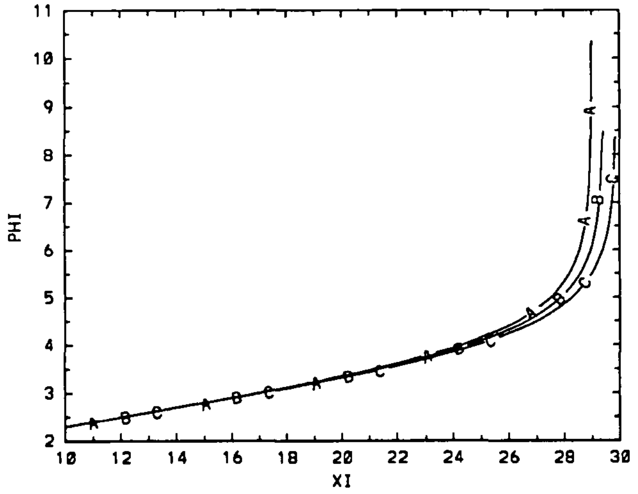

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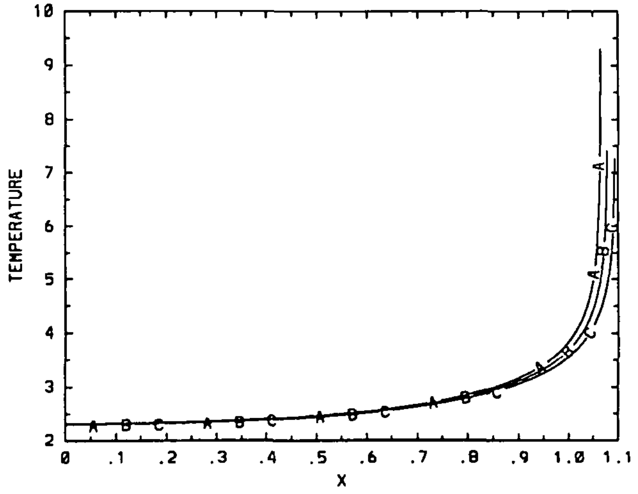

(b)

FIGURE 11. Variation of $\phi(\xi)$ and $T(x, t)$ for (5.3) for the case $\gamma > \delta$ and $\gamma < 0$ ($\gamma = -1$, $\delta = -2$).

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On substitution of (5.3) into (1.8), after some rearrangement we find

$$\frac{d\phi}{d\xi} = \frac{1}{a^2}\left\{\frac{d}{d\xi}\left[\xi e^{\gamma\phi}\left(\xi\frac{d\phi}{d\xi}+\frac{1}{\gamma}\right)\right]+\alpha a^2 e^{\delta\phi}\right\}, \tag{5.6}$$

and since this equation appears not to admit any simple first integrals, it must also be solved numerically. As previously described in Section 2 we replace (5.6) by a pair of first-order ordinary differential equations and assume that both $T(x, t)$ and $\frac{\partial T}{\partial x}(x, t)$ are prescribed on the boundary $x = 0$ at some fixed time $t = t_1$. In the numerical results, we adopt precisely the values given by (2.3), and consider the usual three values of α, namely $\alpha = 1/2(A)$, $\alpha = 1(B)$ and $\alpha = 3/2(C)$. The nature of the solution depends on whether a is positive or negative and in either case there are two possibilities.

For $a > 0$ there are two possibilities $\gamma > \delta$ and $\gamma > 0$ or $\gamma < \delta$ and $\gamma < 0$ which are shown in Figures 8 and 9 (see pp. 310–311) respectively. In the first case ($\gamma = 2$ and $\delta = 1$), $\phi(\xi)$ increases steadily as ξ decreases from $\xi = t_1$. This behaviour is shown in Figure 8(a) while the corresponding temperature variation is shown in Figure 8(b). We note that because of the different scales employed in these two figures, they are actually consistent with each other, which is not apparent from inspection. In the second case ($\gamma = -1$ and $\delta = 1$), $\phi(\xi)$ becomes unbounded and tends to minus infinity as ξ tends to zero, and therefore the initial condition (5.4) is not defined in this case. The behaviour is shown in Figure 9.

For $a < 0$, the two possibilities are $\gamma < \delta$ and $\gamma > 0$, and $\gamma > \delta$ and $\gamma < 0$ which are shown in Figures 10 and 11 (see pp. 312–313) respectively. In the first case ($\gamma = 1$ and $\delta = 2$), there exists some $\xi_0 > t_1$ such that both $\phi(\xi)$ and $\frac{d\phi}{d\xi}(\xi)$ rapidly approach minus infinity as ξ tends to ξ_0 indicating a moving front with $\phi(\xi) = -\infty$ for $\xi > \xi_0$. This behaviour is not clear from the portion of the curves shown in the figure. In the second case ($\gamma = -1$ and $\delta = -2$) both $\phi(\xi)$ and $\frac{d\phi}{d\xi}(\xi)$ tend to infinity as ξ tends to some ξ_0, as is clearly apparent from Figure 11. In this case the trivial solution for $\xi > \xi_0$ is $\phi(\xi) = \infty$.

6. Exponential dependence with $\delta = \gamma$

If $\delta = \gamma$ then $a = 0$ and the one-parameter group (5.1) becomes

$$x_1 = x, \qquad t_1 = e^{\varepsilon}t, \qquad T_1 = T - \varepsilon/\gamma, \tag{6.1}$$

in which case two invariants are x and $T + \gamma^{-1}\log t$ and therefore the functional form of the solution becomes

$$T(x, t) = -\log t/\gamma + \phi(x). \tag{6.2}$$

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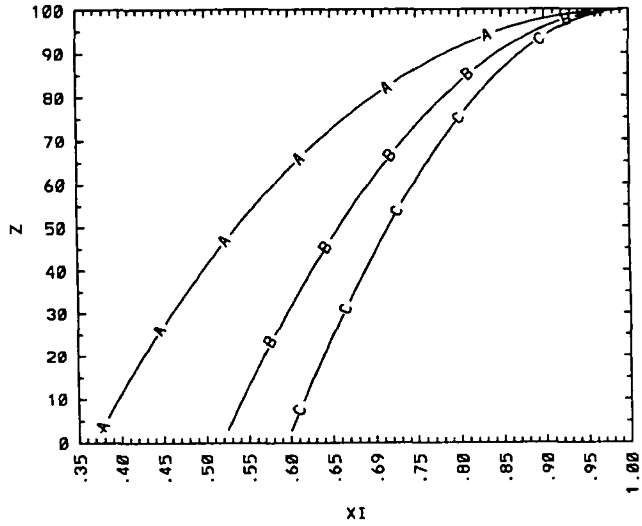
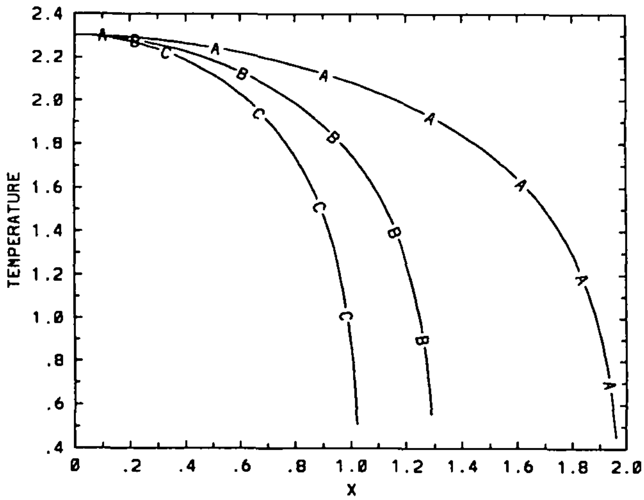


(a)



(b)

FIGURE 12. Variation of $z(\xi)$ and $T(x, t)$ for (6.2) at time $t_1 = 1$ and with $\beta = 1.0$ and $\gamma = 2.0$.

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This solution has the form $T(x, t) = f(x) + g(t)$ and we can readily show that the condition $\delta = \gamma$ is an essential condition for the existence of solutions of this form. Substitution of (6.2) into (1.8) with $\delta = \gamma$ readily yields

$$\frac{d}{dx}\left(e^{\gamma\phi}\frac{d\phi}{dx}\right) + \alpha e^{-\beta x}e^{\gamma\phi} = -\frac{1}{\gamma}, \tag{6.3}$$

and the substitution $z = e^{\gamma\phi}$ gives

$$\frac{d^2 z}{dx^2} + \alpha\gamma e^{-\beta x} z = -1. \tag{6.4}$$

This equation can be solved in a routine manner by means of the transformation $\xi = e^{-\beta x/2}$, which gives an inhomogeneous Bessel's equation, namely

$$\frac{d^2 z}{d\xi^2} + \frac{1}{\xi}\frac{dz}{d\xi} + \frac{4\alpha\gamma}{\beta^2} z = -\frac{4}{\beta^2\xi^2}, \tag{6.5}$$

which has the general solution

$$z(\xi) = C_1 J_0(\eta) + C_2 Y_0(\eta) + \frac{2\pi}{\beta^2}\int_\eta^\infty [J_0(s)Y_0(\eta) - Y_0(s)J_0(\eta)]\frac{ds}{s}, \tag{6.6}$$

where C_1 and C_2 denote arbitrary constants and $\eta = 2(\alpha\gamma)^{1/2}\xi/\beta$.

Figure 12 (see p. 315) shows the typical variation of $z(\xi)$ and temperature profile $T(x, t)$ which are obtained by numerical integration of (6.5), rather than use of (6.6), and the following values have been employed:

$$\beta = 1, \quad \gamma = 2, \quad t_1 = 1, \quad T(0, t_1) = 2.3, \quad \frac{\partial T}{\partial x}(0, t_1) = 0, \tag{6.7}$$

and as usual A, B and C designate the three reference values of α, namely $1/2$, 1 and $3/2$ respectively. These results again indicate the appearance of a moving front with $T(x, t) = -\infty$ as the appropriate value of the temperature outside the front.

7. Exponential dependence with γ zero

If γ is zero, then on introducing a new temperature variable $\overline{T}$ defined by

$$\overline{T}(x, t) = \delta T - \beta x, \tag{7.1}$$

we see that (1.8) becomes

$$\frac{\partial \overline{T}}{\partial t} = \frac{\partial^2 \overline{T}}{\partial x^2} + \delta e^{\overline{T}}, \tag{7.2}$$

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which is formally identical to the simple model used by Hill and Smyth [4], which does not incorporate spatial exponential decay of the source term. In this case (7.2) remains invariant under the one-parameter group of transformations

$$x_1 = e^{\varepsilon} x, \qquad t_1 = e^{2\varepsilon} t, \qquad \overline{T}_1 = \overline{T} - 2\varepsilon, \tag{7.3}$$

for which the invariants are $xt^{-1/2}$ and $\overline{T} + 2\log x$, so that the functional form of the solution becomes

$$\overline{T}(x, t) = -2\log x + \phi(\xi), \qquad \xi = xt^{1/2}. \tag{7.4}$$

We observe that in terms of the original temperature variable T, this solution necessarily has an initial condition of the form

$$T(x, 0) = \{\beta x - 2\log x + \phi(\infty)\}/\delta. \tag{7.5}$$

On substituting (7.4) into (7.2) we can readily deduce the second-order nonlinear ordinary differential equation,

$$\xi^2\phi'' + \tfrac{1}{2}\xi^3\phi' + 2 + \alpha\delta e^{\phi} = 0, \tag{7.6}$$

which again needs to be solved by a numerical scheme. We observe that if we use

$$\overline{T}(x, t) = -\log t + \psi(\xi), \qquad \xi = xt^{-1/2}, \tag{7.7}$$

instead of (7.4), then in this case we obtain from (7.2) the slightly simpler differential equation

$$\psi'' + \frac{\xi\psi'}{2} + 1 + \alpha\delta e^{\psi} = 0, \tag{7.8}$$

which can be reconciled with (7.6) by means of the relation

$$\phi(\xi) = \psi(\xi) + 2\log\xi. \tag{7.9}$$

Altogether we have from (7.1) and (7.4) that the temperature $T(x, t)$ is given by

$$T(x, t) = \frac{1}{\delta}\left\{\beta x - 2\log(x + x_0) + \phi\left(\frac{x + x_0}{(t + t_0)^{1/2}}\right)\right\}, \tag{7.10}$$

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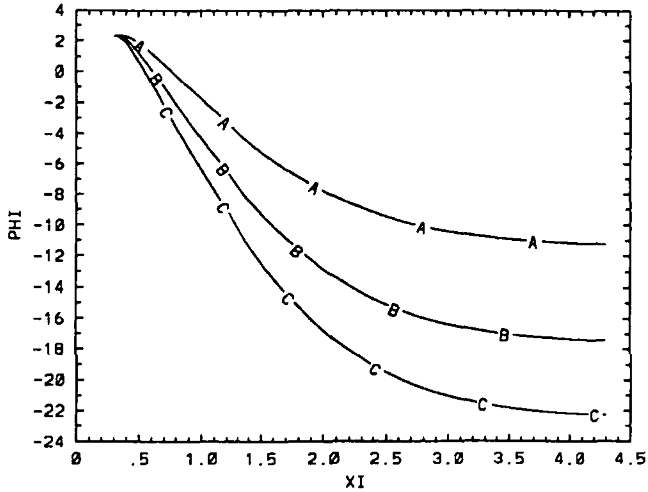

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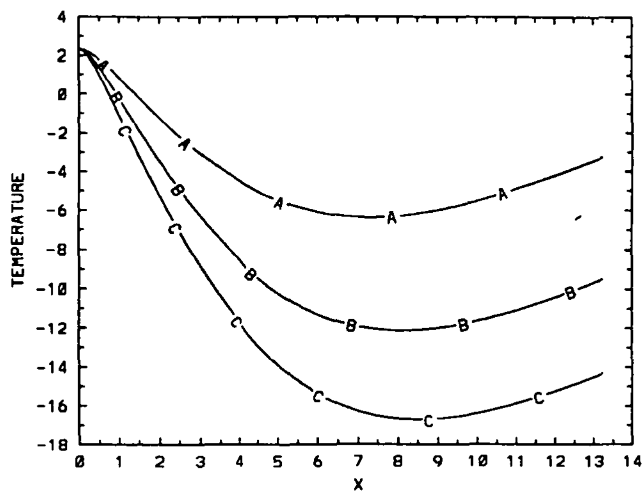

(b)

FIGURE 13. Variation of $\phi(\xi)$ and $T(x, t)$ for (7.10) at time $t_1 = 10$ and with $\beta = \delta = x_0 = t_0 = 1$.

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for arbitrary constants x_0 and T_0. Figure 13 shows the variation of $\phi(\xi)$ and $T(x, t)$ for (7.10) assuming the following values

$$t_1 = 10, \quad \beta = \delta = x_0 = t_0 = 1, \quad T(0, t_1) = 2.3, \quad \frac{\partial T}{\partial x}(0, t_1) = 0, \tag{7.11}$$

and as usual A, B and C designate the three reference values of α.

8. Conclusion

We have attempted to model the microwave heating of a infinite slab by equations of the form (1.7) and (1.8), where α and β designate positive constants. These models incorporate a heat source term which decays exponentially with distance, and increases with increasing temperature. We have examined simple similarity temperature profiles for special cases of the models. These solutions enable the partial differential equations to be reduced to ordinary differential equations for which we have obtained numerical solutions. Since for the similarity solutions under discussion we are not at liberty to impose arbitrary boundary and initial conditions we have adopted the strategy of solving the various ordinary differential equations assuming that both the temperature $T(x, t)$ and the temperature gradient $\frac{\partial T}{\partial x}(x, t)$ are prescribed at the boundary $x = 0$ at some fixed time $t = t_1$, and we are then able to display the temperature profile at this fixed time t_1. These numerical results indicate the appearance of moving fronts. For the model (1.7) with both m and n positive, the fronts move into a region of zero temperature which is a valid trivial solution of the governing equation. This model is therefore entirely consistent with the observed characteristics of microwave heating of materials which are known to exhibit all the classical phenomena associated with nonlinear diffusion, such as "blow-up" which is referred to as "hot-spots" and "waiting-time" phenomena. That is, materials are known to remain at the initial temperature for a finite time when subjected to microwave radiation and then suddenly the temperature starts to increase. Moreover, certain materials are known to be either completely transparent to microwave radiation or respond after the application of convential heating, and these characteristics are also embedded in the model (1.7). However, although the model (1.8) also predicts moving fronts and typical phenomena associated with nonlinear diffusion, the associated trivial solution of (1.8) for γ and δ both positive is $T = -\infty$ while for γ and δ both negative it is $T = \infty$. These may be physically unrealistic and in addition, because $T = 0$ is not a trivial solution of (1.8) the model may not admit the possibility that certain materials can be transparent to microwave radiation.

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